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Optimization of Social Security Systems Under Uncertainty

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Abstract

The aim of this paper is to develop optimization-based approaches for modeling multi-agent and multi-regional social security systems under demographic and economic uncertainties. Conceptually, the proposed model deals with the production and consumption processes coevolving with "birth-and-death" processes of the participating agents. Uncertainties concern fertility, life expectancy, migration and such economic and health variables as rate of return, incomes and disability rates. The goal is to satisfy a reasonable and secure consumption of agents. There is considerable similarity between the decisions involved in the optimization of social security systems and the production planning processes: in both cases "savings" are taken in periods of low demand and "dissavings" when the demand turns high. The significant difference of our problem is that decisions on savings and dissavings may have large-scale effects on the whole economy, in particular, they effect returns on savings through investments and capital formation. The model tracks incomes and expenditures of agents, their savings and dissavings, as well as intergenerational and interregional transfers of wealth. Robust management strategies are defined by using such risk indicators as ruin, shortfall and Conditional-Value-at-Risk (CVaR). The adaptive Monte Carlo optimization procedure is proposed to derive optimal decisions. Numerical experiments and possible applications to catastrophic risk management are discussed.

Key Words: Multi-agent system, production, social security, risk management, Monte Carlo, stochastic optimization.

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About the Author

Tatiana Ermolieva, ermol@iiasa.ac.at, is a Research Scholar in the Social Security Reform Project at IIASA. Her major research interests are in modeling, policy analysis and robust management of complex socio-economic, environmental, and financial systems under uncertainties and risks, especially of an extreme catastrophic nature. Methodological interests include methods for robust strategies, in particular, stochastic optimization techniques. She is furthermore interested in the development of integrated approaches that combine geographically explicit (GIS-based) catastrophe and socioeconomic models. In 1997, she was awarded the Kjell Gunnarson's Risk Management Prize of the Swedish Insurance Society for her paper on "Spatial Stochastic Model for Optimization Capacity of Insurance Networks under Dependent Catastrophic Risks", and the Dr. Aurelio Peccei Award of IIASA for her research and paper on "Design of Optimal Insurance Decisions in the Presence of Catastrophic Risks" during the 1997 Young Scientists Summer Program (YSSP).

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Tatiana Ermolieva

1. Introduction

Social security models deal with overlapping generations of "workers" and "former workers", or pensioners. Workers produce economic outputs during working years, which are consumed by all individuals. Workers also accumulate assets during working years and then "dissave" in retirement, in addition to which intergenerational transfers between working and retired individuals are mediated through the social security system. The goal is to analyze reasonable (and secure) consumption of individuals. Definitely, the ability of workers to maintain appropriate consumption levels depends on demographic instability and uncertainties imposed by processes prevailing in the population, such as "birth-and-death" processes, migration, as well as on instabilities and uncertainties of the production processes. For simple cases (Section 2) Samuelson [17] and Aaron [1] showed that if the sum of the rates of growth of population and incomes exceeds the rate of returns on investments, then the consumption of pensioners can be maintained by the transfer of incomes from workers to pensioners. Such transfer increases the welfare of all generations, i.e., the "paradox of social insurance" states that in this form of savings (the participation in the so-called "pay-as-you-go" pension scheme), provided the condition above is met, an individual will receive a higher rate of return than by saving with further investments (or participating in a funded pension scheme).

However, it has turned out to be difficult to use the Samuelson-Aaron logic for realistic models with overlapping generations such, as, e.g., discussed in Section 3. The paradox disappears even in simplest cases with uncertainties (Section 2). The significant feature of the social security decisions on savings and dissavings is connected with possible large-scale effects on investments and capital formation of the whole economy, that can "move the market", i.e., economic conditions depend, to some extent, on the composition of the saving schemes chosen. Besides, pension schemes involve different risks associated with the volatility of financial markets, mismanagement of social security funds, inflation, early disabilities, migration and aging processes.

All this should lend itself well to the formulation of an optimization problem to maximize social welfare by fine-tuning the mix of pay-as-you-go and funded social security schemes. A main problem expected in the nearest future for developed countries is the decrease of the "supply" by workers resulting from low fertility rates and the rise of "demand" in the generation of pensioners due to the formerly high level of fertility and progress in life expectancy. In this situation the general idea of the solution seems to be similar to the well-known production planning strategies, that is (see discussion in [4]) to save when demographic and economic conditions are good and dissave when they turn bad, which provides a justification for the development of funded pension schemes.

Blanchet and Kessler [4] examined the optimal saving mix of funded and transfer-based pension schemes in a simple deterministic model with identical individuals, where returns and wages are endogenous, i.e., they depend on saving decisions. The optimal policy rules were characterized by abrupt shifts, as a consequence of deterministic models with perfect foresights, unlikely to be realizable in the real world. In this paper we take another look at the question of optimal mixes. We formulate a stochastic optimization problem using an economic-demographic simulation model developed at IIASA [11], [12].

Section 2 illustrates the main problem by using a stylized economic-demographic model with identical individuals. Special attention is paid to the case of uncertainties and the choice of possible risk measures. Section 3 outlines the proposed multi-agent and multi-regional Monte Carlo model. It has a rich demographic core generating the life histories of different population cohorts. It also simulates production processes, incomes of different “agents” (individuals, governments, etc.), capital accumulation, employment, savings, capital flows, the private and the public sector, including firms, governments, public and private pensions and health care. In other words, it provides the fine-grained balances of different age-specific stocks and flows. Section 4 describes the extensions of this model to address explicitly issues of optimization, in particular, the optimal social security composition. Section 5 outlines the main ideas of the adaptive Monte Carlo optimization procedure developed. Section 6 discusses some numerical experiments. In particular, it is evident that an increase in any source of uncertainty leads to more precautionary saving, arising, e.g., from uncertainty about future employment and income perspectives, future life span and associated pension benefits. The proposed model may have different applications. Section 7 concludes with the discussion of important applications to catastrophic risk management.

2. A Model with Identical Individuals

In order to better understand our model let us consider a simple version, which synthesizes, in a sense, models analyzed by Blanchet and Kessler [4], and Belan and Pestiean [3]. The core of this model is a simple “birth-and-death” process for agents (individuals) involved in the production – consumption processes. It is assumed that each individual lives for two periods. All N_t, identical individuals born at time t, comprise generation t of “workers”, whereas N_{t-1}, individuals born at time $t-1$, comprise the generation of “former workers” or pensioners. Generation N_t produces outputs and receives an income (a wage) W_t per worker, which is subdivided into its current consumption c_t, savings $S_t = W_t s_t$ for retirement at $t+1$, and the contribution $W_t \tau_t$ to support the generation N_{t-1} in return to an intergenerational transfer P_{t+1} at time t, where $0 \le s_t \le 1$, $0 \le \tau_t \le 1$. Since individual contributions τ_t of generation N_t finance generation N_{t-1}, $N_{t-1} P_t = N_t W_t \tau_t$. Denoting n_t as the growth rate of population between $t+1$ and t, this leads to

$$P_t = (1+n_t)W_t\tau_t,\ n_t = (N_t - N_{t-1})/N_{t-1} \quad (1)$$

for $t = 1,2,...$. Assuming that savings S_t for retirement at $t+1$ are invested in a pension fund, then consumption c_t, d_{t+1} of an individual in the first t and second $t+1$ period is

$$c_t = W_t(1 - s_t - \tau_t), \quad (2)$$

$$d_{t+1} = (1+r_{t+1})W_t s_t + P_{t+1} = (1+r_{t+1})W_t s_t + (1+n_{t+1})W_{t+1}\tau_t,$$

where r_t is the interest rate. We can rearrange d_{t+1} by using the growth rate of wages $w_t = (W_t - W_{t-1})/W_t$. Since the product $w_t n_t$ is small in real problems, this leads to

$$d_{t+1} = W_t\left[(1+r_{t+1})s_t + (1+n_{t+1}+w_{t+1})\tau_{t+1}\right]. \quad (3)$$

From this simple relations follows the simplification used by Samuelson [17] and Aaron [1], the so-called "social security paradox": if the sum $n_{t+1} + w_{t+1}$ exceeds the rate of returns r_{t+1} on investments, then an individual will receive a higher rate of return from intergenerational transfer τ_t through the social security system than by savings s_t with further investments. The Samuelson-Aaron law also states that both schemes are equivalent for $r = n + w$.

The limitations of this law are evident. Let us illustrate possible complexity emerging from explicit introduction of uncertainties and risks. Consider a simple situation, when n, r, w and s, τ do not depend on t and $W_t = 1$, i.e., the consumptions for two periods are defined as $c = 1 - s - \tau$, $d = \left[(1+r)s + (1+n+w)\tau\right]$, where w, r, n are random variables. For the sake of simplicity let us assume that these are mutually independent variables. Then, the expected consumptions are

$$\bar{c} = 1 - s - \tau,\ \bar{d} = \left[(1+\bar{r})s + (1+\bar{n}+\bar{w})\tau\right],$$

where $\bar{c}$, $\bar{d}$, $\bar{r}$, $\bar{n}$, $\bar{w}$ are expected values of c, d, r, n, w. The real consumptions c, d may deviate from their expected values $\bar{c}$, $\bar{d}$. In particular, this may lead to a difference between the consumption in the first period c and the consumption d at retirement, which calls for the introduction of appropriate risk measures. Let us consider a risk measure similar to the measure used by Markowitz [13]. We can formulate a two-criteria optimization problem, where the first criterion is the total expected consumption for two periods $\bar{c} + \gamma\bar{d} = 1 - s - \tau + \gamma\left[(1+\bar{r})s + (1+\bar{n}+\bar{w})\tau\right]$ with a weight $\gamma > 0$, and the second criterion is the expected least square deviation $E(c - \gamma d)^2$ between consumptions $c, \gamma d$. Assume that $\gamma = 1$ and let us examine the Pareto efficient solutions by maximizing

$$\bar{c} + \bar{d} - \alpha E(c-d)^2 \to \max_{s,\tau\ge 0, s+\tau\le 1} \quad (4)$$

for a given $0 < \alpha \le 1$ subject to $s + \tau \le 1$, $s \ge 0$, $\tau \ge 0$. Constrain $s + \tau \le 1$ is clear because the sum of contribution rates into the pension systems out of wages can not be greater than 1. This leads to the maximization of the following function

$$F = \left[1 + \bar{r}s + (\bar{n}+\bar{w})\tau\right] - \alpha E\left[1 - (2+r)s - (2+n+w)\tau\right]^2. \quad (5)$$

Assume that $s > 0$, $\tau > 0$, $s + \tau < 1$, for the optimal solution. In this case the optimality condition yields

$$\frac{\partial F}{\partial s} = \bar{r} + 2\alpha\left[(2+\bar{r}) - E(2+r)^2 s - (2+\bar{n}+\bar{w})(2+\bar{r})\tau\right] = 0\,, \tag{6}$$

$$\frac{\partial F}{\partial \tau} = (\bar{n}+\bar{w}) + 2\alpha\left[(2+\bar{n}+\bar{w}) - (2+\bar{n}+\bar{w})(2+\bar{r})s - E(2+n+w)^2\tau\right] = 0\,. \tag{7}$$

Proposition 1. Suppose that $\bar{r} = \bar{n} + \bar{w}$, i.e., the assumptions of the Samuelson-Aaron law are fulfilled on average. Then the superiority of funded or a pay-as-you-go schemes depends on $Var(r)$ and $Var(n+m)$. If $Var(r) > Var(n+m) > 0$, then partially funded schemes are optimal, i.e., $s > 0, \tau > 0$ for the optimal solution. If $Var(r) = Var(n+m)$, then both schemes are equivalent. This is a stochastic version of the Samuelson-Aaron law. If $Var(r) < Var(n+w)$, then the funded scheme becomes preferable, i.e., optimal $s > \tau$. Otherwise, a pay-as-you-go scheme becomes preferable, i.e., optimal $\tau > s$.

Let us demonstrate this. If $\bar{r} = \bar{n} + \bar{w}$, but $s > 0$, $\tau > 0$, $s + \tau < 1$ for optimal solutions, then it follows from (6), (7) that

$$\frac{\bar{r}}{2\alpha} + (2+\bar{r}) - \left[E(2+r)^2 s + (2+\bar{n}+\bar{w})^2\tau\right] = 0\,, \tag{8}$$

$$\frac{\bar{r}}{2\alpha} + (2+\bar{r}) - \left[(2+\bar{r})^2 s + E(2+n+w)^2\tau\right] = 0\,. \tag{9}$$

This yields

$$\frac{s}{\tau} = \frac{E(2+n+w)^2 - (2+\bar{n}+\bar{w})^2}{E(2+r)^2 - (2+\bar{r})^2}\,, \text{ or } \frac{s}{\tau} = \frac{Var(n+w)}{Var(r)}\,. \tag{10}$$

Thus, if $\bar{r} = \bar{n} + \bar{w}$, $Var(n+w) = Var(r)$, then both schemes are equivalent. We also see from (10) that $Var(r) < Var(n+w)$ implies $s > \tau$ and *vice versa*. But still we need to show that for optimal s, τ, $s > 0, \tau > 0$.

Since $0 < s + \tau$ for optimal solutions, (0,0) cannot be an optimal solution. Let us show that $s = 0, \tau > 0$ or $s > 0, \tau = 0$ cannot be an optimal solution either, assuming $s + \tau < 1$. If $s = 0, \tau > 0$, then, by substituting $(\bar{n}+\bar{w}) = \bar{r} + 2\alpha(2+\bar{r})$ from equality (7) into (6), we get $\partial F/\partial s = 2\alpha\left[E(2+n+w)^2 - (2+\bar{n}+\bar{w})^2\right]\tau = 2\alpha\tau Var(\bar{n}+\bar{w})$. Hence, $\partial F/\partial s > 0$ at $s = 0$, $\tau > 0$, which contradicts the standard optimality conditions requiring $\partial F/\partial s < 0$. Similarly, if $s > 0, \tau = 0$, then it follows from (7) that $\partial F/\partial\tau = 2\alpha\left[E(2+r)^2 - (2+\bar{r})^2\right]s = 2\alpha s Var(r)$, which contradicts $\partial F/\partial\tau < 0$.

The aim of problem (5) is , roughly speaking, to maximize the total consumption over the lifetime by keeping, in a sense, the balance between random present (c) and future (d) consumptions using a penalty parameter α. Definitely, unconditional optimization of (5) for small enough α cannot guarantee appropriate balances and may even lead to infeasible solutions s and τ with $s+\tau>1$. The requirement $s+\tau<1$ is automatically ensured, e.g., by using the additional penalty function in the form of $\min\{0, s+\tau-1\}^2$, or by using the Lagrange multipliers. It is also possible to show, e.g., that $s+\tau<1$ for optimal solutions, if $\alpha \geq \bar{r}/4$. The same conclusion as in Propositions 1 can be achieved by applying the Markowitz mean-variance approach directly taking into account the balance equation between present and future consumption $\bar{c} < \gamma\bar{d}$ as the additional constraint:

$F = \bar{c} + \bar{d} - \alpha Var(c+d)$, subject to $\bar{c} = \gamma\bar{d}$, $s, \tau \geq 0$.

By using the Langrange multiplier it is again possible to show (10) and $s+\tau<1$ for optimal solution for all $\alpha>0$ (by calculating optimal s, τ directly).

Let us now illustrate the next essential feature of the simulation model described in the following sections. Namely, how decisions on savings s_t affect w_t, r_t through investments and capital accumulation. The production sector of the economy is described by a production function $Y_t = F(K_t, N_t)$, where K_t is the capital stock at time $t = 0,1,\ldots$. Wages W_t and returns r_t are calculated in the standard way as

$$W_t = F_N(K_t, N_t), r_t = F_K(K_t, N_t), \tag{11}$$

where F_N, F_K are derivatives of F with respect to N, K. The capital stock K_t changes over time as

$$K_{t+1} = (1-\delta)K_t + I_t, t = 0,1,..., \tag{12}$$

where δ denotes the depreciation rate of capital stock, I_t is total investments over the population, which are usually equal total savings. An important case is the Cobb-Douglas production function

$$Y_t = A_t K_t^b N_t^{1-b}, \tag{13}$$

where $0<b<1$ and A_t is a positive parameter. This parameter can reflect the productivity of workers, technological progress. Random components of A_t allow us to model economic instabilities. If we denote per capita levels of output and capital $y = Y/N$, $k = K/N$, then for function (10) the production of outputs and the accumulation of capital can be characterized to lie between per capita levels y, k only, i.e.,

$$y_t = A_t k_t^b, W_t = (1-b)A_t k_t^b, r_t = bA_t k_t^{b-1}, \tag{14}$$

$$(1+n_t)k_{t+1} = (1-\delta)k_t + s_t, \tag{15}$$

These equations significantly modify the simple linear relations (2), (3). In Section 4 we formulate stochastic optimization problems with s_t, r_t treated as decision variables.

3. The Multi-regional and Multi-agent Simulation Model

The economic-demographic simulation model developed at IIASA has the same basic features as the simplest model described above. This section provides only a general description of the model. Detailed analyses can be found in [11].

The model presented here is a compromise between the two extremes of a purely actuarial approach and an overlapping generations computable general equilibrium model. The actuarial models contain detailed demographic projections but very little representation of the economy. On the other hand, the traditional economic models lack sufficient demographic detail. Besides, the larger part of traditional economic modeling is centered around perfect markets in the state of equilibrium without paying attention to transition paths, adequate treatment of uncertainty, and the rich variety and complexity of dynamic interactions between different relevant “agents”.

Conceptually, the model is composed of entities, which can be called agents. These are region-specific households subdivided into single-year age groups, firms, governments and financial intermediaries, including pension systems, banks, insurance, mutual funds. The agents are involved in production of outputs, their distribution, transfer, savings, accumulation and consumption. The model tracks incomes and outcomes of households by single-year age groups, as well as intergenerational and interregional transfers of resources. Households accumulate assets during working years and then dissave in retirement, in addition to which intergenerational transfers between the working and the retired populations are mediated through the pay-as-you-go public pension system.

Production processes are characterized by a Cobb-Douglas production function. Other forms of production functions can be used as well. Rates of return and wages are endogenous as in (11). A special procedure creates age-specific wage-rate profiles. The sources of household income are wages, rents from residential capital, dividends distributed from earnings on capital operated by firms, public social security system benefits, and private pension benefits. All taxation is assumed to occur at the level of incomes.

The population is divided into age groups $t = \overline{0,T}$, $T \leq 100$ (for population projections, see [10]). There are different options to modeling aging processes. In the simplest option, a single deterministic or stochastic demographic scenario consisting of population by age group is input from another source. In the second option, a deterministic or stochastic population scenario is produced within the model using the age-specific mortality rate and the flow of net migrants.

Apart from demographic uncertainties, social security may be significantly affected by economic, social, and political uncertainties. It is widely practiced (see, for example, [5]) to represent variables such as GDP, investment, government consumption, and price indices using ARCH-type processes. Thus, in our experiments (Section 6) the scale parameter A_t of the production function (similar to Section 2) is assumed to follow an ARCH-M process, in which the mean of the variable depends on its own conditional variance. The process is modeled as $A_t = \mu(t) + \varepsilon(t)$, where $\mu(t) = A^* + \delta h(t)$,

$h(t) = \sqrt{\alpha_0 + \sum_{i=1}^{q} \alpha_i \varepsilon^2 (t-i)}$, $\delta > 0$, $A^* = 1200$ is the baseline parameter and we assume $\varepsilon(t) \sim N(0, 0.05A^*)$. In empirical analysis, the parameters δ, α_0, and α_i must be econometrically estimated according to region-specific data. We assumed parameters $\delta = 1, \alpha_0 = 0$, $\alpha_1 = 0.65$ (see, for example, [5]) suitable for modeling developed economies.

Capital is either residential or non-residential. The latter is further subdivided into capital operated by private unincorporated enterprises and capital operated by firms, i.e., corporate enterprises. Residential capital is installed entirely in the home region and is held by households directly. Capital operated by corporate enterprises is installed either at home or abroad.

Financial claims on this capital are held on behalf of households by institutions which collect and distribute dividends. These institutions comprise the private pensions system and other financial institutions such as banks and mutual funds. Foreign investment can consist either of portfolio claims or foreign direct investment.

Persons above the age of eligibility for social security benefits are entitled to public pension system benefits calculated on the basis of their years of employment, the number of years they have been retired, the degree of indexation of pension benefits to real wages, and the evolution of wages since their retirement. Private pension system benefits represent the sale of financial assets. The public pension system is assumed to be financed on a pay-as-you-go basis. In the classic pay-as-you-go system total contributions equal total benefits; there is neither accumulation of a surplus nor a deficit to be financed out of general government revenues. However, there are cases where nominally pay-as-you-go systems are currently running surpluses in order to accumulate resources to deal with demographic instabilities. This is possible in our model.

The assumption is made that, when wealth is inherited, it is converted to cash, some of which is allocated to consumption and the remainder is allocated among residential or non-residential forms of capital. Consumption comes out of income, out of the proceeds of asset sales, sales of inherited assets and retirement dissaving. Household net saving is the difference between real income and consumption.

Firms operate capital installed at home and abroad; they earn profits and pay out direct taxes and dividends. In the case of portfolio investment abroad, profits are credited to firms in the foreign region; in the case of foreign direct investments, earnings are credited to firms in the home region.

The government consumes a share of GDP, makes interest payments on public-sector debt, collects taxes and social security contributions and pays social security benefits.

This short summary of the Monte Carlo simulation model illustrates its complexity and rich variety of dynamic interactions between different agents. The model simulates in time random trajectories of different variables, in particular, paths of age-specific consumption. Conceptually, this is easy to see from equations (2), (3), (11-15). The age-specific profile of the population is in this case (N_{t-1}, N_t) generated by the population growth model (or by some scenarios). K_t, capital available at time t, and

N_t allow to calculate incomes W_t and returns r_t by (11). Given decisions s_t, τ_t allow to calculate savings S_t, consumptions c_t, d_t, investments I_t and capital K_{t+1} by (2), (3), (11-15), etc.

4. The Stochastic Optimization Problem

Section 2 illustrates possible effects of stochastic models. A more general model can be formulated by using equations (2), (3), (11-15) as follows. Assume that A_t, n_t are time dependent random variables describing economic and demographic uncertainties ("shocks"). In the general case these may be mutually dependent variables with distributions dependent on k_t, which is essential for modeling catastrophic shocks [7]. Due to the randomness of n_t, variables c_t, d_t are also random, i.e., they may drop below a given target level of the life standard. This calls for the use of appropriable risk indicators (measures). The choice of the risk measure in the simplest model of Section 2 is motivated primarily by its analytical tractability. Unfortunately, this indicator is suitable only for normal distributions of c_t, d_t which cannot be expected from non-linear path-dependent transformations (13-15). Instead, it is important to impose "survival" constraints on the probability of consumptions c_t, d_t, $t = 1,2,...,T$, to drop below target ("ruin") levels $\underline{c}_t$, $\underline{d}_t$, i.e.,

$$\text{Prob}\left[c_t \geq \underline{c}_t, d_t \geq \underline{d}_t, t = 1, \ldots, T\right] \geq 1 - p\,. \tag{16}$$

Here p is a given "safety" level, say, the "ruin" may occur only once in 100 years. The goal can be formulated as the maximization of the expected weighted intergenerational consumption. It could also be formulated as the maximization of the expected "utility" (welfare) function

$$E\, u(c_1, d_1, c_2, d_2, \ldots, c_T, d_T)\,, \tag{17}$$

if this function can be defined in a meaningful way. For example, we can think of the maximization of expected weighted minimal consumption for periods $t = 1,2,\ldots,T$,

$$E \min_{1 \leq t \leq T} \{\beta_t (c_t + \gamma_t d_t)\}, \tag{18}$$

where $0 < \beta_t < 1, 0 < \gamma_t < 1$.

There is a significant complexity involved in dealing with constraints (16). They prevent the frequency of the ruin and do not pay attention to the size of shortfall. This is why the function defined by the left-hand side of (16) is not concave and may be a discontinuous function, even in the case of random variables c_t, d_t linearly dependent on decision variables $x = (s_1, \tau_1, \ldots, s_{t-1}, \tau_{t-1})$. To deal with this complexity let us consider the maximization of the function

$$F(x) = E\, u(c_1, d_1, c_2, d_2, \ldots, c_T, d_T) + \alpha \sum_{t=1}^{T} E \min\{0, c_t - \underline{c}_t, d_t - \underline{d}_t\}, \tag{19}$$

where $\alpha > 0$ is the risk factor, providing a trade-off between the expected utility of the consumptions and the risk associated with the violation of leaving standards $\underline{c}_t$.

Proposition 2. If α is large enough, then maximization problem (19) approximates arbitrarily well the maximization problem (16), (17), i.e., a solution of problem (19) can be chosen as a solution of problem (16), (17).

The details of this proposition and its proof can be found in [6]. Let us only mention that problem (19) preserves important concavity property inherent to functions c_t, d_t linear in x, whereas the problem (16), (17) transforms them into highly nonlinear, non-concave and possibly discontinuous functions through constraints (16).

Proposition 2 is a key for the solution procedure proposed in the next Section for the following general problem. The simulation model of Section 3 generates in time t, $t = \overline{1,T}$, a path of age-specific consumptions, which we define as $C_m^t(x^t, \omega^t)$. It is indexed by age m, $m = \overline{1,M}$, and it depends on decisions $x^t = \{x(0), x(1), \ldots, x(t)\}$, where components of vector $x(t)$ denote the contribution rate to the public pension system and the contribution rate to the private pension system (variables of type τ_t, s_t, as in Section 2). The set of decisions may also include taxes and borrowings as is discussed in [3]. The consumption $C_m^t(x^t, \omega^t)$ depends also on "shocks" ω^t until time t.

Denote $x := x^T$, $\omega := \omega^T$, $C(x,\omega) := \{C_m^t(x^t, \omega^t), m = \overline{1,M}, t = \overline{0,T}\}$, and let $u(C(x,\omega))$ be the utility of consumption $C(x,\omega)$, $U(x) = E\, u(C(X,\omega))$. For example, $u(C(x,\omega)) = \sum_{t=1}^{T} \sum_{m=1}^{M} \gamma_m^t u_m^t (C_m^t(x,\omega))$, where $u_m^t(\cdot)$ is the utility of consumption at t of a cohort m, γ_m^t are the welfare weights specified for each cohort m at time t. The function $U(\cdot)$ can also be defined similar to (19). Assume that the leaving standards are defined by threshold curves $\underline{C}_m^t$. A rather general problem (similar to (19)) is to maximize function

$$
\begin{aligned}
&F(x) = E\ f(x,\omega), \\
&f(x,\omega) = u(C(x,\omega)) + \alpha \sum_{t=1}^{T} \sum_{m=1}^{M} \gamma_m^t \min\left\{0, C_m^t(x,\omega) - \underline{C}_m^t\right\}
\end{aligned}
\tag{20}
$$

The discussion of risk measures similar to (16), (17), (19), the so-called Value-at-Risk (VaR) and Conditional-Value-at-Risk (CVaR), can be found in [2], [16].

The main challenge of optimization of F(x) is the lack of exact information on $F(x)$ for a given x. Each run of the simulation model generates random outcomes of the sample performance function $f(x,\omega)$, which can be used in the search of a desirable decision by stochastic optimization techniques proposed in the next section.

5. Optimization Procedures

There are three main optimization approaches which can be used in Monte Carlo simulation models. One family of procedures to optimize $F(x)$ implicitly given by (20) is known as *Response Surface Methods*. In these methods, the function $F(x)$ is approximated by a quadratic function (or other easily estimated) $Q(x)$ in the neighborhood of a current solution x by using random values $f(x,\omega)$. Then, standard optimization techniques are used for $Q(x)$ to produce a new solution. Again, random values $f(x,\omega)$ are generated for the new x, leading to a new response surface $Q(x)$, and so on. This method requires estimation and updating of function $Q(x)$ at each step of the search procedure, which may become practically impossible when the number of decision and stochastic variables increases.

An alternative method is to use an approximation of the function $F(x)$ in the whole feasible set. The most important approximation of $F(x)$ is defined as the sample mean, which is obtained by preliminary simulation of S paths ("life" histories) $\omega_1,\ldots,\omega_S$:

$$F^S(x) = \frac{1}{S}\sum_{s=1}^{S} f(x,\omega_s).$$

The use of this approach is restricted to cases when the sample function $f(\cdot,\omega)$ has a well defined analytical structure, which does not apply our case. It also requires smoothness of $F(x)$ which may not be the case with $F(x)$ defined by (20).

The proposed Adaptive stochastic optimization procedure is based on Stochastic Quasigradient Methods (see general description, e.g., in [8], [9]), which do not require a preliminary approximation of $F(x)$. The adjustment of decisions x takes place in an adaptive way simultaneously with the sampling of ω. The principal idea is the following. Let x^0 be an initial solution and x^s an approximate solution after s steps. The direction of movement from x^s to the next x^{s+1}, $s = 0,1,\ldots$, is a statistical estimate ξ^s of the gradient $\nabla F(x^s)$ (or its analogue for non-smooth function $F(x)$), i.e.,

$$x^{s+1} = x^s + \rho_s \xi^s,\ s = 0,1,\ldots, \tag{21}$$

where ξ^s is called a stochastic quasi-gradient of the function F at x^s. For example, we can use

$$\xi^s = \frac{f(\tilde{x}^s + \Delta_s h^s, \omega^s) - f(\tilde{x}^s, \omega^s)}{\Delta_s} h^s\ , \tag{22}$$

where $\Delta_s > 0$, $\Delta_s \to 0$, $\tilde{x}^s$ is uniformly distributed in the α_s-vicinity of $\tilde{x}^s$, $\alpha_s \to 0$, and h^s, with independent components uniformly distributed on [-1, 1].

Convergence of the sequence x^s, $s = 0,1,\ldots$ to optimal solution is ensured by appropriate choice of ρ_s, and Δ_s in particular, and $\rho_s = 1/s$, $\Delta_s = 1/s$ is possible. General convergence conditions on algorithm parameters have the form of:

$\sum_{s=0}^{\infty} \rho_s = +\infty, \quad \sum_{s=0}^{\infty} \rho_s^2 < +\infty, \quad \frac{\Delta_s}{\alpha_s} \to 0, \quad \alpha_s \to 0$ for the convex case, and additionally

$\frac{\rho_s}{\alpha_s} \to 0, \quad \frac{|\alpha_s - \alpha_{s+1}|}{\rho_s} \to 0, \quad EL^2(\omega) < \infty$ for a general case of the Lipschitz function $f(x,\omega)$ with Lipschitz constant $L(\omega)$ (see [14]).

The procedure (21), (22) is easily incorporated into the model of Section 3, allowing for adaptive optimal "tuning" of decisions simultaneously with Monte Carlo simulations. Namely, at each step s, $s = 0,1,\ldots$, random variables ω^s, h^s and outcomes $f(x^s, \omega^s)$, $f(x^s + \Delta_s h^s, \omega^s)$ are simulated, vector ξ^s is calculated, current x^s is adjusted to x^{s+1} by (21), (22); new ω^{s+1}, h^{s+1} and corresponding outcomes are simulated again, and so on. The procedure (21) is rather flexible to take into account available analytical "blocks" of the model for designing more efficient stochastic quasigradients ξ^s. For example, if the goal is similar to (18), i.e., $u(C(x,\omega)) = \min\{C_m^t(x,\omega), 1 \le t \le T, 1 \le m \le M\}$, then we can use

$$\xi^s = \left[C_{m_s}^{t_s}(\tilde{x}^s + \Delta_s h^s, \omega^s) - C_{m_s}^{t_s}(\tilde{x}^s, \omega^s)\right] / \Delta_s -$$
$$\alpha \textstyle\sum_{t.m}^{-} \gamma_m^t \left[\left[C_{m_s}^{t_s}(\tilde{x}^s + \Delta_s h^s, \omega^s) - C_{m_s}^{t_s}(\tilde{x}^s, \omega^s)\right] / \Delta_s\right]$$

where m_s, t_s are the worst-case responses in $u(C(x,\omega))$ for $x = x^s$, $\omega = \omega^s$, and $\sum_{t,m}^{-}$ denotes the sum over these t, m, where $\min\{0, C_m^t(x^s, \omega^s) - \underline{C}_m^t\} < 0$. The applicability of vector ξ^s in (21) can be derived from the general results for stochastic minimax (maximin) problems [8], [9].

6. Numerical Experiments

In this study the model is calibrated for a country with GDP per capita on average equal to 36,000 USD per annum, and a capital to output ratio equal to 2.6. The model was implemented on a Pentium 4 PC. The computational time for solving the problem discussed here is approximately 30 minutes.

The function to be maximized is defined by (20). The decision variables are contribution rates, τ, to a pay-as-you-go (PAYG) and private funded pension systems (s). Another decision variable is the so-called replacement ratio (initial pension relative to average wage during the 3 years prior to retirement) involved in pensions assignment at the retirement age. The choice of the decision variables is driven by the dominating discussion of fairness in the reform of social security. In particular, scarce labor force is no longer able to finance the growing number of retirees. The smooth tradeoff between necessary measures to increase the contribution rate and to decrease the benefits can be achieved through a well-balanced combination of contribution and replacement rates. Implicit in the constraint (16) implied by (20) is that the optimal PAYG contribution rate and the replacement rate will be consistent.

Improvements are tracked with respect to the consumption of workers and retirees in per capita terms. Stochastic shocks (as discussed in Section 3) administered to the model cause heavily tailed distributions. Therefore, to track the distributions of the main variables, we present the results in the form of some important percentiles. Table 1 shows initial and optimal values of decisions variables. The initial solution was chosen from a preliminary analysis of the model (see, for example, [12]).

	Contribution Rate Funded System	Contribution Rate PAYG	Replacement Ratio PAYG
Initial	0.045	0.085	0.1
Optimal	0.06	0.035	0.12

Table 1. Initial and optimal values of decision variables.

At a first glance the changes to the decision variables are not significant: the contribution rate to the PAYG system decreased from 8.5% to 3.5%, the replacement ratio increased from 0.1 to 0.12, and the contribution rate to the private defined contribution pension system changed from 4.5% to 6%. On the other hand, at the end of the simulation time horizon Figures 1-4 indicate an increase of the 5-th percentile workers' consumption by 1% and of retirees by 1.5%. Consumption of both age groups consists not only of wage income (see Section 3), but also of capital related earnings – dividends, unincorporated enterprises earnings, etc. Because of the implied growth model, wages increase with the increase of capital, e.g., when savings within the funded pension system increase. Increased savings, on the other hand, cause a decrease of the capital return rate, and, therefore, capital related earnings. Thus, the overall increase of consumption occurs only when an appropriate balance between savings within two pension systems is found.

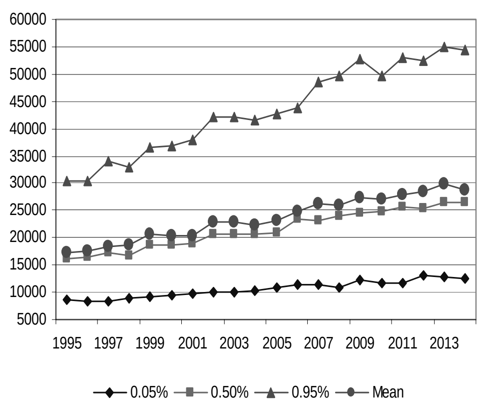

Fig. 1. Initial Consumption of Workers.

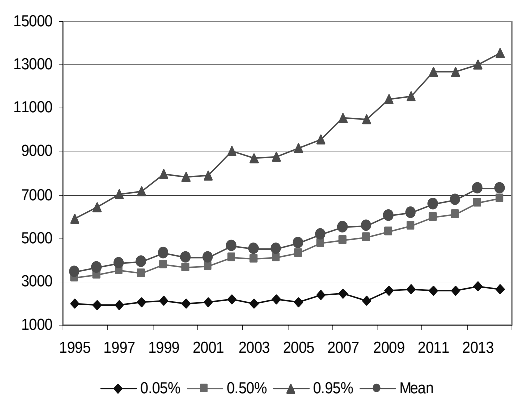

Fig. 2. Initial Consumption of Retirees.

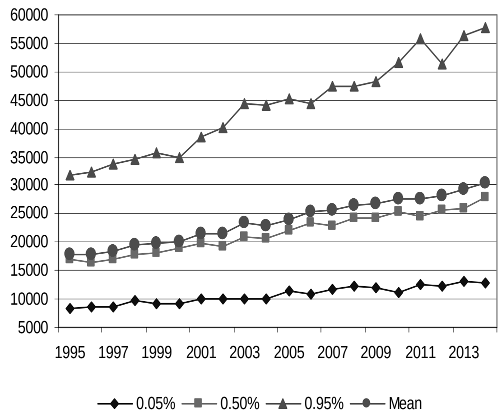

Fig. 3. Optimal Consumption of Workers.

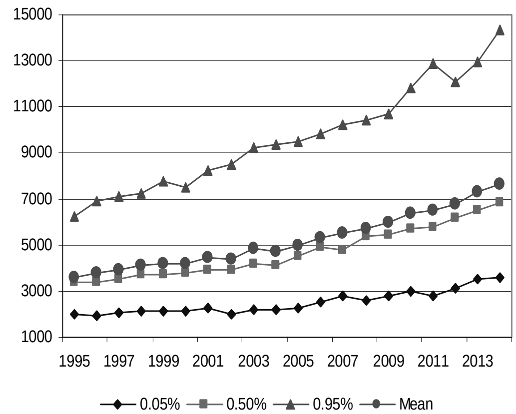

Fig. 4. Optimal Consumption of Retirees.

7. Concluding Remarks

While the proposed model is suitable for a wide range of applications dealing with long-term economic growth, it is especially designed to simulate effects of population growth and aging. An important area of applications may be connected with the analysis of macroeconomic impacts of catastrophes. The increasing catastrophic losses [15] are mainly due to "path-dependencies" in movements of people and capital to risk-prone areas. The rich demographic core and spatial features of the model enable us to evaluate regional vulnerability and the necessary measures for reducing and spreading catastrophic losses. In particular, explicit introduction of region-specific risks may show the necessity of the "capital flight" (including human capital) from risk-prone areas, i.e., the interventions in alarming path-dependent capital accumulation processes.

In dealing with "low probability - high consequences" catastrophic risks it may be important to modify the objective function (17) to a more "conservative" one (with respect to risk indicators):

$$f(x,\omega) = u(C(x,\omega)) + \alpha \min\left\{0, \min_{t,m} \gamma_m^t C_m^t(x,\omega) - \underline{C}\right\},$$

where $\underline{C}$ is a (normalized by γ_m^t) "critical" level. The occurrence of catastrophes in time and space can be modeled by using a similar approach to the one developed in [6]. Our quite different numerical experiments show the feasibility of the proposed approach. To take into account abrupt shocks and other significant ex-post information, which may appear during the given time horizon, we used (in the numerical experiments) the concept of the so-called two-stage dynamic stochastic optimization models [9] with rolling horizon. This allows us to avoid the multistage models, which require explicit analytical (linear) structures and a small number of random parameters.

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