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ON DUALISTIC EQUILIBRIUM AND TECHNICAL CHANGE IN A SIMPLE HUMAN SETTLEMENT MODEL

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Preface

This paper is the fourth in a series on 'Regional Development and Land-Use Models'. The purpose of this series is to consider the application of optimizing and behavioural land-use models as tools in the study of regional development. The present paper considers the problem of the impact of economic growth on regional land-use patterns. A theoretical model of a simple spatial economy is developed. The model can be used to trace out the implications of different kinds of economic growth. The 'saturation' principle identified there is a useful, and potentially very important, concept which should be included in more applied models of regional development. This is viewed as the first in a group of papers concerned with models of private sector behaviour in regional development.

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PAPERS IN THE REGIONAL DEVELOPMENT AND LAND-USE MODELS SERIES

(1) John R. Miron, "Regional Development and Land-Use Models: An Overview of Optimization Methodology", RM-76-27. April, 1976.

(2) Ross D. MacKinnon, "Optimization Models of Transportation Network Improvement: Review and Future Prospects", RM-76-28. April, 1976.

(3) John R. Miron, "Alternative Land-Use Policy Tools for Green Area Preservation in Regional Development", RM-76-29. April, 1976.

(4) John R. Miron, "On Dualistic Equilibrium and Technical Change in a Simple Human Settlement Model". RM-76-30. April, 1976.

Abstract

A formal micro-economic model of a simple dualistic spatial economy is outlined. The equilibrium solution to this model includes a measure of population density whose level is dependent on the technical parameters of the economy. Numerical experiments with the model highlight the importance of a saturation principle in determining how economic growth affects the spatial pattern of population density. Although the model is abstract, the saturation principle is seen to be an important concept for future applied behavioural models of regional development.

ON DUALISTIC EQUILIBRIUM AND TECHNICAL CHANGE IN A SIMPLE HUMAN SETTLEMENT MODEL

John R. Miron

Settlement system policies can not be designed without models and theories of the processes which underly the current spatial pattern of population. The scarcity of empirical models of settlement systems would therefore seem to be incomprehensible given the current near-universal concern with such policies. However, this paucity reflects somewhat an inadequate theoretical basis for a policy-relevant applied model. The lack of a firm theory has in part been attributed to the complexity of dealing with a system whose elements usually have considerable locational flexibility. Where, for instance, can one find a model or theory which explains relocation processes caused by economic growth and technological change?

The purpose of this paper is to make a contribution towards a theory of a settlement system. A model of a simple spatial economy is presented in which a spatial equilibrium distribution of population can be defined. This model is based on classical micro-economic theory and presumes a competitive land market wherein Ricardian land rents preserve the equilibrium. This model can be pursued in several interesting ways. Here, emphasis is placed on the implications for spatial density patterns of those parameter variations which might represent technical change and economic growth. It is shown that this formal micro-economic model gives several interesting deductions about the changes in spatial behaviour associated with economic development. The model emphasizes differences among market areas and specifically the role played by a kind of market 'saturation'.

The model used in this paper contains two kinds of economic units. The first is a factory, occupying a spaceless point, and producing a single output. The second is a set of farms with each using land and producing two goods (one land-using and the other a perfect substitute for the factory good). These farms are each capable of varying the amount of land they use in response to local economic factors and it is the resulting spatial density of farms which is of central concern in this paper. Two sub-models are defined; one for the factory and one for the farm at a given distance from the factory. These sub-models are linked by prices and market equilibria conditions. The effect of technological change (represented as parameter variations) in either sub-model on the spatial density of farms can thus be examined.

It might be argued that a model based on farming units is not very useful to the analysis of settlement systems which are overwhelmingly urban in nature. The Losch-Christaller models of an urban system for example emphasize multiple market-thresholds in defining hierarchies of urban centers. However, these same models base all urban structure on a rural hinterland which is assumed to be uniformly dense through space. The present model can be viewed as the replacement of this assumption by a model of the hinterland and a lowest-threshold factory which endogenously determines the spatial pattern of farm densities. At the same time, this model can in a general sense represent the effect of a centre anywhere in the urban hierarchy on its whole hinterland including dominated lower-order centres. Under either of these two interpretations, the present model is a contribution toward a better understanding of settlement systems.

I. THE STUCTURE OF THE MODEL

(a) THE FARM SUB-MODEL: ASSUMPTIONS

The farm sub-model used in this paper has been developed by the author in an earlier paper.[1] The assumptions, definitions, and hypotheses from that paper are reviewed quickly here. The solution to this model is extended to cover a special corner solution case of some subsequent interest.

Begin by assuming three sets of actors associated with a very large homogenous plane; (i) a single factory at some fixed space-less point on the plane, (ii) a very large set of farms occupying the remainder of this plane at an everywhere finite density, and (iii) a large set of absentee (from the plane) landlords. Each landlord is identical in certain respects. Each resides outside the region and also spends his land rent income there. Each attempts to maximize the rent received for his unit of land. However, each landlord behaves competitively in that there are no collusive agreements and no landlord possesses enough land to behave monopolistically.

Each farm behaves as a unit maximizing its well-behaved utility function. Each has the same total amount of labour, h, which it allocates among activities in fulfilling this goal. Further, each farm (or labour unit as it might equally be referred to) can locate wherever it chooses if its bid rent is the highest offered for that site. Any relocation is itself assumed to be costless. Every farm is also free to choose the amount of land (L) to be occupied by it. Each has the same, strictly-convex, utility function (U) defining its preference orderings over consumption of two infinitely divisible goods; soap (X) and food (Y).[2] Each produces a gross output of food, (Q) using labour (h_y) and the land area of the farm as inputs with decreasing returns to scale. This gross output can be divided into rental payments, RL, and a remainder termed net food output, Y_1. The farm also produces an output of soap, X_1, which is tied solely to its labour input, h_x, with constant returns to scale. Finally the farm can also allocate labour services to the factory in the amount of h_z units although the total labour constraint must not be exceeded.

It is assumed that the factory offers to trade its own soap for farm food at a given mill price, P_b, where food is the numeraire. With freight costs proportional to distance, the delivered price, P(s), increases with distance 's' from the factory. The farm at distance 's' offers to purchase an amount of soap, X_2, from the factory for which it gives up $P(s)\ X_2$ units of food. The factory also offers employment at a wage of w_b units of food per unit of labour. Distance-proportional commuting costs decrease the net wage w(s), received by the farm.

Finally, it is assumed that all farms are in a state of equilibrium such that there is no incentive for any farm to alter its location or production-consumption combinations. Given that the farm optimizes its production-consumption bundle at any location, the Ricardian rent level, R, is such as to permit each household to achieve at best the same level of utility at every location.

The specific structure of the farm sub-model is outlined in Table 1. Note that all variables are assumed to be non-negative. Further, there is assumed to be a uniform positive land rent, R*, which forms a floor for all land rents (i.e., $R(s) \geqslant R^*$).

Table 1: Structural Equations in the Farm Sub-Model.

$U = X^{\alpha} Y^{1-\alpha}$	$(0 < \alpha < 1)$
$Q = b h_y^{\beta} L^{\gamma}$	$(b, \beta, \gamma > 0 \;\; ; \;\; \beta + \gamma < 1)$
$Y_1 = Q - RL$	
$X_1 = c h_x$	$(c > 0)$
$P(s) = P_b + ts$	$(t > 0)$
$w(s) = w_b - rs$	$(r > 0)$
$h = h_x + h_y + h_z$	$(h > 0)$
$X = X_1 + X_2$	
$Y = Y_1 + w(s)\, h_z - P(s)\, X_2$	

Source: See text

(b) THE FARM SUB-MODEL: SOLUTION

The assumptions of the model render it an 'open' model in the sense of Wheaton (1974). The uniform rent R*, occurring without the factory, continues to exist in the presence of the factory although only outside its market areas. The factory is small enough that it has no effect on these outlying areas even though density and rent levels change within its market areas. Because the factory has no effect on conditions at or beyond its largest market boundary and because the total number of farms within its market areas is variable, the model is said to be open.[3]

As discussed in Miron (1975, pp. 155-156), there are three equilibrium solutions to this sub-model ignoring a possible corner solution. These solutions correspond to different distance ranges from the factory. A distance s^* can be found beyond which the delivered price of factory soap is so high that each farm moves to a state of autarky (i.e., $X_2=0$). Another distance s^Δ can be defined beyond which the farm chooses not to allocate any labour to factory work (i.e., $h_z=0$). For simplicity of presentation, it is assumed that $s^\Delta < s^*$. Thus, we may define an autarky solution where $s \geqslant s^*$, an M_1 solution when $s^\Delta \leqslant s \leqslant s^*$, and an M_2 market solution where $0 \leqslant s \leqslant s^\Delta$.

These solutions are discussed extensively in Miron (1975; pp. 156-162) and are summarized here in Tables 2,3, and 4. The principal differences between the M_1 and M_2 solutions occur because of the constant marginal productivity of labour in factory work and in on-farm soap production. Because these are constant, the farm engages in at most one of these two activities at any given distance from the factory.[4] These are factory labour in the M_2 zone and on-farm soap production in the M_1 zone. Thus in the M_2 solution it is noted that h_x and X_1 are zero while h_z is zero in the M_1 solution.

A very significant difference between the M_2 and M_1 market solutions can now be established. In both markets, the total demand (X) for soap by any farm is inelastic with respect to the delivered (or even the mill) price.[5] In the M_2 market, the demand for purchased soap (X_2) is also inelastic because the lack of farm soap production makes purchased demand equivalent to total demand. In the M_1 market, there is some on-farm production of soap. The demand for purchased soap here is more elastic than in the M_2 market because of the possibility of substitution between soap purchases and production on the farm. Another way to express this is to say that the

Table 2: Autarky Solution ($s \geq s^*$).

$$U = U^*$$
$$X = c(1 - k_1)\ h$$
$$Y = (1 - \gamma)\ b(\gamma b/R^*)^{\gamma/(1-\gamma)}(k_1 h)^{\beta/1-\gamma}$$

$$h_x = (1 - k_1)\ h$$
$$h_y = k_1 h$$
$$h_z = 0$$

$$L = (k_1 h)^{\beta/(1-\gamma)}(\gamma b/R^*)^{1/(1-\gamma)}$$
$$R = R^*$$

$$Q = b(k_1 h)^{\beta/(1-\gamma)}(\gamma b/R^*)^{\gamma/(1-\gamma)}$$
$$Y_1 = (1 - \gamma)\ b(k_1 h)^{\beta/(1-\gamma)}(\gamma b/R^*)^{\gamma/(1-\gamma)}$$

$$X_1 = c(1 - k_1)\ h$$
$$X_2 = 0$$

$$P = b^{1/(1-\gamma)}(\beta/c)(\gamma/R^*)^{\gamma/(1-\gamma)}(k_1 h)^{-(1-\gamma-\beta)/(1-\gamma)}$$

where

$$k_1 = \left[\beta(1 - \alpha)\right]/\left[\beta(1 - \alpha) + \alpha(1 - \gamma)\right]$$
$$U^* = \left[c(1 - k_1)\ h^{1/k_1}\right]^{\alpha}\left[(1 - \gamma)\ b^{1/(1-\gamma)}(\gamma/R^*)^{\gamma/(1-\gamma)}k_1^{\beta/(1-\gamma)}\right]^{1-\alpha}$$

Source: Miron (1975; pp. 156-157).

Table 3: M_1 Market Solution ($s_2^{\Delta} \leq s \leq s^*$).

$$U = U^*$$
$$X = \alpha k_2 P(s)^{\alpha-1}$$
$$Y = (1 - \alpha)\ k_2 P(s)^{\alpha}$$

$$h_x = \left[(1 - \gamma)\ chP(s) - \beta k_2 P(s)^{\alpha}\right] / \left[(1 - \beta - \gamma)\ cP(s)\right]$$
$$h_y = \left[\beta g_a(s)\right] / \left[cP(s)\right]$$
$$h_z = 0$$

$$L = b^{-1/\gamma} \left[cP(s)/\beta\right]^{\beta/\gamma}\ g_a(s)^{(1-\beta)/\gamma}$$
$$R = b^{1/\gamma}\ \gamma \left[cP(s)/\beta\right]^{-\beta/\gamma}\ g_a(s)^{-(1-\beta-\gamma)/\gamma}$$

$$Q = g_a(s)$$
$$Y_1 = (1 - \gamma)\ g_a(s)$$

$$X_1 = \left[(1 - \gamma)\ chP(s) - \beta k_2 P(s)^{\alpha}\right] / \left[(1 - \beta - \gamma)\ P(s)\right]$$

$$X_2 = \left[(1 - \alpha)/[(1 - \beta - \gamma)(1 - k_1)]\right]\left[\alpha k_2 P(s)^{\alpha-1} - c(1 - k_1)\ h\right]$$

$$P(s) = P_b + ts$$

where

$$g_a(s) = \left[k_2 P(s)^{\alpha} - chP(s)\right] / \left[1 - \beta - \gamma\right]$$
$$k_2 = U^* \alpha^{-\alpha} (1 - \alpha)^{-(1-\alpha)}$$

Source: Miron (1975; pp. 157-160).

Table 4: M_2 Market Solution ($0 \leq s \leq s_1^{\Delta}$).

$$U = U^*$$
$$X = \alpha k_2 P(s)^{\alpha-1}$$
$$Y = (1 - \alpha)\ k_2 P(s)^{\alpha}$$

$$h_x = 0$$
$$h_y = \beta g_b(s)/w(s)$$
$$h_z = \Big[(1 - \gamma)\ hw(s) - \beta k_2 P(s)^{\alpha}\Big] / \Big[(1 - \beta - \gamma)\ w(s)\Big]$$

$$L = b^{-1/\gamma}[w(s)/\beta]^{\beta/\gamma}\ g_b(s)^{(1-\beta)/\gamma}$$
$$R = \gamma b^{1/\gamma}[w(s)/\beta]^{-\beta/\gamma}\ g_b(s)^{-(1-\beta-\gamma)/\gamma}$$

$$Q = g_b(s)$$
$$Y_1 = (1 - \gamma)\ g_b(s)$$

$$X_1 = 0$$
$$X_2 = \alpha k_2 P(s)^{\alpha-1}$$

$$P(s) = P_b + ts$$
$$w(s) = w_b - rs$$

where

$$g_b(s) = \Big[\ k_2 P(s)^{\alpha} - w(s)\ h\Big]/[1 - \beta - \gamma]$$

Source: Miron (1975; pp. 160-162).

M_2 area is saturated because the factory supplies all the soap consumed there while it supplies only a portion of the soap market in the M_1 area.

(c) THE FARM SUB-MODEL: CORNER SOLUTIONS

Two problems emerge with this model by way of corner solutions. The first occurs where the farm allocates all labour to food production (i.e., $h_y=h$). In the M_2 market, it is noted that h_y is an increasing function of 's' while, in the M_1 area, it is decreasing. Thus if h_y equals h anywhere, it occurs in a band of s-values around s^Δ. In particular, this corner solution emerges when $s_1^\Delta < s < s_2^\Delta$ where the limits are defined by[6]

(1.a) $s_2^\Delta = \left[\beta k_2/\left(ch(1-\gamma)\right)\right]^{1/(1-\alpha)}/t \quad - \quad P_b/t$

(1.b) $w(s_1^\Delta) = \left[\beta k_2/\left(ch(1-\gamma)\right)\right] \; P(s_1^\Delta)^\alpha$

and such that

(1.c) $s_2^\Delta \geq s^\Delta$ and $s_1^\Delta \leq s^\Delta$

The two conditions in (1.c) can be shown to be equivalent.

We can define an M_3 market area, when (1.c) holds, for $s_1^\Delta < s < s_2^\Delta$. Such an area is illustrated in Figure 1. The solution for each optimized variable can be determined in a manner similar to that outlined in the earlier paper. The derived optimal solutions are presented in Table 5. Note that the inelastic demand for factory soap by a farm is a feature shared with the M_2 market solution. In this sense, both the M_2 and M_3 areas can be thought to have saturated demands for soap.

An M_3 market area need not exist in this model. It is just one possibility. However, for simplicity of presentation, we shall assume that it always exists. In the case where it doesn't, the values s_1^Δ and s_2^Δ can be thought to converge to s^Δ so that the M_3 market vanishes.

The measure of population density used in this paper is h/L. This can be interpreted as the total number of labour units (e.g., man-hours) per unit area. This, of course, is proportional to the number of farms per unit area. An example of how h/L behaves in the different market areas is displayed in Figure 2. This example illustrates the monotonically-declining densities with distance

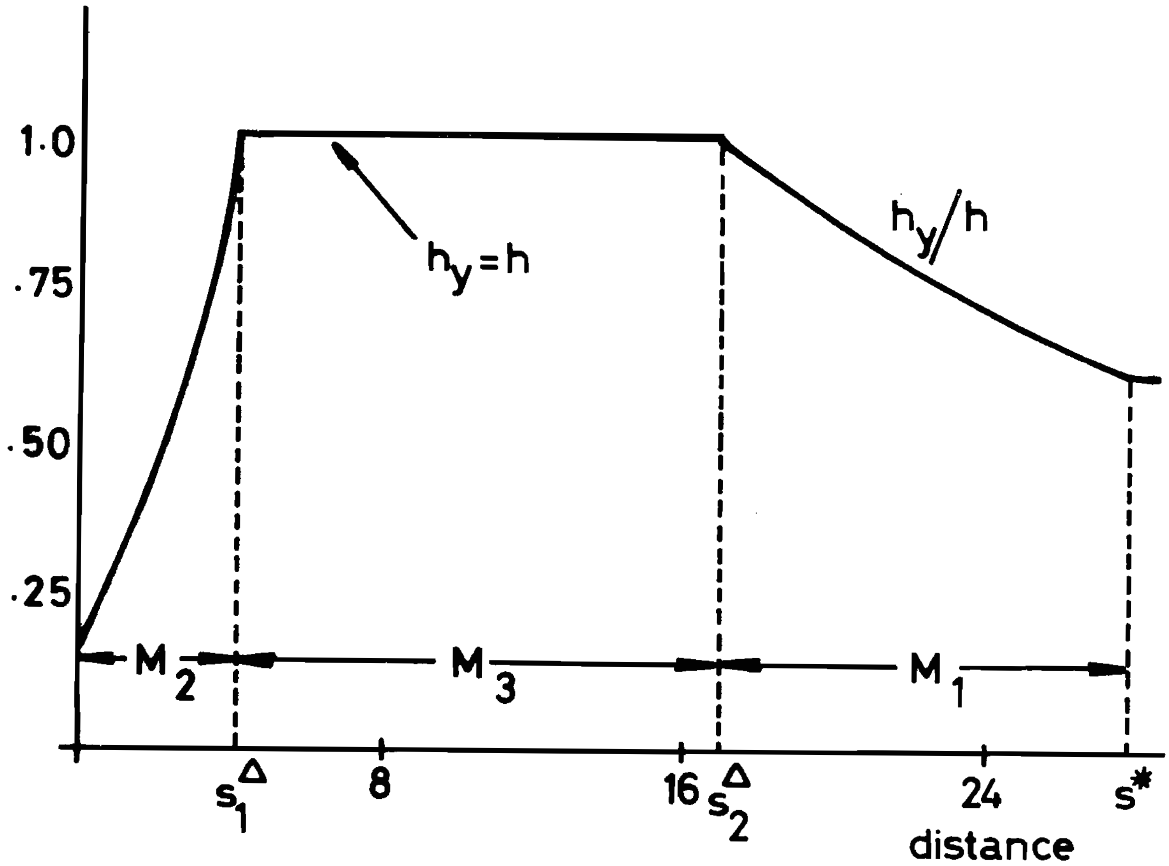

Figure 1. Food labour input as a function of distance from the factory; base run solution.

Table 5: M_3 Market Solution $(s_1^{\Delta} \leq s \leq s_2^{\Delta})$.

$$U = U^*$$
$$X = \alpha k_2 P(s)^{\alpha-1}$$
$$Y = (1 - \alpha)\ k_2 P(s)^{\alpha}$$

$$h_x = 0$$
$$h_y = h$$
$$h_z = 0$$

$$L = (bh^{\beta})^{-1/\gamma}\ g_c(s)^{1/\gamma}$$
$$R = \gamma(bh^{\beta})^{1/\gamma}\ g_c(s)^{-(1-\gamma)/\gamma}$$

$$Q = g_c(s)$$
$$Y_1 = (1 - \gamma)\ g_c(s)$$

$$X_1 = 0$$
$$X_2 = \alpha k_2 P(s)^{\alpha-1}$$

$$P(s) = P_b + ts$$

where

$$g_c(s) = \left[k_2/(1 - \gamma)\right]\ P(s)^{\alpha}$$

Source: See text.

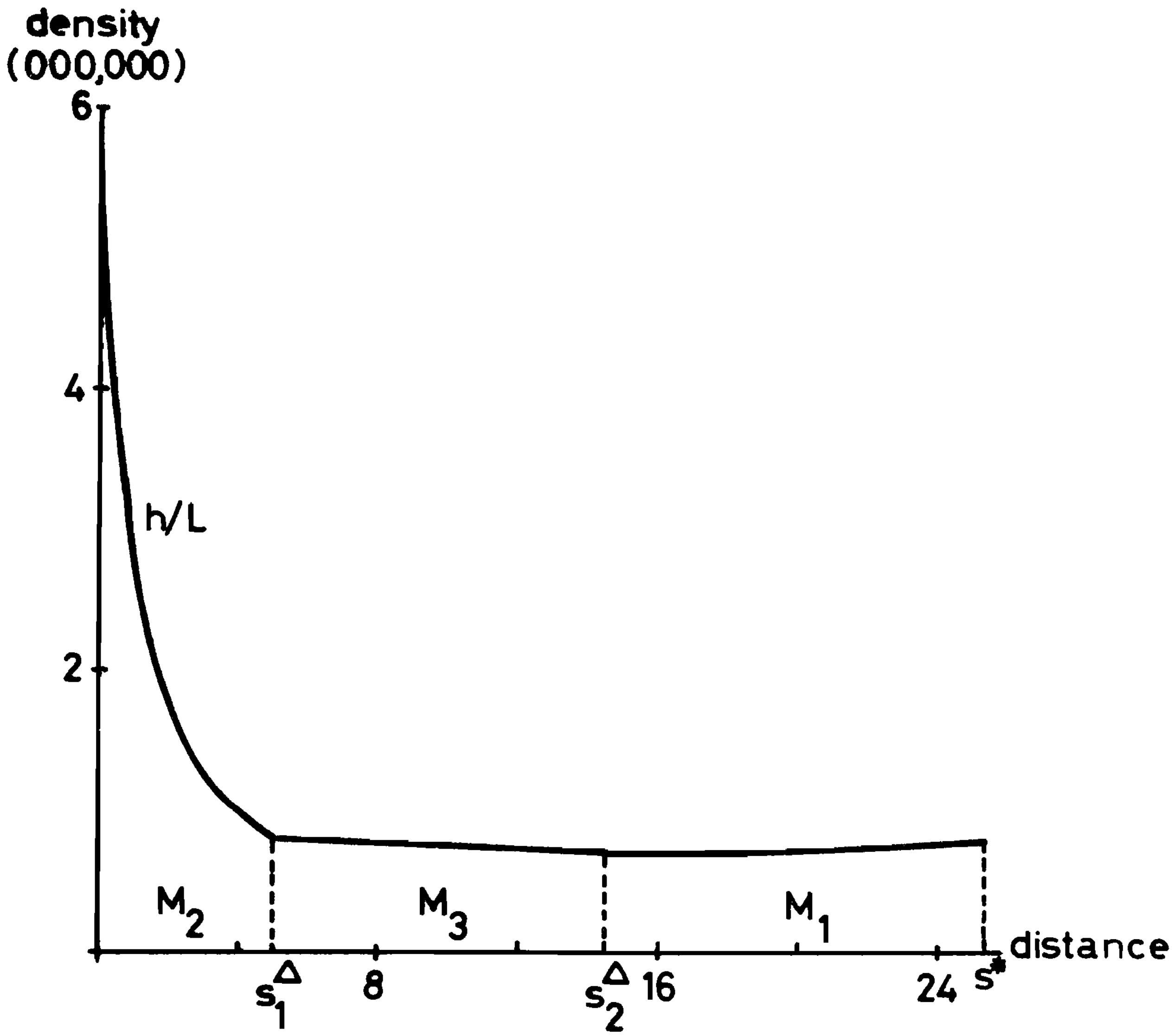

Figure 2. Population density as a function of distance from factory when $k_o = 5.0$.

found in the M_2 and M_3 areas in contrast to the M_1 zone where the density is increasing. As argued in the earlier paper, density always increases with distance for at least some range of distances adjacent to s* in the M_1 market.

The second corner solution problem is much more difficult to reconcile. This problem occurs when the farm allocates all labour to the factory. This is significant because farms near the factories would no longer consume any land, the density of population would then become infinite, and a utility equilibrium could not be maintained by a scheme of Ricardian land rents.[7] The model loses its interpretability in such a situation. In the results described in the remainder of this paper, attention was paid to ensure that such a solution was not approximated.[8]

(d) THE FACTORY SUB-MODEL

In specifying a model of producer behaviour in space, two important choices have to be made. In both cases, the simpler option has been chosen. The first choice concerns an assumption about the existence of competition. Two equilibrium cases are usually considered in the literature; pure monopoly and spatial monopolistic competition. The former assumes the factory to be the only one on the plane while the latter assumes many identical factories (equally spaced) with the smallest overlapping areas such that all consumers are served.[9] The difference involves both (i) treatment of the firm's boundary which moves from a circle in pure monopoly to a hexagon in monopolistic competition and (ii) the determination of optimal behaviour at this boundary. The pure monopoly case is assumed here because of its relative simplicity although the Christaller and Loschian models, for example, are based on monopolistic competition.

The second choice to be made concerns the pricing behaviour of the firm. Two common alternatives are to assume either that the firm sets a fixed mill price or that it engages in spatial price discrimination. Beckman (1968) shows that the latter is usually more profitable as a pricing strategy. However, the mill pricing case is assumed here because of its simplicity and its frequent occurrence in reality.

To introduce producer behaviour, we need only add production and profit functions to the above assumptions. A simple production function relating factory output, Z, to labour input, N, is used. Further, the profit level (π) is the difference between the output sales and the sum of fixed costs (C_o) and labour costs. The relationship between the mill wage-price combination and the output-demand and labour-supply levels is determined from the farm sub-model. Such relationships, presuming circular markets, are described in Table 6.

The factory has the freedom to choose only one of the four variables (P_b, w_b, Z, and N) within its domain in maximizing profits. The constraints A, C, and D in Table 6 simultaneously determine all three remaining variables. Further, it is noted from Tables 3, 4, and 5 that both Z and N in Table 6 are each jointly dependent on both the mill wage and the price of the factory. Changes in P_b for example affect both the demand for soap and the labour supply made available.

Table 6: The Factory Sub-Model

A. Production

$$Z = K_o N^{\delta} \qquad (o < \delta < 1; \quad K_o > o)$$

B. Profits

$$\pi = P_b Z - C_o - w_b N$$

C. Demand

$$Z = 2\pi \int_o^{s_1^\Delta} (sX_2/L)\, ds + 2\pi \int_{s_1^\Delta}^{s_2^\Delta} (sX_2/L)\, ds + 2\pi \int_{s_2^\Delta}^{s^*} (sX_2/L)\, ds$$

D. Labour Supply

$$N = 2\pi \int_o^{s_1^\Delta} (sh_Z/L)\, ds$$

Source: See Text

(e) NUMERICAL RESOLUTION OF THE MODEL

If one attempts to analytically determine optimal factory behaviour in this sub-model, a problem emerges in the evaluation of the integrals in Table 6. Consider the demand equation as an example. Substituting from Tables 3, 4, and 5, the output demand equation reduces to that shown in Table 7. The third and final integral term in this expression poses an immediate problem. It can be broken into a sum of integrals whose typical structure each is as follows

$$\text{(2.a)} \qquad \int X^m \, (a + bx^n)^p \, dx$$

where m, n, and p are here constants.
According to Gradshteyn and Ryzhik (1965; page 71), no general mathematical solution exists for such an integral.[10]

A numerical procedure must be drawn upon to approximate at least part of the integral in Table 7. The generalized Simpson's Rule has been used for the numerical results described below.[11] This method is used to approximate the first and third (corresponding to the M_2 and M_1 markets respectively) integrals in Table 7 as well as the labour supply integral (corresponding to the M_2 area) of Table 6.[12]

An example of the discrepancy between the actual integral and the approximation is presented in Figure 3. Here the labour supply integral is estimated by a sum of 10 rectangular blocks as shown. The error in approximation usually appears to be on the order of 2.0 to 2.5 per cent.[13]

Given the necessity of numerical analysis, a systematic method for evaluating the virtually-infinite array of possible parameter combinations is required. The method used here begins by defining a base run set of parameter values. Then, experiments can be defined in which one or more of these parameters are varied while the rest are held at their base run values.

Table 7: Reduced form for the output demand equation.

$$Z = k_3(1 - \beta - \gamma)^{(1 - \beta)/\gamma} \int_o^{P(s_1^\Delta)} (P - P_b) P^{-(1 - \alpha)} \, w(P)^{-\beta/\gamma} [\alpha k_2 P^{-(1 - \alpha)} - w(P)h]^{-(1 - \beta)/\gamma} dP$$

$$+ k_3[(1 - \gamma)/k_2]^{1/\gamma} \int_{P(s_1^\Delta)}^{P(s_2)} P^{-\alpha(1 - \gamma)/\gamma} - P_b P^{-(1 + \alpha(1 - \gamma)/\gamma)} dP$$

$$+ k_3 \frac{(1 - \alpha)\, c^{-\beta/\gamma} (1 - \beta - \gamma)}{(1 - k_1)}^{(1 - \beta - \gamma)/\gamma} \int_{P(s_2^\Delta)}^{P^*} (P - P_b)[P^{\alpha - 1} - (c(1 - k_1)h/\alpha k_2)]P^{-1/\gamma} [k_2 P^{\alpha - 1} - \alpha ch]^{-(1 - \beta)/\gamma} dP$$

where $k_3 = 2\Pi\alpha k_2 b^{1/\gamma} \beta^{\beta/\gamma}$

and $w(P) = (tw_b + rP_b)/t - (r/t)P$

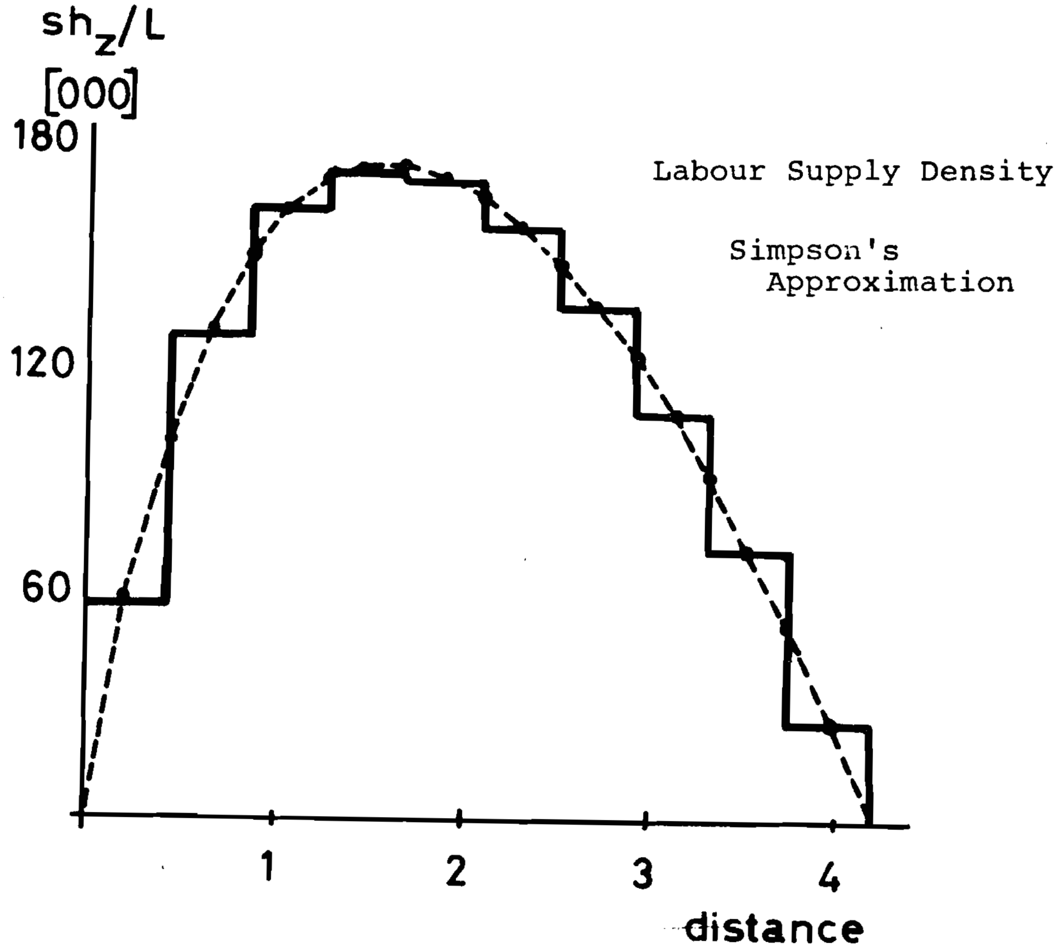

Figure 3. Approximation of factory labour supply; base run solution.

(f) THE BASE RUN AND ITS INTERPRETATION

Such an approach makes the choice of a base run quite important because all the experiments described below are based on variations of it. The particular values used in the base run are described in Table 8 and the resulting solution is summarized in Table 9. An intuitive feel can be lent to this base run by examining some of these values. It is noted, for instance, that the low value for α suggests that farms consume a large portion of their real income in food. Further, the value of γ indicates that each farm spends half of its gross food output in land rent charges. The value of h is set as the average number of man-hours worked by two persons in one year. The autarky land rent, R*, is set as $9600 per square mile (or about $37 per hectare) per year.

In the solution, these parameters generate a relatively small labour shed radius of 4.19 miles (6.7 km) and an output market area radius of 27.85 miles (44.8 km). Further, about two-thirds of the demand for factory soap is concentrated in the M_2 and M_3 markets where, it has been noted, the individual farm's demand for soap is inelastic. Finally, the average density (total man-hours in the market area divided by total market area) differs markedly from area to area. In autarky the farm consumes .0499 square miles (12.9 hectares) so that the average density there is 80,160 man-hours per square mile (or 309.4 man-hours per hectare). Thus, the average density in the M_3 and M_1 areas is lower than the autarky density while the M_2 average density is considerably higher.

The base run solution emphasizes a primarily rural economy in an early phase of development. Farm rent payments are high relative to incomes. Freight costs restrict the market area of the factory quite significantly. Food consumption accounts for well over half of the total budget of each farm. This kind of early-development solution is the most appropriate given the assumptions underlying the farm sub-model in subsection I(a) above.

Table 8: Base Run Parameter Values

A. CONSUMER PREFERENCES.

$\alpha = 0.20$

B. FOOD PRODUCTION

$b = 900$ $\beta = 0.20$

$\gamma = 0.50$

C. DOMESTIC SOAP PRODUCTION

$c = 0.10$

D. FREIGHT AND COMMUTING COSTS

$t = 0.02$ $r = 0.01$

E. OTHER FARM PARAMETERS

$R^* = 9600.$ $h = 4000.$

F. FACTORY SOAP PRODUCTION

$K_o = 10.0$ $\delta = 0.90$

Table 9: Solution for Base Run

A. The factory and its market areas

Optimal Price = 0.222 Optimal Wage = 0.0916

		Market Area			
		M_2	M_3	M_1	Total
Outer Radius		4.19	16.96	27.85	----
Output Demand	(000's)	676	3958	2305	6939
Factory Labour	(000's)	3133	----	----	3133
Total Labour	(000's)	7542	65675	116210	189428
Average Density	(000's)	136.5	77.4	75.8	77.8

B. The Farm in Autarky

$U = 381.9$
$X = 153.8$
$Y = 479.3$

$h_x = 1538.5$
$h_y = 2461.5$
$h_z = 0.0$

$L = 0.0499$
$R = 9600.$

$Q = 958.7$
$Y_1 = 479.3$

$X_1 = 153.8$
$X_2 = 0.0$

$P = 0.779$

II. EXPERIMENTAL RESULTS AND INTERPRETATIONS

It is possible to vary any of the parameters in either of the two sub-models, measure the change in the optimal mill wage-price combination, and determine the implications of this for the average density within the factory's various market areas. Here, we choose to examine variations only in three parameters; K_o, δ, and b. The first two of these are scale and labour elasticity parameters respectively in the factory's production function. Changes in them might reflect changes in the firm's capital stock or technology. The final one is a scale parameter in the farm's production function reflecting its level of agricultural capital stock or technology.

In all cases, the experimental results are described in a similar two-part manner. First, the effects of parameter variations on the optimal wage-price combination are described and explained. Then, the implications of these wage-price variations for the spatial pattern of population are discussed. This procedure emphasizes the fact that it is only through the wage and price variables that the farm sub-model (and thus population density) reacts to the factory sub-model.

(a) VARIATION OF THE FACTORY SCALE PARAMETER: WAGE PRICE EFFECTS

Variations in K_o have interesting effects on the levels of wage and price chosen by the firm. Consider the following experiment where all parameters are given their base run values with the exception of K_o whose value is varied from 1.0 to 20.0[14]. The resultant wage-price combinations as a function of the value of K_o are displayed in Figure 4. As shown in that figure, the profit-maximizing price declines monotonically with increasing K_o although $\partial^2 P_b / \partial K_o^2$ is positive. The wage level, however, is at first an increasing, then a decreasing, function of K_o. What causes P_b to decline so quickly at first and then almost level out? What does this have to do with the non-monotonic behaviour of w_b?

To answer these questions, it is useful to estimate the price elasticity of demand, ϵ.

$$(3.a)\quad \epsilon = -\left(\frac{P_b}{Z}\ \frac{\partial 7}{\partial P_b}\right)$$

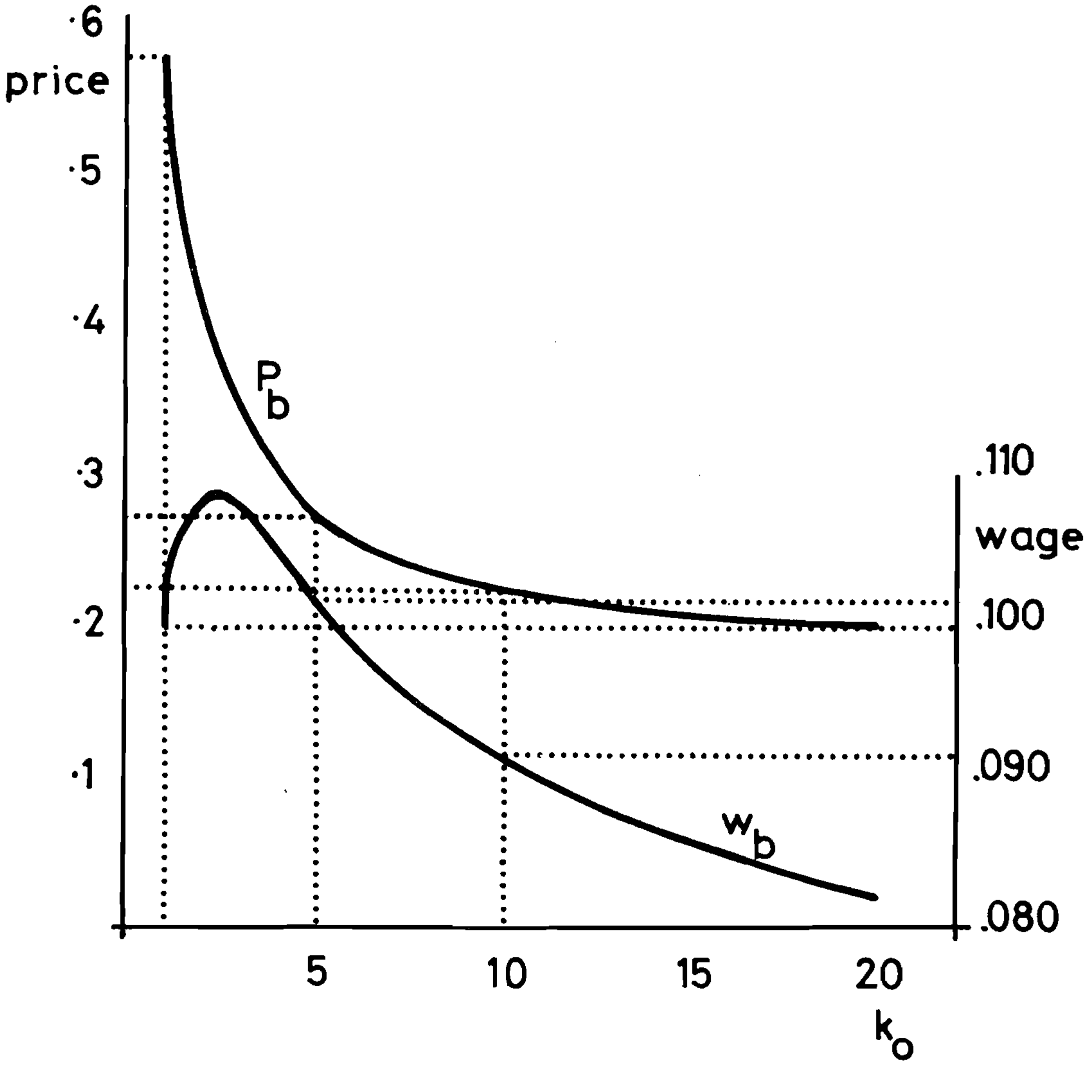

Figure 4. Effect of changes in K_o on the wage-price combination.

This elasticity is a function of both the wage and price chosen. Using the base-run parameters in the farm sub-model, this elasticity has been estimated for a net of P_b - w_b values where $0.18 \leqslant P_b \leqslant 0.56$ (increments of 0.01) and $0.075 \leqslant w_b \leqslant 0.110$ (increments of 0.001). The estimated elasticities are presented graphically in Figure 5 where the mill price is measured along Oa, the mill wage along Ob and the elasticity along bc.[15] The estimates range from a low of 0.900 ($P_b = 0.18$, $w_b = 0.077$) to a high of 6.29 ($P_b = 0.56$, $w_b = 0.095$). From Figure 5, it is seen both that at a high mill price (regardless of the wage level) demand is very elastic and that at lower prices demand becomes increasingly inelastic.

The effect of increasing K_o on the factory's wage-price choice can now be intuitively explained. As the firm experiences an increase in K_o, it can be expected to react in a combination of two ways; decreasing its price and increasing revenues or decreasing its wage and its costs to take advantage of its enhanced productiveness. Initially, when P_b is high, an increment in K_o is reflected primarily in a price reduction because demand is quite elastic. As K_o becomes larger and P_b smaller, increments to K_o are not reflected in sizable decreases in P_b because demand is inelastic at these P_b values. Instead, the wage offer is reduced.

What causes demand to become inelastic at these lower P_b values? The answer lies in the shifting composition of the factory's output market. The elasticity of demand observed by the factory is partly dependent on the relative number of farms in the M_2 and M_3 areas as compared with the M_1 area. As indicated earlier each farm in the former area has an inelastic demand while each in the latter has a more elastic demand. The proportion of output demand in the M_2 and M_3 areas is displayed in Figure 6 for the case of the present experiment. This proportion, $\bar{m}_1$, declines initially until about $K_o = 1.75$ and then increases monotonically. The shape of this curve reflects the monotonically declining proportion of the market in the M_2 area and the monotonically increasing proportion of the M_3 area above $K_o = 1.75$ (where it initially appears). In effect, with an increasing value of K_o, the factory

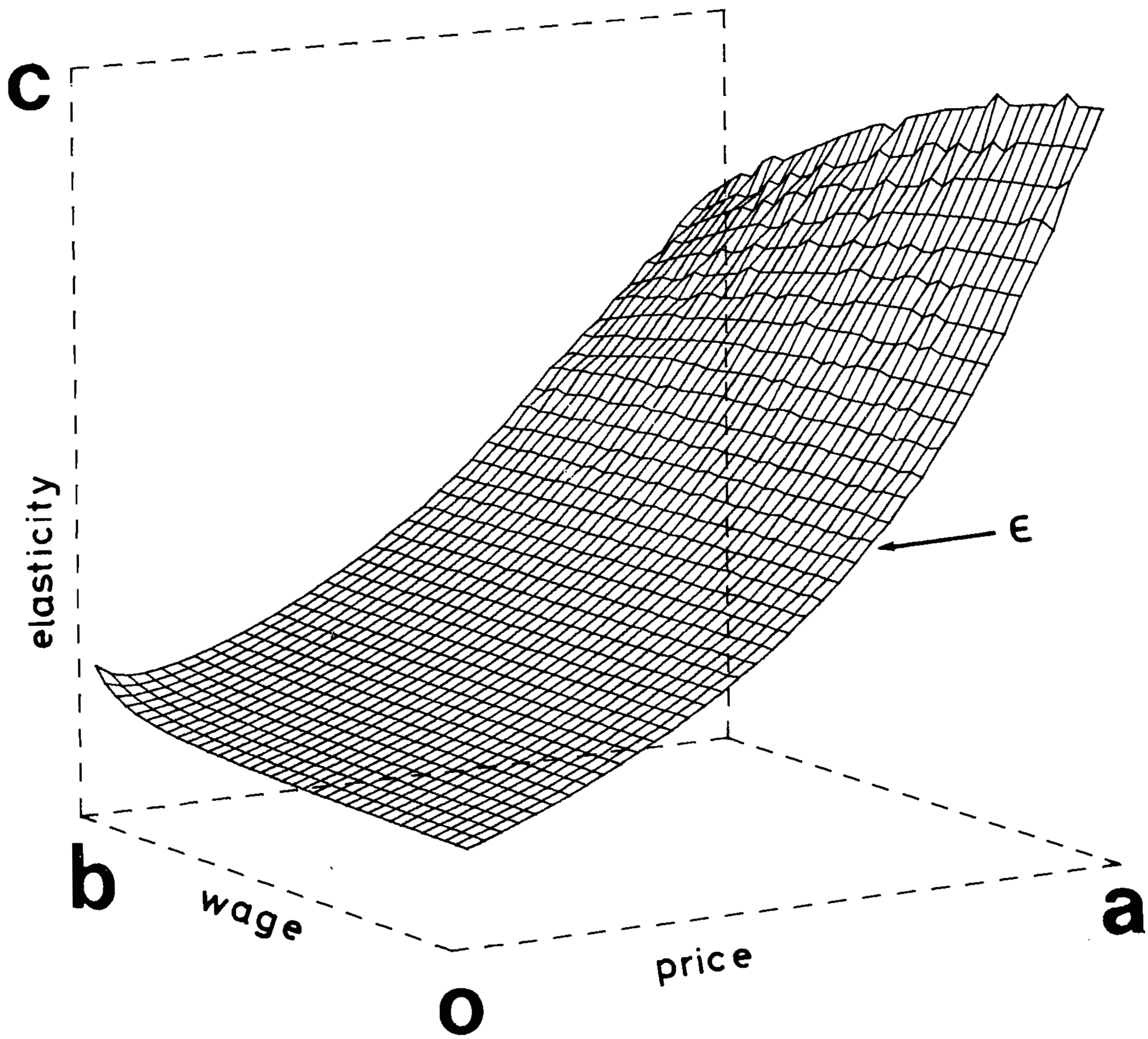

Figure 5. The elasticity of demand (ε) as a function of price (oa) and wage (ob) using base-run parameters.

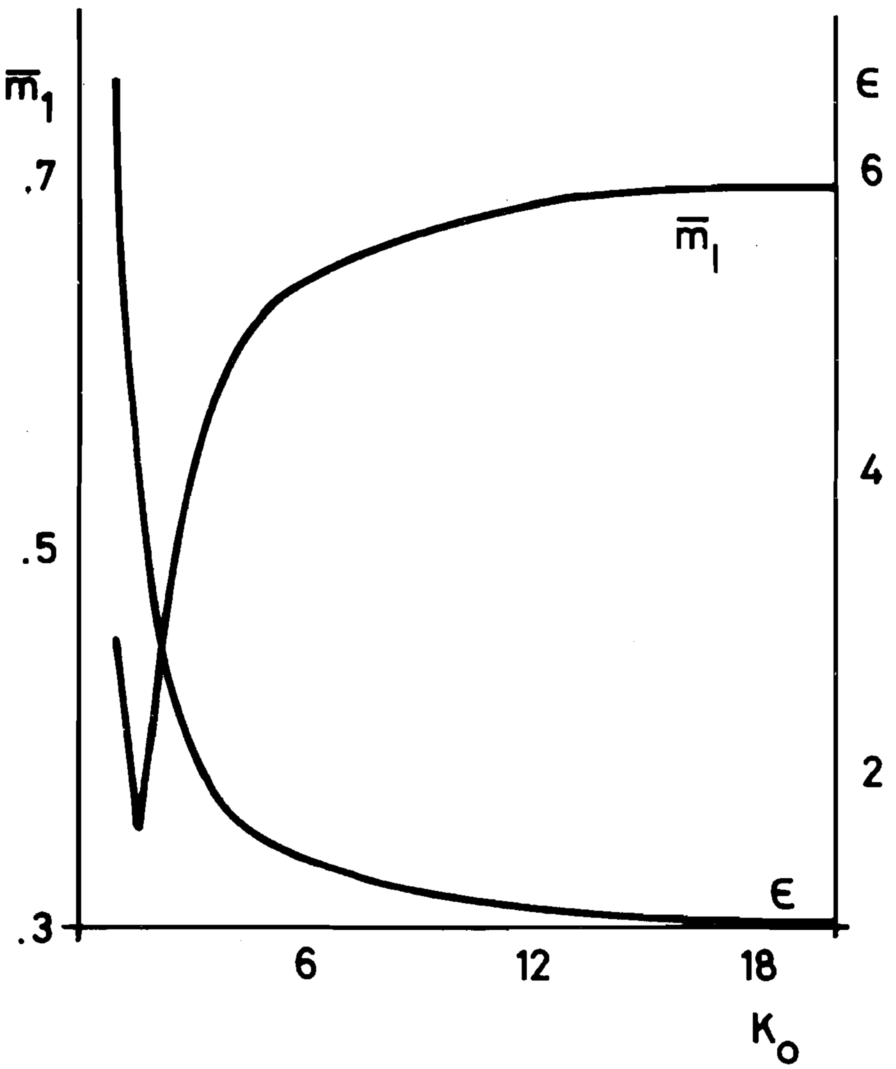

Figure 6. Proportion of market within M_2 and M_3 areas ($\bar{m}_1$) and the elasticity of demand (ϵ) as functions of K_o.

finds its output market increasingly 'saturated' with individually-inelastic farms. This accounts for its increasingly inelastic market demand.

(b) VARIATION OF THE FACTORY SCALE PARAMETER: WAGE-PRICE INTERACTION

We have specified in an intuitive manner how the wage and price offers of the firm are inter-related as K_o is increased. To make a more specific or formal statement about their inter-connections, it is useful to consider momentarily a somewhat simpler model than that found in Table 6. Specifically, let us consider the model outlined in Table 10.

This simple model differs from the one of Table 6 in that the output demand and labour supply relations have been made more tractable. In Table 6, Z responds to the mill price but not with a fixed elasticity as found in Table 10. Also, Z responds somewhat to w_b in the full model and this effect is ignored in the simple model. The simple model posits that both the mill price and wage have an effect on the labour supply as is implicit in Table 6. However, constant elasticities have been used when these are variable in the full model. Finally, the wage elasticity of labour supply is assumed to be larger than the price elasticity ($\beta_1 > \beta_2$) in absolute value and this is in keeping with numerical results obtained in experiments with the full model.

The solution to the full model can be expressed in terms of the effects of K_o. We concentrate here on Z and w_b. From Table 10, it is seen that the optimal output level is a monotonically increasing function of K_o regardless of the parameter values chosen. From the output demand equation, this implies that P_b is monotonically declining with respect to K_o.

However, the behaviour of the mill wage is not so clear. The exponent of K_o here has a sign which depends on the value of

(4.a) $(\alpha - 1) - \beta_2$

Here, w_b is an increasing or decreasing function of K_o as $(\alpha-1)-\beta_2$ is greater or less then zero. For w_b to be an increasing function of K_o, the output demand elasticity (α) must be sufficiently large to offset the price elasticity of labour supply (β_2). In other words, if the labour supply is very responsive to the output

Table 10: A simple model of monopolistic behaviour in interdependent markets

A. THE MODEL

(i) MAXIMIZE: PROFIT LEVEL

$$\pi = P_b Z - w_b N$$

(ii) SUBJECT TO:

(a) OUTPUT DEMAND

$$Z = a_o P_b^{-\alpha} \qquad (a_o > 0;\ \alpha < 1)$$

(b) PRODUCTION FUNCTION

$$Z = K_o N^{\delta} \qquad (K_o > 0;\ 0 < \delta < 1)$$

(c) LABOUR SUPPLY

$$N = b_o w_b^{\beta_1} P_b^{-\beta_2} \qquad (\beta_1 > \beta_2 > 0)$$

B. SOLUTION

(i) OPTIMAL OUTPUT

$$Z = C_o K_o^{\alpha(1+\beta_1)/\gamma_o}$$

(ii) OPTIMAL WAGE

$$w_b = c\ K_o^{[\alpha-1-\beta_2]/\gamma_o}$$

where $\gamma_o = \alpha + \alpha(1-\delta)\beta_1 + (\beta_1-\beta_2)\ \delta\ > 0$

and c_o, c_1 are constants.

price, the firm may find that a lower wage offer will still gain enough labour to produce the required output at the new, lower, optimal price.

Two observations can now be made with respect to the full model. First, since the elasticities are not fixed in the full model, there may be regions of the P_b - w_b space in which $\alpha - 1 - \beta_2 > 0$ and regions in which it is not. Experimental work tends to suggest that α increases rapidly with P_b but that β_1 varies only slightly so that $\alpha - 1 - \beta_2 < 0$ can ususally be expected where P_b is low.[16] Secondly, the simple model indicates that it is strictly this feedback effect of the mill price on the labour supply which generates the observed non-monotonic relationship between the mill wage and K_o.

(c) VARIATION OF THE FACTORY SCALE PARAMETER: DENSITY EFFECTS

To understand the implications of Figure 4 for the density of farms, it is necessary to re-examine Tables 3 through 5. Within the three market areas, the distance-specific density levels are as follows

$$(5.a) \quad h/L = hb^{1/\gamma}\left[w(s)/\beta\right]^{-\beta/\gamma}\left[g_b(s)\right]^{-(1-\beta)/\gamma} \qquad 0 < s < s_1^\Delta \qquad (M_2)$$

$$(5.b) \quad h/L = h^{(\gamma+\beta)/\gamma}\, b^{1/\gamma}\left[g_c(s)\right]^{-1/\gamma} \qquad s_1^\Delta < s < s_2^\Delta \qquad (M_3)$$

$$(5.c) \quad h/L = hb^{1/\gamma}\left[\dot{c}P(s)/\beta\right]^{-\beta/\gamma}\left[g_a(s)\right]^{-(1-\beta)/\gamma} \qquad s_2^\Delta < s < s^* \qquad (M_1)$$

First, note that variations in the mill wage affect only the densities in the M_2 area and the boundary between the M_2 and its adjacent market area. Density can be seen to be an increasing function of the mill wage in the M_2 market.[17] In the M_2 and M_3 markets, density is a decreasing function of the mill price. In the M_1 market, an increase in the mill price may increase or decrease the density level at any distance as argued in Miron (1975; pp. 158-159). Thus the implication of Figure 4 for density levels may well be different in each of the three market areas facing the firm.

Three distance-specific density functions (corresponding to K_o values of 1.0, 5.0, and 10.0) are illustrated in Figure 7. As seen from Figure 4, relative to the w_b value at $K_o = 1.0$, w_b is larger at $K_o = 5.0$ and smaller at $K_o = 10$. The mill price at the same time is monotonically decreasing. The changes in density are in accordance with the wage-price changes. In the M_2 market, the density level increases as K_o changes from 1.0 to 5.0 but then declines for $K_o = 10$. An M_3 area does not exist for $K_o = 1.0$ but density is increasing in this area from $K_o = 5.0$ to $K_o = 10.0$. Within the M_1 area densities are shifting downward (and to the right in Figure 7) as K_o increases.

Figure 7 presents an awkward format for the solution to this model when K_o is given a number of different values. It is useful to summarize the solution in terms of the average density within each market. The average density $\{h/L\}_i$ in market i is defined as follows

$$(6.a) \qquad \{h/L\}_i = \left[2\int_{\ell_{1,i}}^{\ell_{2,i}} sh/L \, ds\right] \Big/ \left[\ell_{2,i}^2 - \ell_{1,i}^2\right]$$

Here, $\ell_{1,i}$ and $\ell_{2,i}$ are the inner and outer radii of the i'th market. In Figure 8 are presented the average densities in each market as functions of K_o.

It is noted that the average density in the M_2 market peaks near K_o=6. However, from Figure 4, the mill wage peaks near K_o=1.8. The reason that $\{h/L\}_2$ continues to increase for K_o between 1.8 and 6.0 is that the rate of price decline has an effect on density levels which more than offsets effects of the rate of wage decline. Thus, one might expect in general that $\{h/L\}_2$ will always peak at a higher value of K_o than does the mill wage.

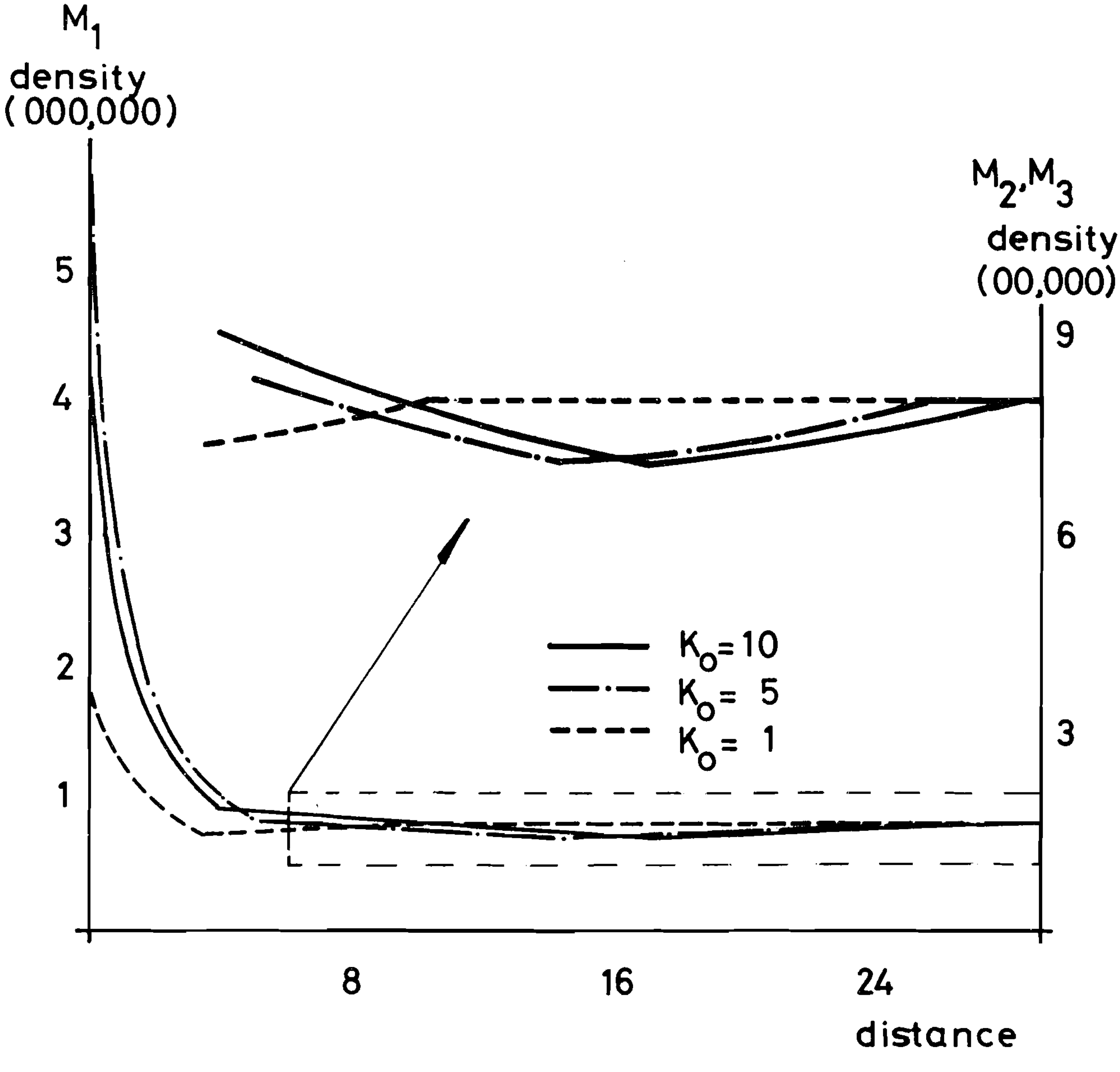

Figure 7. Effect of alternate K_o values on the equilibrium density of population by distance.

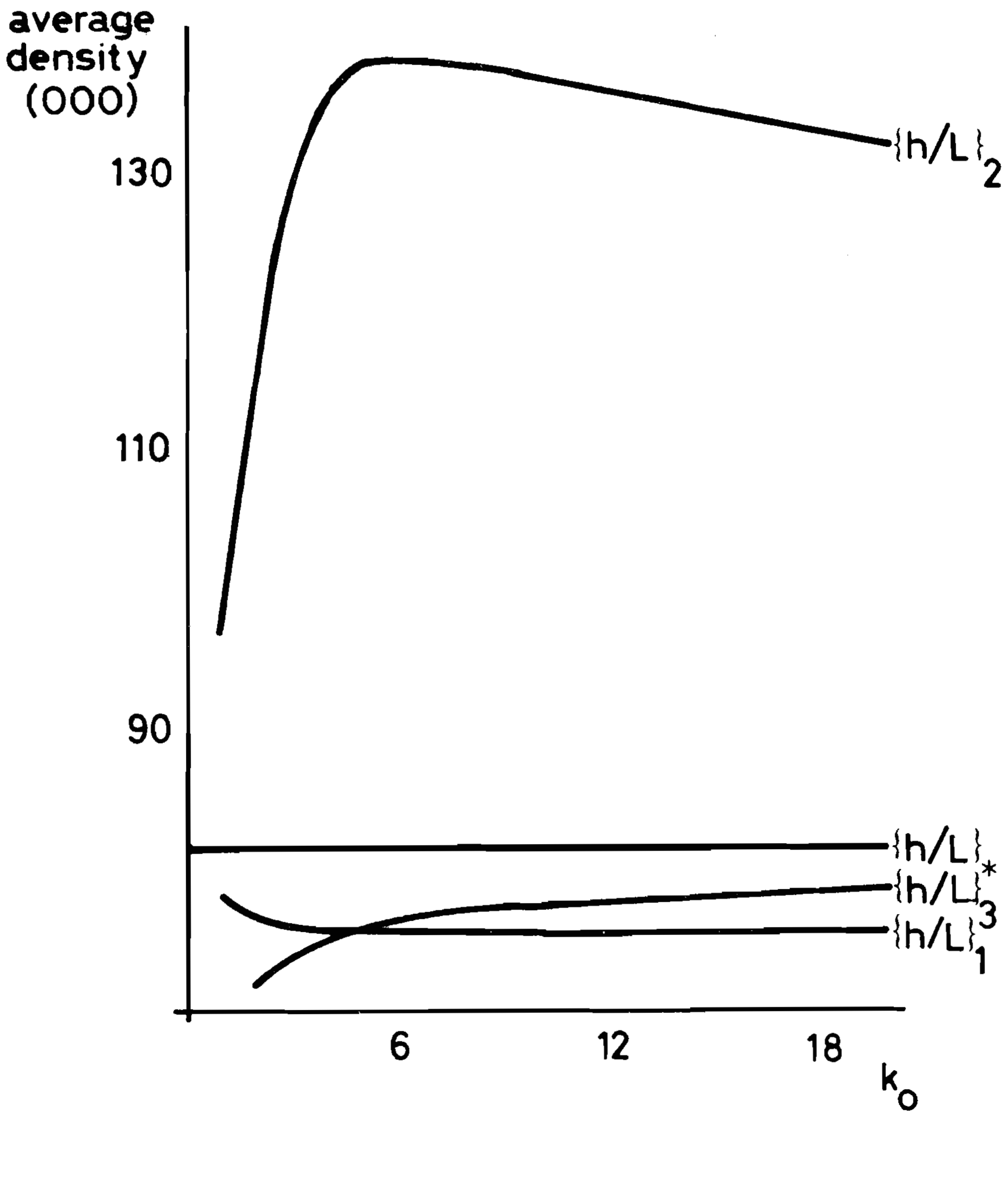

Figure 8. Average density in M_2, M_3, and M_1 markets as a function of K_o.

It is also noted that $\{h/L\}_3$ and $\{h/L_1\}$ have a monotonic behaviour with respect to K_o that could be expected from Figure 4. Both appear to be approaching asymptotic limits for large values of K_o. Also, these variables move in opposite directions.

It seems reasonable to conclude from these experimental results that the effect of a scale change in the factory's production function on the population density pattern is a complex one. Within the different market areas of the factory, different kinds of behaviour can be expected. Further, these effects may or may not be monotonic. However, these effects are all explicable within the terms of the model used.

(d) VARIATION OF THE FACTORY LABOUR ELASTICITY PARAMETER

The second set of experiments conducted with this model concern the effect of variations in δ. A range of δ-values from 0.65 to 1.06 are used and for each value the optimal wage-price combination is found.[18] The wage and price levels for each δ are depicted in Figure 9 where all other parameters have been held to the base run values. These solutions bear a substantial correspondence to those for the K_o variations in Figure 4. The most interesting difference concerns the second derivatives of P_b and w_b in the present case. Here, P_b has an inflection point whereas earlier it did not while w_b has two such points compared to one earlier.

These price and wage characteristics have important conclusions for the associated density patterns. The M_2 market is most affected by the slowly changing wage and price levels near δ=0.65. In Figure 10, the average density in the M_2 zone is seen to increase very slowly at first. For δ between about 0.75 and 0.85, there is a rapid growth in $\{h/L\}_2$ reflecting the quickly rising w_b and quickly falling P_b. Thus, if the P_b-w_b pattern of Figure 9 is representative, it indicates that there will be a small intermediate range of δ values over which the M_2 density will be very sensitive to variations. In the M_1 market, density is affected only by P_b and the solution depicted in Figure 10 has a shape similar to that of the mill price solution in Figure 9. The M_3 area density, on the other hand, has a monotonically increasing density over the range of δ values for which it exists.

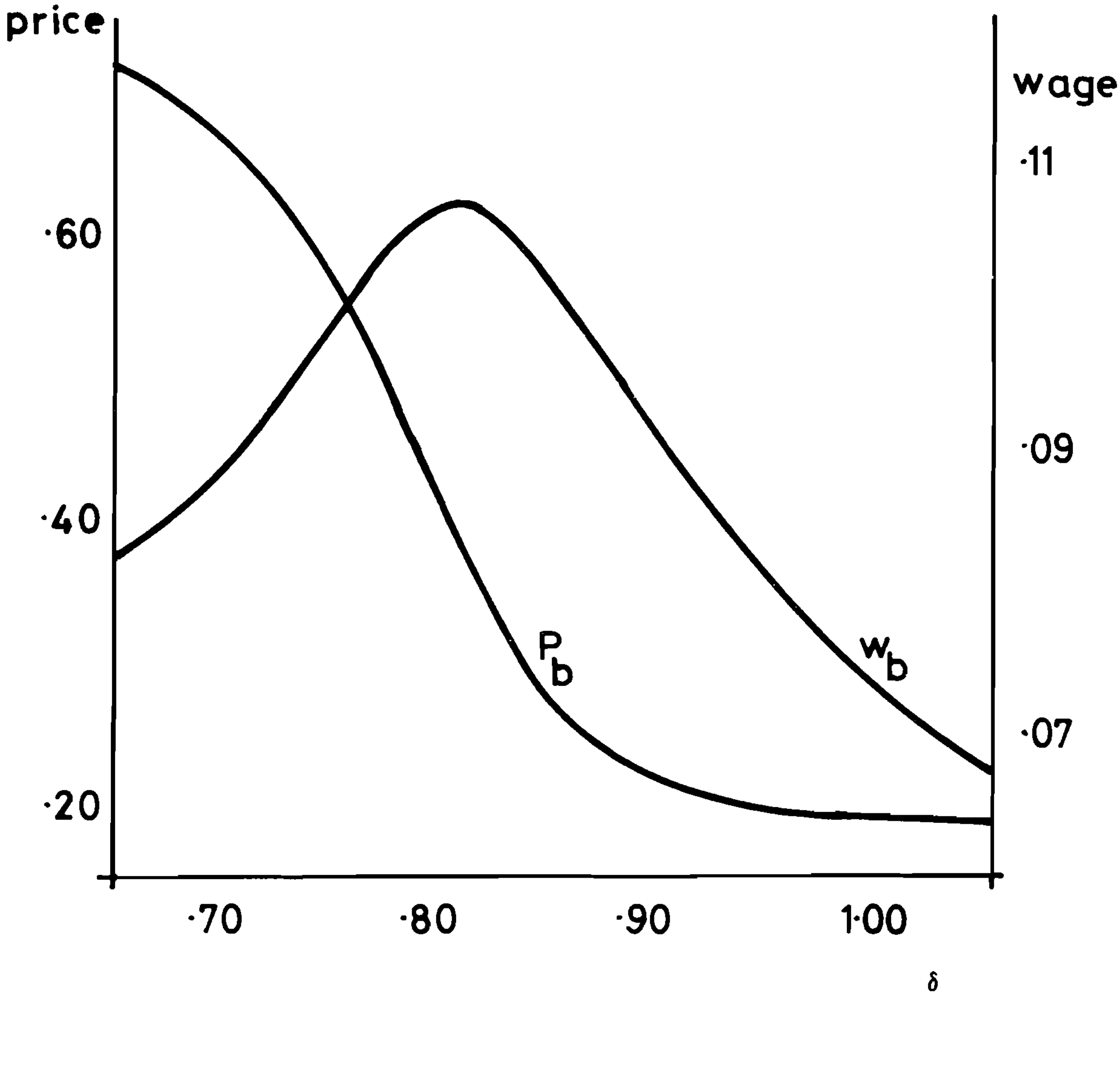

Figure 9. The optimal wage-price combination when δ is varied from 0.65 to 1.06.

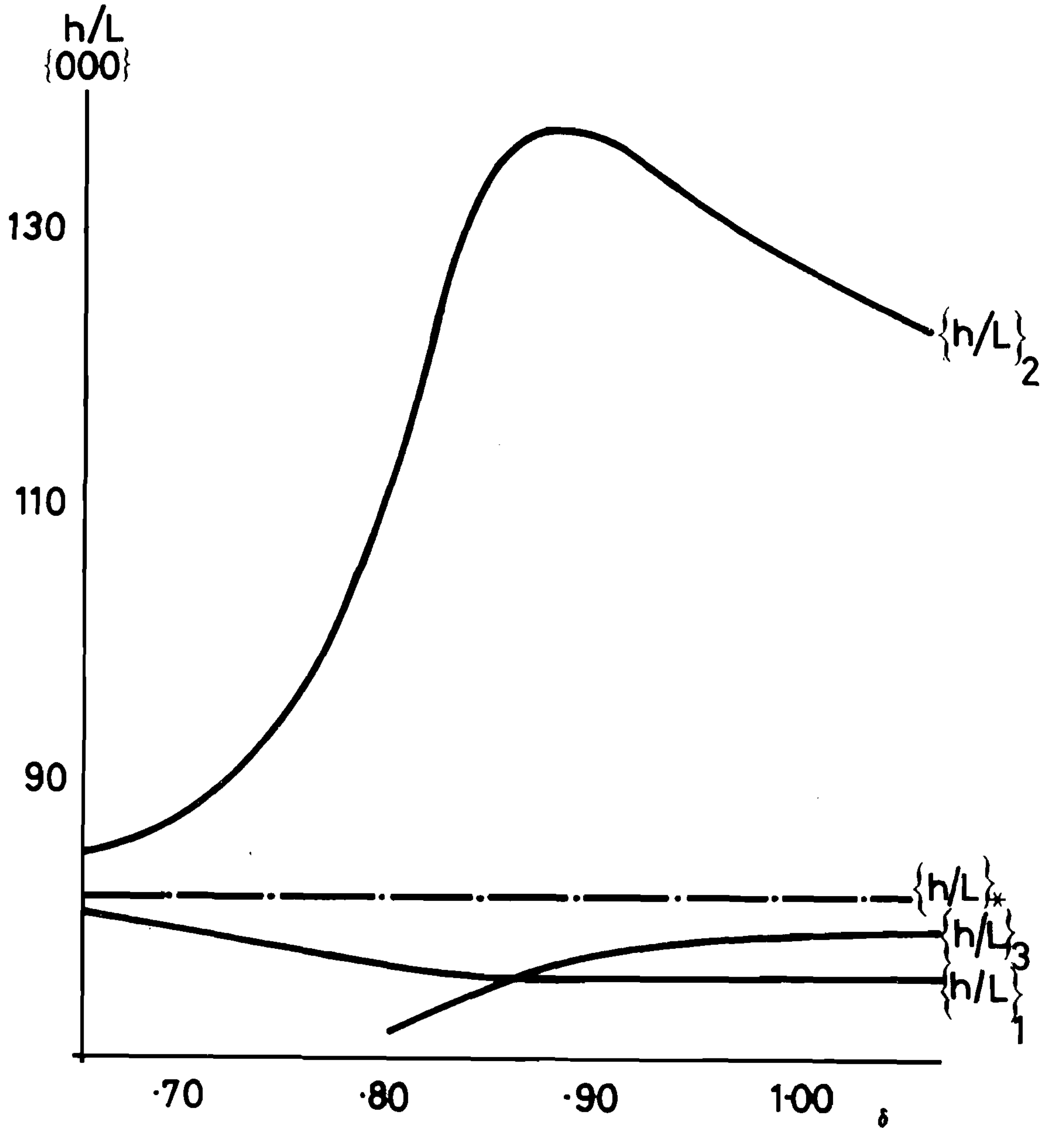

Figure 10. Average density in the three market areas as δ is varied from 0.65 to 1.06.

With the exception of their second derivative behaviour, these density solutions are very similar to those of Figure 8 where K_o is varied. In both cases, $\{h/L\}_2$ is the most volatile density; at first increasing and then decreasing. The M_3 area density is monotonically increasing in both cases while the M_1 density is monotonically declining.

(e) SIMULTANEOUS VARIATION OF δ AND K_o

Given the effect of varying δ and K_o individually in the previous two experiments, it is interesting to investigate the effect of varying both parameters simultaneously. One method of implementing this experiment is to carry out several sets of experiments similar to the first one where K_o is varied and in which δ is varied from one set to the next. These experimental results are summarized in Figure 11 which depicts the optimal mill wage-price combination as a function of K_o for δ=0.8 and δ=0.9.

The impact of a change in δ while K_o is increasing can now be seen. Larger values of δ cause w_b to become more sharply peaked as a function of K_o. Further, w_b achieves a larger maximum while the mill price falls more rapidly (as a function of K_o) for higher δ values. The resultant average densities in Figure 12 reflect these wage-price patterns. An increase in δ causes $\{h/L\}_2$ to peak sooner and to have a larger maximum value. Further, this increase causes $\{h/L\}_1$ and $\{h/L\}_2$ to reach the same asymptotes as before but at a faster rate.

(f) VARIATION OF THE FARM FOOD SCALE PARAMETER

To this point, all experiments have been concerned with technical change at the factory level. It is reasonable to ask if changes in the farm's productive capacity have analogous effects on densities. Before such an experiment can be carried out, it is noted that a change in one of the farm's parameters changes the nature of the autarky solution as well as the various market solutions. However, we are primarily interested in the market solutions relative to the autarky solution. Therefore, we standardize here for autarky changes by considering a relative price (P_b/P^*), a relative wage (w_b/cP^*), and a relative average density $\left(\{h/L\}_i / \{h/L\}_*\right)$.[19]

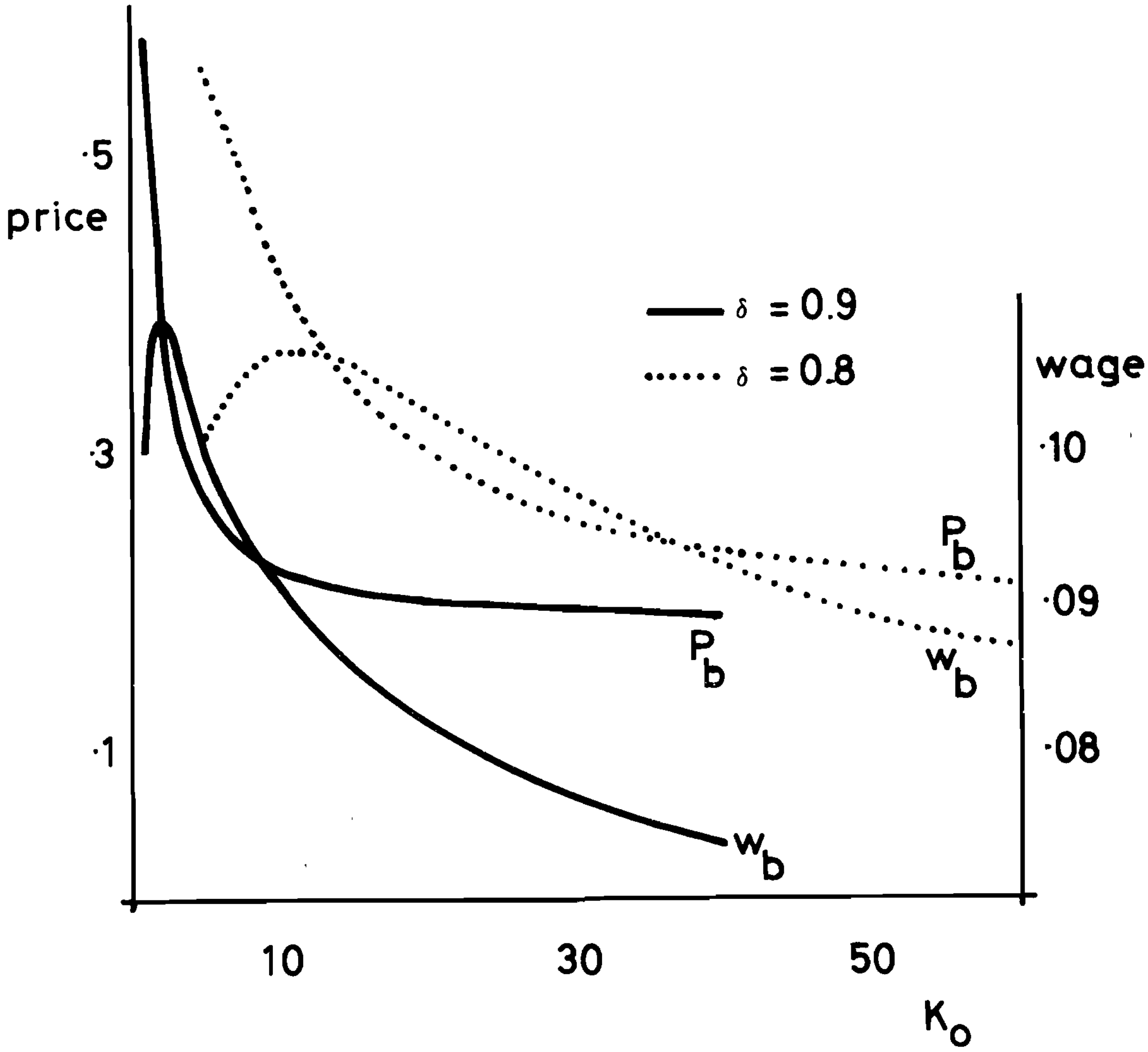

Figure 11. Price and wage as functions of K_o given $\delta = 0.8$ and $\delta = 0.9$.

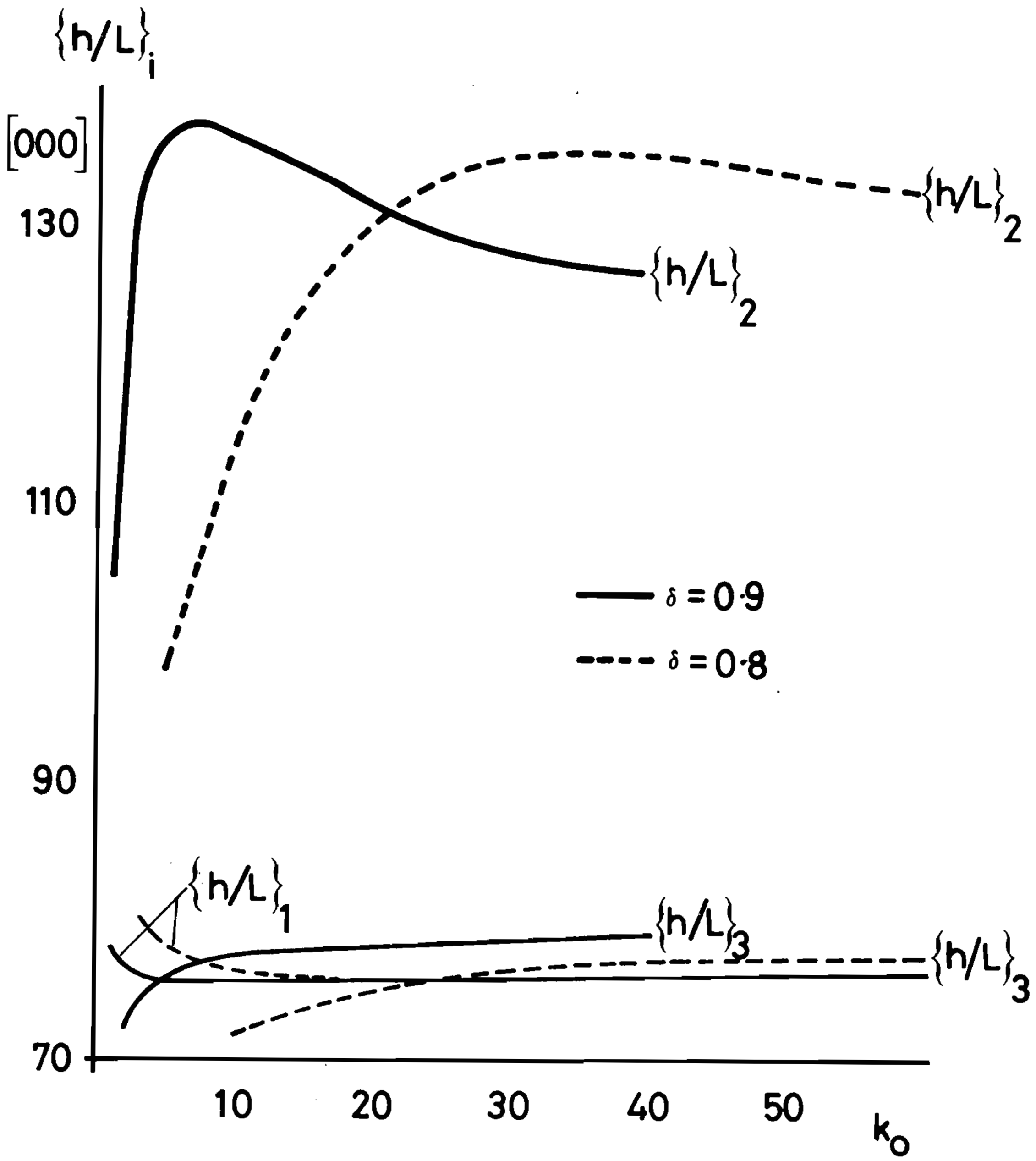

Figure 12. Average Density in the M_2, M_3, and M_1 markets as a function of K_o; $\partial = 0.8$ and $\partial = 0.9$.

In the base run solution, b is set at 900.0. An experiment is undertaken in which b is varied from 400.0 to 1700.0.[20] The resultant relative wage and price are depicted in Figure 13. It is immediately noted that there is very little change in these solutions as b is altered. The effects of these solutions on the average relative densities are displayed in Figure 14. These indicate that the small relative wage-price changes have very little effect on densities in any of the market areas relative to what is happening in autarky.

This is more surprising in view of the large absolute effect b-variations have on both the autarky solution and the market area solutions. For example, the lot size in autarky, $\dot{L}^*$, grows exponentially with b as shown in Figure 14 even though the relative densities change little. It should be concluded that b-variations have significant effects on solutions both inside and outside the factory's market area but appears to have little differential (or relative) effect between the two.

III. CONCLUSIONS

A formal micro-economic model of a single dualistic spatial economy has been outlined. The equilibrium solution to this model includes a measure of population density whose level is dependent on, among other coefficients, the technical parameters of the economy. Because the model is solvable only through numerical approximation, several experiments have been described in which various model technical parameters are changed. The experimental findings bear out the view that population density does not change uniformly through space with respect with changes in these parameters. These findings suggest that it is difficult to make simple statements about the effect of technical change on spatial patterns of population density without a careful specification of the kind of market areas involved.

The model presented is of a very particular structure but the essentials of it are germane to almost any model, empirical or theoretical, of spatial population patterns. In this model, an improved factory technology enables the factory to expand its market areas. This is done to some extent by stripping away local

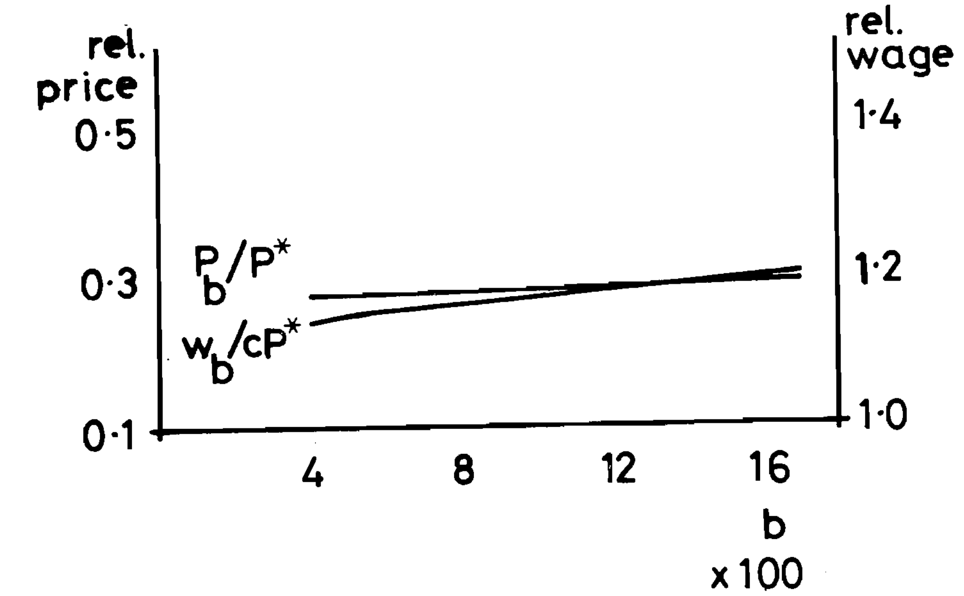

Figure 13. Optimal wage and price as b is varied from 400 to 1700.

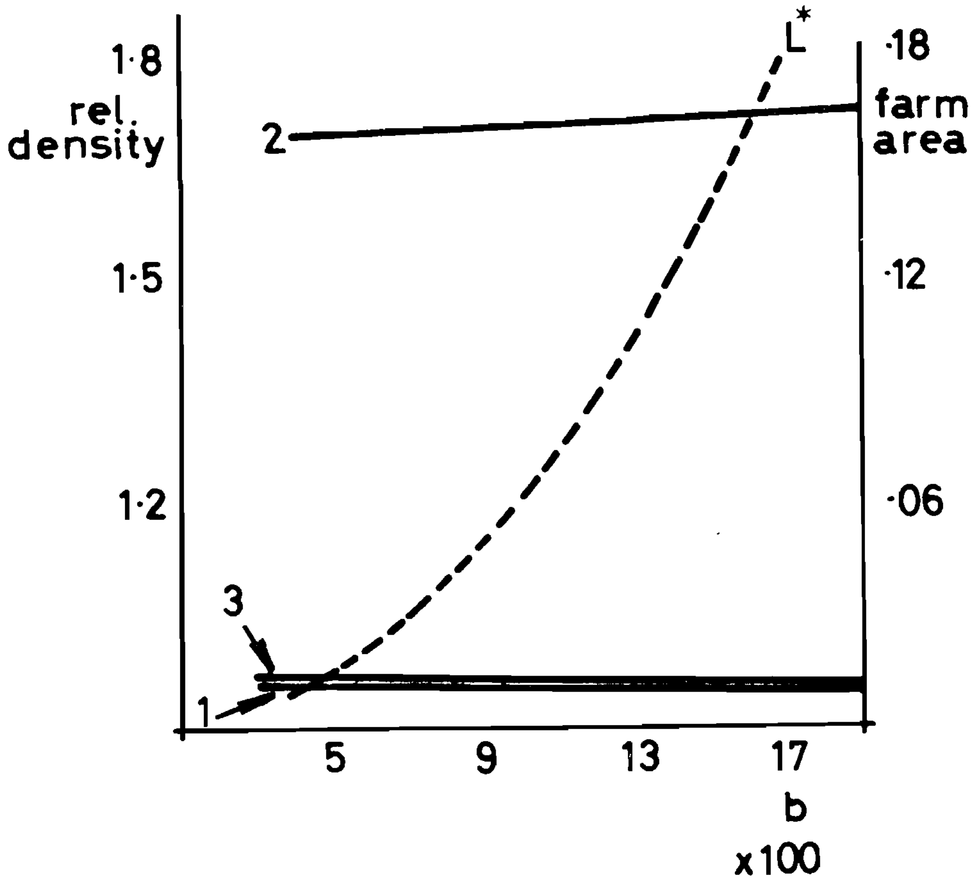

Figure 14. Relative density of population and farm size as b is varied from 400 to 1700.

production in hinterland areas and encouraging more specialization in agriculture in these areas. As this process continues, the central factory faces a dimminishing elasticity of demand. The factory's demand is most elastic when its market is dominated by farms who are not completely specialized in agriculture. As the factory decreases its mill price, its output market becomes more and more saturated (and inelastic) in that a larger proportion of farms are completely specialized. It is this aspect of spatial economic development which is emphasized in the model and which is relevant to virtually any economic model of spatial population patterns.

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1. Miron (1975)

2. The utility function is strictly convex in that the associated indifference curves are strictly convex with respect to the origin.

   The utility function does not include land area as an argument. Thus there exists a limit to the interpretability of this model which is discussed below.

3. The model is also open in the sense that land rent (and, later, factory profits), are leakages from it.

4. Further, the correspondence between the M_2 and M_1 solutions are seen when it is appreciated that the marginal value product of labour is w(s) in the former and cP(s) in the latter.

5. From the total soap demand equations in Tables 3 and 4, the elasticity in both markets is $\alpha-1$. This inelastic demand reflects the specific utility function used.

6. Condition (1.b) is a non-linear equation which is solved numerically for s_1^{Δ}.

7. These problems can be avoided by introducing residential land explicitly into the farm's utility function. This has not been undertaken in the present work because it greatly increases the difficulty of deriving numerical solutions without a correspondingly better insight into the model.

8. The condition the L > o everywhere requires, from Table 4, that

   $$k_2 \; P_b^{\alpha} > w_b h \quad .$$

   This condition will always be satisfied. As the factory raises its mill wage toward $k \; p_b^{\alpha}/h$, it finds that any desired level of employment can be generated. The main concern in experimentation was not that L would actually drop to zero but that it would simply become 'too' small.

9. Refer to Stern (1972) for a discussion of producer behaviour in either case given a uniform spatial demand.

10. Although solutions are available where one of the following is an integer; p, (m + 1)/n, ((m + 1)/n) + p .

11. See Davis and Rabinowitz (1975; pp 45-48) for a description of the method.

12. There are 10 intervals used in estimating the M_2 area integrals for output demand and labour supply and 20 intervals used in estimating the M_1 market integral for output demand. (The middle integral of Table 7 has a simple exact analytical solution). The difference here reflects the usual relative contributions of the two areas to the total output demand.

13. This error can be reduced by increasing the number of approximation interval. Only 10 or 20 are used in the present study because of the extensive computing time required to solve the model for a given set of parameter values. In the computer program used, a pre-defined set of mill prices are scanned to find the optimal price. For each mill price, the corresponding appropriate wage must be found via an iterative procedure involving usually about 10 iterations. Since usually about 25 mill prices are scanned before an optimal one is found, the output demand and labour supply integrals are each evaluated about 250 times. Given the number of sets of parameter values to be examined here, the time involved in such computations is very substantial and must be weighed against the insight gained from additional accuracy.

14. The lower limit for k_o of 1.0 represents a point at which the factory faces a negligible demand at its optimal wage-price combination. The upper limit is chosen to illustrate the phenomenon involved.

15. The 'roughness' of the elasticity surface for large values of P_b reflects the error introduced in numerical approximations.

16. It is noted in passing that since α and β_1 also enter into c_o and c_1 in Table 10, the combination of mill price and wage at which $\alpha - 1 - \beta_1 = 0$ may only approximate the point at which the mill wage crests.

17. $$\frac{\partial(h/L)}{\partial w_b} = \left[\frac{h}{L}\right]\left[\frac{w(s)h - \beta k_2 P(s)^{\alpha}}{\gamma w(s) g_b(s)}\right]$$

This is always positive since, in the M_2 area, $h_z \geq 0$ which requires $w(s)h \geq \beta k_2 P(s)^{\alpha}/(1 - \gamma) > \beta k_2 P(s)^{\alpha}$.

18. As in the case of k_o, the lower limit on δ represents the point at which the factory's optimal output becomes negligible. The upper limit again is arbitrarily set to illustrate the phenomenon involved. Since δ is greater than one in some experiments, increasing returns to scale are allowed for.

19. Note that P* and $\{h/L\}_*$ refer to autarky solutions of Table 2. The product cP* is the marginal value product of labour in autarky and this is used to deflate the mill wage.

20. At b below 400.0, factory soap demand becomes negligible. The uppler limit is chosen arbitrarily.