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AGGREGATION AND DECOMPOSITION IN POPULATION PROJECTION

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Preface

A central concern of the Human Settlements and Services research group at IIASA has been the analysis of the dynamics of multiregional population growth and distribution. Recently this activity has stimulated a concerted effort to extend and expand the applicability of mathematical demographic models in the study of such dynamics. This paper, the fifth of a series addressing the general topic of spatial population dynamics, considers the fundamental problem of reducing the dimensionality of large-scale population projection models by means of aggregation and decomposition.

A fuller version of this paper will be forthcoming in Environment and Planning.

Papers in the Spatial Population Dynamics Series

1. Andrei Rogers and Frans Willekens, "Spatial Population Dynamics," IIASA RR-75-24, (Laxenburg, Austria, International Institute for Applied Systems Analysis, 1975); forthcoming in Papers, Regional Science Association.

2. Andrei Rogers and Jacques Ledent, "Multiregional Population Projection," IIASA internal paper, forthcoming in Proceedings, 7th I.F.I.P. Conference, 1976.

3. Andrei Rogers and Jacques Ledent, "Increment-Decrement Life Tables: A Comment," IIASA internal paper, forthcoming in Demography.

4. Andrei Rogers, "Spatial Migration Expectancies," IIASA RM-75-57 (Laxenburg, Austria, International Institute for Applied Systems Analysis, 1975).

5. Andrei Rogers, "Aggregation and Decomposition in Population Projection," IIASA-RM-76-11 (Laxenburg, Austria, International Institute for Applied Systems Analysis, 1976).

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Abstract

During the past two decades social scientists have come to model dynamic socioeconomic systems of growing size and and complexity. Despite a heavy reliance on ever more sophisticated high-speed digital computers, however, computer capacity for handling such systems has not kept pace with the growing demands for more detailed information. Consequently, it is becoming ever more important to identify those aspects of a system which permit one to deal with parts of it independently from the rest or to treat relationships among particular subsystems as though they were independent of the relationships within those subsystems. These questions are, respectively, those of aggregation and decomposition, and their application toward "shrinking" large-scale population projection models is the focus of this paper.

1. Introduction

Imagine a demographer faced with the problem of projecting, in a consistent manner and in age-specific detail, the future populations of the 265 Standard Metropolitan Statistical Areas (SMSA's) of the contemporary United States. Such a large-scale system is beyond the data processing capabilities of his digital computer and, moreover, would be needlessly cumbersome in light of certain observed weak connectivities between several subsystems of SMSA's. Consequently, it is important to find methods for "shrinking" the computational requirements for such a task, and the two most obvious methods for effecting such a reduction are aggregation and decomposition.

1.1 Aggregation

The need to use aggregates arises out of sheer necessity in most social science research involving large systems. Theoretical abstract reasoning and numerical empirical computation both rely on the conceptual clarity and efficient manipulation of variables afforded by aggregation. In economic modelling, for example, the many producers and consumers of a national or regional economy are aggregated into a relatively small number of sectors, and the interaction among these sectors is then studied as though it were free of influences arising from intrasectoral interaction. A typical example of this occurs in input-output analysis, and indeed it was the increasing world-wide numerical application of such models that first stimulated much of the interest in aggregation among social scientists (e.g., Ara, 1959; Fisher, 1969; Rogers, 1969).

Aggregation generally introduces inconsistencies between the outputs of the disaggregated and aggregated models. The conditions for aggregation without such inconsistencies, that is, for perfect aggregation, are very severe and therefore are almost never met in practice. However, since any model is at best only an approximate description of reality, we remain interested in establishing the conditions under which perfect aggregation may be carried out. These conditions suggest the criteria, or rules, for selecting which variables to aggregate and help to identify the circumstances under which such an aggregation will yield results that are consistent with those of the original disaggregated model.

Aggregation of large scale problems, therefore, has two fundamental aspects. The first is the process of consolidation itself. Here the two sets of variables that are connected by a system of relations are grouped into aggregates and a new smaller system of relations is developed which connects the two sets of aggregates. The second fundamental aspect of the aggregation process is the selection of the consolidation scheme that most closely satisfies the conditions necessary for perfect aggregation, while at the same time meeting whatever informational requirements and additional constraints that may have been specified a priori. In short, consolidation is an operation that expresses a set of "new" variables as weighted averages of the set of original "old" variables, such that there are fewer new variables than old variables. Criteria for perfect aggregation, on the other hand, are rules that indicate which variables to consolidate, for example, the rule that variables which always move together may be consolidated into a single variable without introducing an aggregation error.

Two particular forms of aggregation are frequently employed in demographic analysis. The first is a consolidation across age groups. When carried out over <u>all</u> age groups, this form of consolidation transforms a cohort-survival model into a components-of-change model (Rogers, 1971, Ch.1). We shall, therefore, refer to aggregations of this sort as <u>components-of-change</u> aggregations. Such aggregations retain the geographical areal units of the original cohort-survival model but sacrifice all age-specific details.

The second form of aggregation that is frequently used is a division of a multiregional population system into two regions: a particular region under study and "the rest of the world." Such consolidations will be called <u>biregional aggregations</u> in this paper. They sacrifice considerable geographical information but preserve details about age compositions. However, if applied in sequence to each and every region of a multiregional system, they permit a <u>collection</u> of aggregated projections to completely preserve the levels of detail found in the original unconsolidated projection.

1.2 <u>Decomposition</u>

The idea of decomposing a large and complex problem into several smaller subproblems in order to simplify its solution is not new and indeed has been used for well over a century in the physical and social sciences, as well as in engineering. However, the development and use of high-speed computers to solve these problems during the past two decades has stimulated a focused interest in decomposition techniques in such various fields of application as process control, structural engineering, systems optimization, electrical network theory, and a wide variety of seemingly unrelated problems in economics, mathematics, design, and operations research (e.g., Himmelblau, 1973; Rose and Willoughby, 1972; Tewarson, 1973; and Theil, 1972).

The central principle of decomposition analysis is that the solution of a large systems problem, involving many interacting elements, often can be broken up and expressed in terms of the solutions of relatively independent subsystem problems of lower dimensionality. The solutions of the subsystem problems then can be combined and, if necessary, modified to yield the solution of the original large-system problem. A well-known illustration of this approach is provided by the Dantzig and Wolfe decomposition algorithm in mathematical programming (Dantzig and Wolfe, 1960).

This algorithm breaks up a large linear programming problem into several smaller linear programming problems and imposes additional constraints on each of the latter in order to ensure that their solutions combine to yield the optimal solution for the large scale problem.

Decompositions of large-scale problems generally proceed in two stages. First there is the partitioning stage in which a large system of variables and relations is rearranged and reordered in a search for disjoint subsystems, that is, subsets of relations which do not contain any common variables. If such subsystems exist, then each one can be treated independently of the rest. In this way the relational structure of the original large-scale problem can be exploited to produce a more efficient solution method.

Systems that can be partitioned into independent (disjoint) subsystems are said to be completely decomposable, and their matrix expression can be transformed into what is known as a block-diagonal form. The rearrangement and reordering of the relations to identify and delineate the disjoint subsystems is called permutation, and the actual separation of the large system into disjoint subsystems is called partitioning.

Partitioning of a large system into disjoint subsystems obviously cannot be accomplished if each relation in the system contains every variable. Such systems are said to be indecomposable. Fortunately, the relations in most mathematical models of socioeconomic phenomena contain only a few common variables. Moreover, when complete decomposition cannot be achieved, a partial decomposition that rearranges and reorders the relations into a block-triangular form may still be possible.

A block triangular structure defines an information flow that is serial and without loops. Causal sequences in such systems, therefore, run one-way and permit feedbacks only upward in the triangular hierarchy. An example of such a structure is afforded by a hierarchy of migration flows in which people migrate only to larger urban regions. If the regions are ordered according to their size in the population projection process, then the growth matrix assumes a block-triangular form.

Once a large system of variables and relations has been either completely or partially decomposed into indecomposable subsystems, further simplification of the problem can only be achieved by a process called tearing. This is the second stage of the decomposition procedure and consists of deleting variables from one or more of the relations in which they appear. Thus tearing represents an attempt to solve a system problem by a "forced" partitioning of that system into supposedly disjoint subsystems. The partitioning is forced because the subsystems are not truly disjoint and are rendered so only through a disregard of certain connecting relationships which are held to be insignificant. If the impacts of these connecting relationships are not completely disregarded but are allowed somehow to affect the solution of the system problem, then we have an instance of <u>compensated tearing</u>.

2. Shrinking by Aggregation

Aggregation in demographic analysis may be carried out by consolidating:

1) <u>population characteristics</u>, for example, combining several sex, color, or age groups;

2) <u>time units</u>, for example, dealing with five-year intervals of time instead of annual ones; and

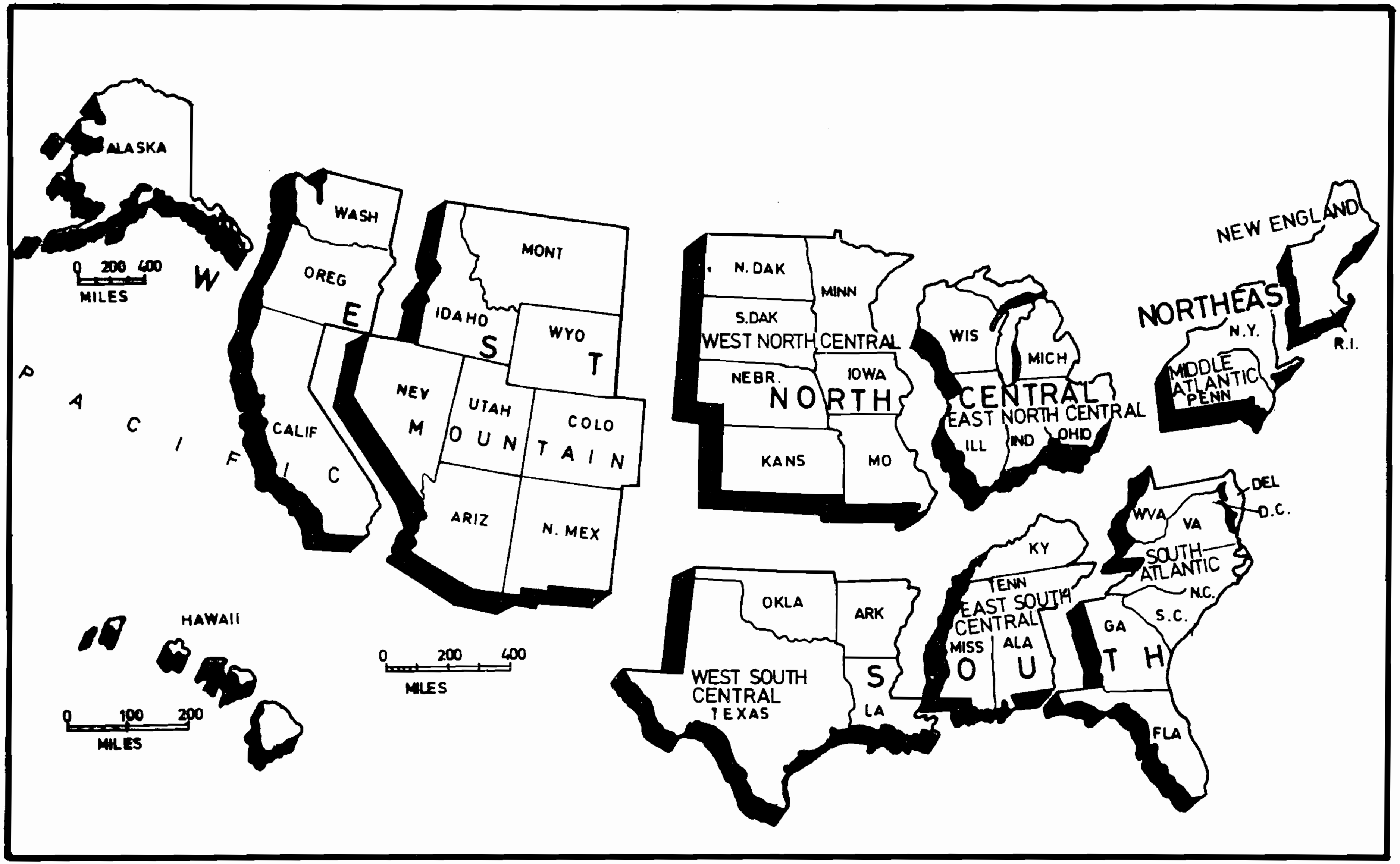
3) <u>spatial units</u>, for example, aggregating the fifty states of the USA into its nine Census Divisions.

In each case, the consolidated projection produces results that are coarser with regard to levels of detail than those provided by the original unconsolidated model.

Consider, for example, the two multiregional population systems illustrated in Figure 1: the nine Census Divisions of the US and the corresponding four Census Regions. A spatial consolidation of the nine Census Divisions into the four Census Regions permits a considerable shrinkage of the original model, but the process introduces some aggregation error and, more importantly, leads to population projections that are less detailed geographically than those obtained from the unconsolidated model. This can be seen by examining Tables 1 and 2, which give the nine-region and four-region projections, respectively, of the total US population in the year 2008 if the age-specific mortality, fertility, and migration schedules of 1958 were to remain unchanged over the fifty-year projection period.

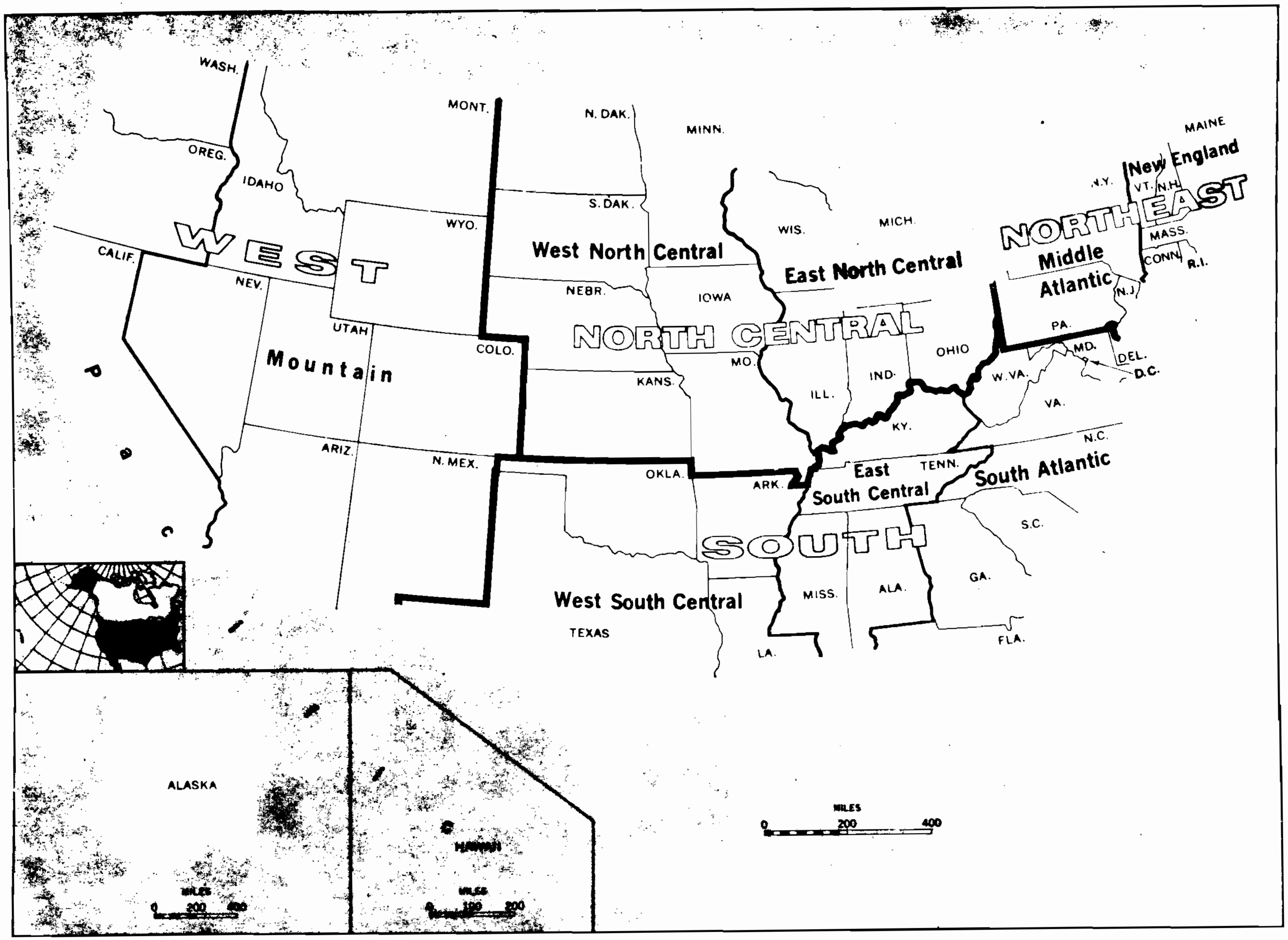
A comparison of the population projections summarized in Tables 1 and 2 indicates the magnitudes of the aggregation errors that are introduced by the consolidation of the nine Divisions into the four Regions. For the US as a whole one finds, for example, that a fifty-year projection of the 1958 population to the year 2008, on the assumption of an unchanging growth regime, produces an over-projection of almost 400,000 people. But, curiously enough, further projection of the same population until stability does not appreciably alter the intrinsic rate of growth (r) of the multiregional system. A difference of 0.00008 is all that distinguishes the intrinsic rate of growth of the four-region projection from that of the nine-region projection.

Aggregation over regions preserves age-specific details at the expense of geographic details. If the latter are of greater interest than the former, one may instead consolidate all age groups into a single variable and retain the original set of geographical areas. The application of such an aggregation



SOURCE: U.S. BUREAU OF THE CENSUS

Figure 1a. Regions and geographic divisions of the United States.



Source: U.S. Department of Commerce, Social and Economic Statistics Administration, Bureau of the Census

Figure 1b. Regions and geographic division of the United States.

Table 1. Multiregional projections to stability: United States total population, 1958, nine-region projection.

Projections and Stable Growth Parameters	DIVISION OF RESIDENCE 1 New England	2 Middle Atlantic	3 East North Central	4 West North Central	5 South Atlantic	6 East South Central	7 West South Central	8 Mountain	9 Pacific	TOTAL
K (1958)	9,911,000	33,181,000	35,763,000	15,114,000	24,749,000	11,769,000	16,117,000	6,349,000	19,141,000	172,154,000
% (1958)	0.0576	0.1927	0.2077	0.0878	0.1438	0.0684	0.0940	0.0369	0.1112	1.0000
K (2008)	21,644,039	59,187,140	80,761,069	31,173,278	68,283,065	24,394,274	40,446,886	22,805,818	73,166,573	421,862,143
% (2008)	0.0513	0.1403	0.1914	0.0739	0.1619	0.0578	0.0959	0.0541	0.1734	1.0000
r (∞)					0.02184					
% (∞)	0.0447	0.1013	0.1719	0.0727	0.1535	0.0492	0.1024	0.0680	0.2362	1.0000

Table 2. Multiregional projections to stability: United States total population, 1958, four-region projection.

Projections and Stable Growth Parameters	REGION OF RESIDENCE 1 NORTHEAST	2 NORTH CENTRAL	3 SOUTH	4 WEST	TOTAL
K (1958)	43,092,000	50,877,000	52,695,000	25,490,000	172,154,000
% (1958)	0.2503	0.2955	0.3061	0.1481	1.0000
K (2008)	80,383,757	112,077,195	132,843,209	96,955,108	422,259,268
% (2008)	0.1904	0.2654	0.3146	0.2296	1.0000
r (∞)	0.02192				
% (∞)	0.1431	0.2491	0.3046	0.3032	1.0000

to the cohort-survival model associated with Table 1 yields a components-of-change projection model that produces the multiregional projections in Table 3.

Table 3 reveals that a components-of-change aggregation of the original cohort-survival model leads to a substantial underprojection of total population growth, but a relatively accurate projection of the spatial distribution of that growth. The total US population in the year 2008, for example, is underprojected by over fifty-one million people, and the intrinsic rate of growth is underprojected by more than six per 1,000. Yet the Pacific Division is allocated approximately 17% of the total population in the year 2008 by both models.

The divergence between the projections in Tables 1 and 3 increases exponentially over time. Figure 2 shows that the two models project similar population totals during the first decade, start to diverge shortly thereafter, and then grow increasingly further apart. This suggests that shrinking by components-of-change aggregation is most effective for short-run projections.

We have seen that aggregation is generally accompanied by loss of detail. This, however, need not always be the case. One can, for example, obtain a biregionally aggregated population projection for every region of a multiregional system and thereby retain the same level of detail in the resulting collection of consolidated projections as originally existed in the single unconsolidated model. By way of illustration, consider the projections summarized in Table 4, which were obtained using nine biregionally aggregated versions of the cohort-survival model that produced the results in Table 1. A comparison of the projections in Table 4 with those in Table 1 suggests that an exhaustive collection of biregional aggregations is a reasonably accurate substitute for a large-scale population projection model.

Although biregional aggregations may be applied with some success to shrink a large model, they can be computationally demanding if it is necessary that they be applied as many times as the number of regions in a multiregional system. In such instances, a more efficient and effective shrinking technique often can be developed using decomposition methods.

3. Shrinking by Decomposition

Decomposition procedures have been used often in demographic analysis, although they have not been specifically identified by that name. Perhaps their most common application is manifested in representations of multiregional population systems by collections of single-region models which assume that each regional population is undisturbed by migration. Such an assumption is, of course, equivalent to the premise that the multiregional population system is completely decomposable into independent single-region subsystems arranged in block-diagonal form. A modification of the no-migration assumption is often

Table 3. Multiregional projections to stability: United States total population, 1958, nine-region components-of-change projections.

Projections and Stable Growth Parameters	DIVISION OF RESIDENCE 1 New England	2 Middle Atlantic	3 East North Central	4 West North Central	5 South Atlantic	6 East South Central	7 West South Central	8 Mountain	9 Pacific	TOTAL
K (1958)	9,911,000	33,181,000	35,763,000	15,114,000	24,749,000	11,769,000	16,177,000	6,349,000	19,141,000	172,154,000
% (1958)	0.0576	0,1927	0.2077	0.0878	0.1438	0.0684	0.0940	0.0369	0.1112	1.0000
K (2008)	17,927,349	53,159,821	68,434,148	25,822,107	62,159,432	21,199,129	35,493,951	19,076,175	61,336,572	364,608,685
% (2008)	0.0492	0.1458	0.1877	0.0708	0.1705	0.0581	0.0973	0.0523	0.1682	1.0000
r (∞)	0.01554									
% (∞)	0.0360	0.0897	0.1516	0.0631	0.1748	0.0490	0.1107	0.0717	0.2533	1.0000

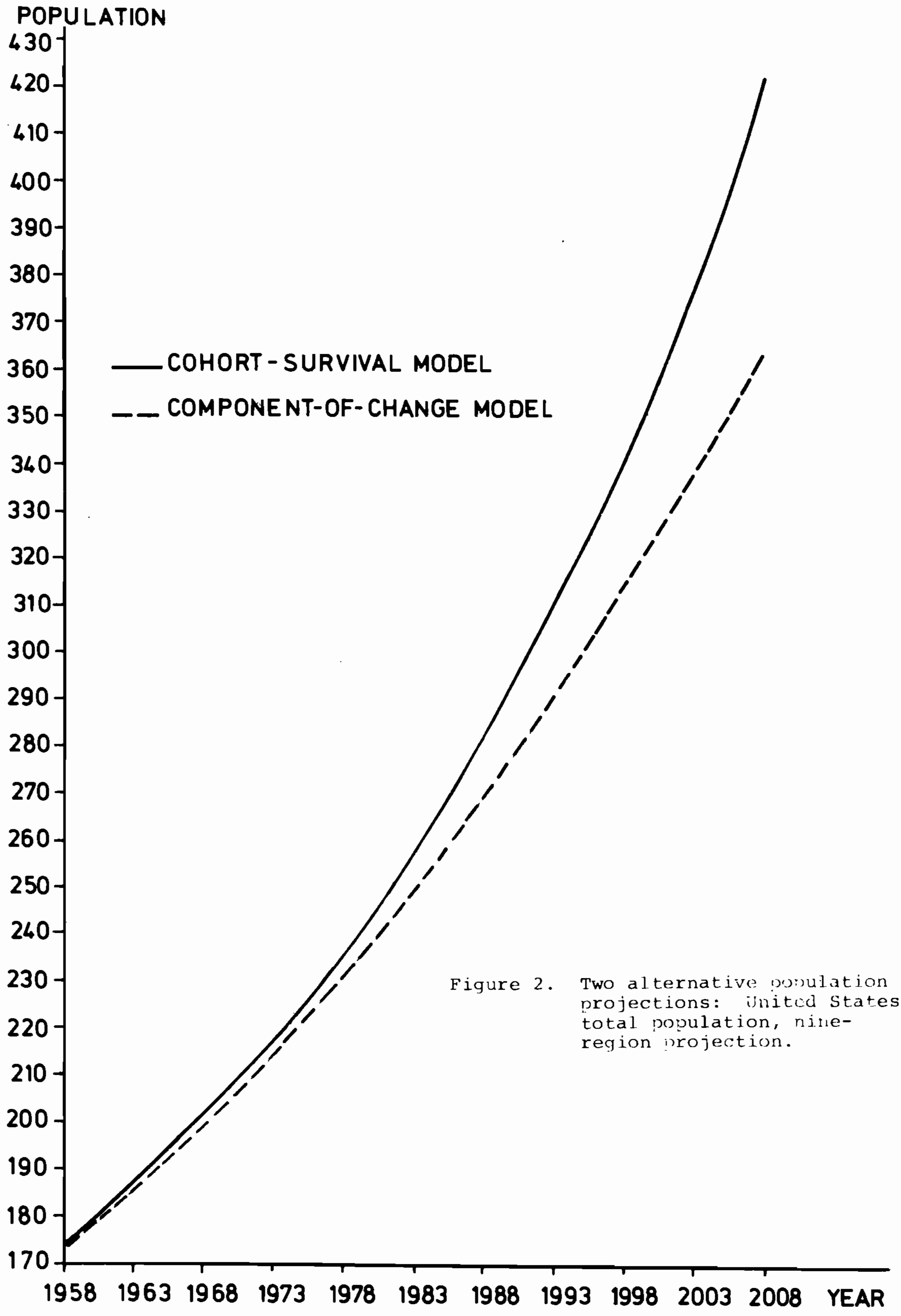


Figure 2. Two alternative population projections: United States total population, nine-region projection.

Table 4. Multiregional projections to stability: United States total population, 1958, nine biregional projections

Projections and Stable Growth Parameters	DIVISION OF RESIDENCE									TOTAL
	1 New England	2 Middle Atlantic	3 East North Central	4 West North Central	5 South Atlantic	6 East South Central	7 West South Central	8 Mountain	9 Pacific	
K (1958)	9,911,000	33,181,000	35,763,000	15,114,000	24,749,000	11,769,000	16,177,000	6,349,000	19,141,000	172,154,000
% (1958)	0.0576	0.1927	0.2077	0.0878	0.1438	0.0684	0.0940	0.0369	0.1112	1.0000
K (2008)	22,420,802	60,240,640	83,052,968	31,136,660	70,878,872	24,837,796	40,472,448	22,355,426	73,141,824	428,537,436
% (2008)	0.0523	0.1406	0.1938	0.0727	0.1654	0.0580	0.0944	0.0522	0.1707	1.0000
r (∞)	0.02157	0.02181	0.02157	0.02154	0.02155	0.02155	0.02157	0.02162	0.02159	---
% (∞)	0.0513	0.1070	0.1890	0.0663	0.1737	0.0513	0.0933	0.0565	0.2118	1.0000

introduced into the single-region model by including the impact of net migration in the survivorship proportions, that is, by treating an out-migrant as a "death" and an in-migrant as a replacement for a death. Such a modification of the complete single-region decomposition was adopted to derive the projections in Table 5.

Table 5 presents the summary results of nine single-region cohort-survival population projections. The regions are those delineated in Figure 1, and the results correspond to the ones set out earlier in Table 1. Thus Table 5 may be viewed as the output produced by a particular shrinking of the "large-scale" population projection model associated with Table 1. The discrepancies between the two sets of results may be attributed largely to the representation of interregional migration as net migration in the decomposed model.

Table 5 reveals that the representation of internal migration as a net flow can introduce serious errors into the population projection process. Net migration is defined with respect to the particular regional population being projected. If that population is currently experiencing an excess of in-migrants over out-migrants, this feature will be built-in as part of the projection process, and its effects will multiply and increase cumulatively over time. The converse applies, of course, to regions experiencing net out-migration. In short, regional populations with a positive net migration rate are likely to be overprojected and those with a negative net migration rate are likely to be underprojected. The projections in Table 5 support this argument. Only the populations of the three Census Divisions that experienced a positive net migration in 1958 are overprojected in the year 2008, that is, the South Atlantic, the Mountain, and the Pacific Divisions); the populations of the remaining six Census Divisions are underprojected.

The original nine-region population projection model and its complete single-region decomposition represent opposite extremes of the decomposition spectrum. A large number of alternatives lie in between, two of which appear in Figure 3.

Figure 3 describes two complete decompositions of the nine-region population system. Both decompositions reflect the particular structure of interregional migration levels observed in the data, and both were defined by an essentially arbitrary decision to delete interregional linkages that exhibited migration levels below 8%.[1] Since in both cases this procedure still did not produce a complete decomposition, four additional migration levels (those lying outside of the block-diagonal submatrices in Figure 3) were also deleted in each decomposition.

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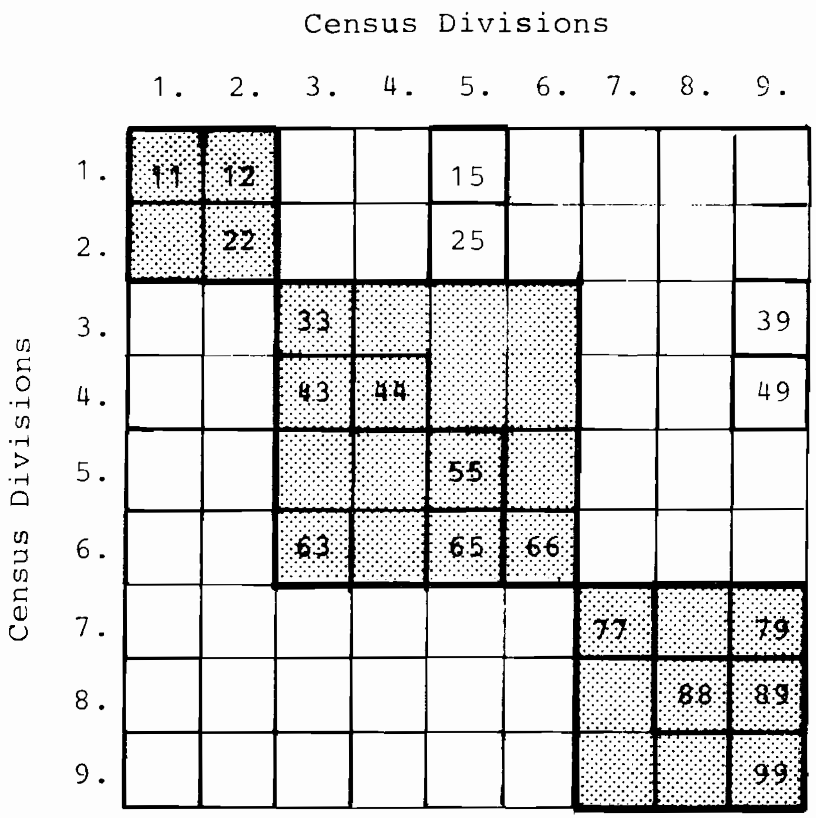
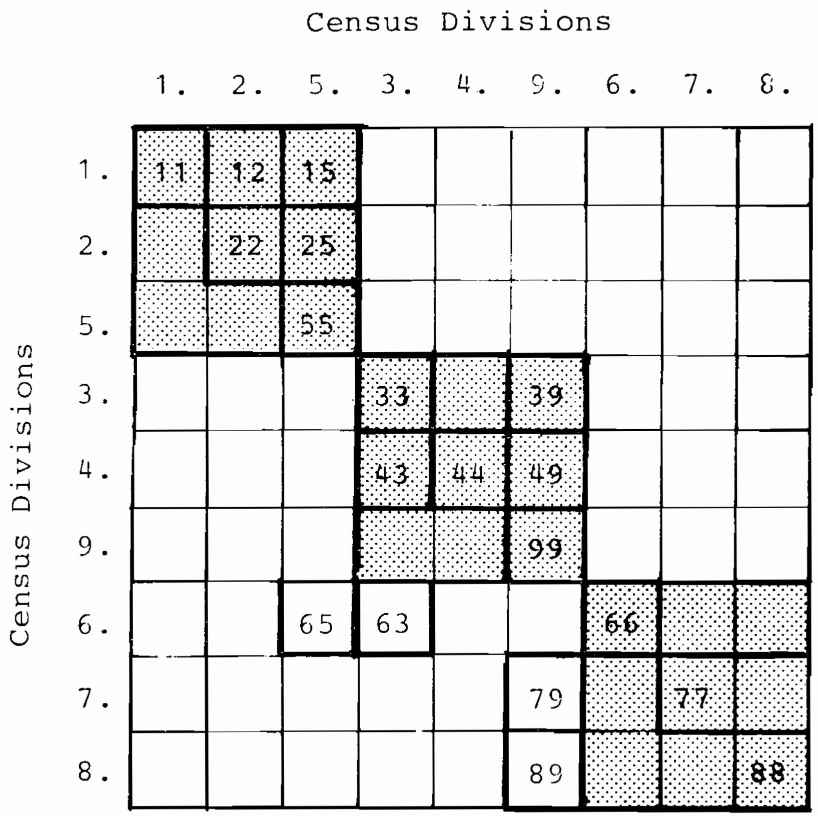
[1]Migration levels were measured as the fraction of an individual's expected lifetime that is expected to be lived in a particular region other than the individual's region of birth (Rogers, 1975).

Table 5. Multiregional projections to stability: United States total population, 1958, nine single-region decompositions with net migration.

Projections and Stable Growth Parameters	DIVISION OF RESIDENCE 1 New England	2 Middle Atlantic	3 East North Central	4 West North Central	5 South Atlantic	6 East South Central	7 West South Central	8 Mountain	9 Pacific	TOTAL
K (1958)	9,911,000	33,181,000	35,763,000	15,114,000	24,749,000	11,769,000	16,177,000	6,349,000	19,141,000	172,154,000
% (1958)	0.0576	0.1927	0.2077	0.0878	0.1438	0.0684	0.0940	0.0369	0.1112	1.0000
K (2008)	21,361,806	54,784,164	80,574,344	27,888,196	72,708,288	21,538,842	38,569,232	27,877,196	105,479,992	450,782,060
% (2008)	0.0474	0.1215	0.1787	0.0619	0.1613	0.0478	0.0856	0.0618	.2340	1.0000
r (∞)	0.02027	0.01451	0.02049	0.01638	0.02379	0.01400	0.02034	0.03207	0.03907	---
% (∞)	0.0543	0.1856	0.2025	0.0861	0.1549	0.0750	0.0976	0.0393	0.1047	1.0000



A. Complete Decomposition A



B. Complete Decomposition B

Figure 3. Two alternative decompositions of a multiregional system.

Figure 3A illustrates a decomposition of the nine-region population model into three smaller multiregional models containing two, four, and three regions, respectively. Internal migration is treated as a place-to-place flow among regions within each diagonal block and as a net flow elsewhere. Thus we have here an example of compensated tearing in which the conceptual approaches at both extremes of the decomposition spectrum are represented. Table 6 summarizes the multiregional population projections produced by this particular model.

Figure 3B depicts an alternative decomposition. In this instance, a permutation of the rows and columns of the migration level matrix and a decision to delete a different set of four migration levels yields a different connectivity structure and associated decomposition. This decomposition partitions the nine-region system into three three-region subsystems and results in the projections set out in Table 7.

The two alternative decompositions both overproject the total US population in 2008. The individual regional shares of this total population follow the general pattern exhibited by the single-region decomposition of Table 5. That is, regional populations experiencing positive net migration in 1958 are accorded a larger than warranted regional share, and vice-versa. This pattern arises out of the particular method of compensated tearing used in the projections, that is, compensation by means of net migration, and reflects the same biases that were found in the single-region decomposition.

Another contributor to the discrepancies between the results of the two decomposed models and those of the original model is the insufficiently weak degree of connectivity between the various sets of multiregional subsystems. Recall that, for illustrative purposes, we arbitrarily deleted internal migration flows associated with migration levels below 8%. It is likely that this is much too high a value for a threshold level, and its adoption undoubtedly contributed something to the overall projection error. That contribution, however, is surely small compared to the one introduced by the representation of internal migration as a net flow. Both sources of error are, of course, interrelated. The level of compensation which is required in the form of net migration is intimately related to the amount of net migration which is to be treated in that way, and this amount in turn depends on the volume of migration that falls below the threshold level.

Aggregation and decomposition techniques are not mutually exclusive methods of shrinking a large-scale population model. They can, of course, be combined in various ways to reduce the dimensionality of such a model without incurring a major sacrifice in accuracy or level of detail in the process. We now turn to an examination of one of the more obvious ways in which they may be combined and compare its empirical performance with that of an equally obvious alternative.

Table 6. Multiregional projections to stability: United States total population, 1958, Decomposition A.

Projections and Stable Growth Parameters	DIVISION OF RESIDENCE 1 New England	2 Middle Atlantic	3 East North Central	4 West North Central	5 South Atlantic	6 East South Central	7 West South Central	8 Mountain	9 Pacific	TOTAL
K (1958)	9,911,000	33,181,000	35,763,000	15,114,000	24,749,000	11,769,000	16,177,000	6,349,000	19,141,000	172,154,000
% (1958)	0.0576	0.1927	0.2077	0.0878	0.1438	0.0684	0.0940	0.0369	0.1112	1.0000
K (2008)	20,818,862	55,406,756	79,776,664	28,969,902	69,440,440	23,452,330	42,158,288	27,528,572	93,899,880	441,451,694
% (2008)	0.0472	0.1255	0.1807	0.0656	0.1573	0.0531	0.0955	0.0624	0.2127	1.0000
r (∞)	0.01664	0.01664	0.02026	0.02026	0.02026	0.02026	0.03289	0.03289	0.03289	---
% (∞)	0.0979	0.1299	0.2036	0.0588	0.2301	0.0544	0.0312	0.0372	0.1570	1.0000

Table 7. Multiregional projections to stability: United States total population, 1958, Decomposition B.

Projections and Stable Growth Parameters	DIVISION OF RESIDENCE 1 New England	2 Middle Atlantic	3 East North Central	4 West North Central	5 South Atlantic	6 East South Central	7 West South Central	8 Mountain	9 Pacific	TOTAL
K (1958)	9,911,000	33,181,000	35,763,000	15,114,000	24,749,000	11,769,000	16,177,000	6,349,000	19,141,000	172,154,000
% (1958)	0.0576	0.1927	0.2077	0.0878	0.1438	0.0684	0.0940	0.0369	0.1112	1.0000
K (2008)	21,162,692	57,420,652	82,082,112	30,588,244	69,149,768	22,266,124	39,261,660	25,469,752	87,833,784	435,234,788
% (2008)	0.0486	0.1319	0.1886	0.0703	0.1589	0.0512	0.0902	0.0585	0.2018	1.0000
r (∞)	0.02018	0.02018	0.02900	0.02900	0.02018	0.02555	0.02555	0.02555	0.02900	----
% (∞)	0.0637	0.1385	0.0566	0.0298	0.2924	0.0111	0.0481	0.0943	0.2656	1.0000

4. Aggregation and Decomposition Combined

The idea that it might be useful to model different parts of a large system at different levels of detail received one of its first formal mathematical treatments two decades ago in a seminal paper read by Herbert Simon and Albert Ando at the meetings of the Econometric Society in December of 1956 and subsequently published in *Econometrica* five years later (Simon and Ando, 1961).[2] The essence of their basic argument is neatly captured by the following physical illustration:

> Consider a building whose outside walls provide perfect thermal insulation from the environment. The building is divided into a large number of rooms, the walls between them being good, but not perfect, insulators. Each room is divided into a number of offices by partitions. The partitions are poor insulators. A thermometer hangs in each of the offices. Suppose that at time t_0 the various offices within the building are in a state of thermal disequilibrium--there is a wide variation in temperature from office to office and from room to room. When we take new temperature readings at time t_1, several hours after t_0, what will we find? At t_1 there will be very little variation in temperature among the offices within each single room, but there may still be large temperature variations *among* rooms. When we take readings again at time t_2, several days after t_1, we find an almost uniform temperature throughout the building; the temperature differences among rooms have virtually disappeared.
>
> A temperature equilibrium *within* each room will be reached rather rapidly, while a temperature equilibrium *among* rooms will be reached only slowly,...as long as we are not interested in the rapid fluctuations in temperature among offices in the same room, we can learn all we want to know about the dynamics of this system by placing a single thermometer in each room--it is unnecessary to place a thermometer in each office [Simon and Ando, 1961, pp. 70-71].

4.1 The Simon-Ando Theorem

Recognizing that complete decomposability is relatively rare in socioeconomic systems, Simon and Ando (1961) examine the behavior of linear dynamic systems with "nearly" completely decomposable subsystems. They show that, in the short-run,

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[2] A recent revival of interest in this fundamental idea has produced several interesting articles, one of which specifically suggests an application to migration modeling (Batty and Masser, 1975).

such systems behave almost as though they were in fact completely decomposable and that, in the middle-run, their behavior can be studied by consolidating the variables of each subsystem into a single variable and ignoring the interrelationships within each subsystem.[3]

The crux of the Simon-Ando theorem is the assertion that the equilibrium of a nearly completely decomposable dynamic linear system may be viewed as a composite growth process which evolves in three temporal phases. During the first phase, the variables in each subsystem arrive at equilibrium positions determined by the completely decomposed system. After a longer time-period, the system enters its second phase, at which point the variables of each subsystem, maintaining their proportional relationships, move together as a block toward equilibrium values established by the third phase of the growth process. In this final phase, all variables approach the rate of growth defined by the largest characteristic root of the matrix associated with the original nearly completely decomposable system.

The Simon-Ando theorem suggests a shrinking procedure for large-scale population projection models that combines aggregation and decomposition in a particularly appealing way. One begins by partitioning the large multiregional system projection model into smaller submodels in a way that effectively exploits any weak interdependencies revealed by indices such as migration levels. The growth of the original multiregional system then may be projected by appropriately combining the results of disaggregated intra-subsystem projections, in which within subsystem interactions are represented at a relatively fine level of detail, with the results of aggregate inter-subsystem projections, in which the between subsystem interactions are modeled at a relatively coarse level of detail. For example, within each multiregional subsystem, the projection model could focus on the full age composition of every regional population and examine its evolution over time; between each multiregional subsystem, the projection model would suppress the regional age compositions and would deal only with total populations. In the short-run, the within subsystem interactions would dominate the behavior of the system; in the long-run, the between subsystem interactions would become increasingly important and ultimately would determine the behavior of the entire system.

4.2 Simple Shrinking by Aggregation and Decomposition

The Simon and Ando theorem suggests the following simple method for shrinking large-scale population projection models.

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[3]In a subsequent paper, Ando and Fisher (1963) extend the Simon-Ando theorem to nearly block-triangular (that is, nearly partially decomposable) linear systems. Although we do not consider such systems in the rest of this paper, it should be clear that our exposition could be appropriately expanded to cover this more general case of near decomposability.

We begin by partitioning a multiregional system into its constituent single regions and projecting their growth and change as if they were independent closed population subsystems undisturbed by migration. The first stage, therefore, corresponds to a single-region decomposition with zero net migration. We then suppress all age-specific details and project the multiregional population using a components-of-change model. The results of the latter stage determine the total multiregional population and its spatial distribution; the results of the first stage define the individual regional age compositions. In this way, within subsystem interactions (that is, changes in age structure) are modeled at a fine level of detail, whereas between subsystem interactions (that is, changes in spatial structure) are modeled at a coarse level of detail. If the original multiregional system is sufficiently close to being nearly decomposable, the approximate (two-stage) projection should produce a reasonably accurate multiregional population projection.

The shrinking procedure described above may be applied to the "large-scale" nine-region population projection model of Table 1. Table 8 sets out the principal results generated by such a shrinking of the original model. The growth of the total population and its spatial allocation are taken from the projection in Table 3; the individual regional age compositions (consolidated into three age groups for ease of presentation) were obtained by recomputing the single-region projections of Table 5 with net migration set equal to zero. The combined results indicate that regional age compositions and regional shares are projected moderately well, but that the total multiregional population is seriously underprojected. (The latter is no surprise since it already was observed and discussed in connection with Table 3.)

In applying the above shrinking procedure we adopted the regional age compositions of the single-region (no-migration) projections and the regional shares of the components-of-change projection. For the total multiregional population we chose the level projected by the latter (364,608,685); we would have done much better to have used that of the former (419,173,278). In the remainder of this paper, therefore, we shall modify the shrinking procedure accordingly and shall define the resulting modified version to be the cohort-components method of simple shrinking. This method adopts the regional age compositions and total multiregional population projected by a collection of single-region cohort-survival models that ignore migration, and then spatially allocates this total population according to the regional shares projected by a components-of-change model.

The accuracy with which the biregionally aggregated models of Table 4 approximated the original projection in Table 1 suggests another method of simple shrinking, one which we shall call the cohort-biregional method of simple shrinking. In this method, the tearing occasioned by complete decompositions of

Table 8. Multiregional projections to stability: United States total population, 1958, nine single-region (no-migration) decompositions with components-of-change aggregation.

	1	2	3	4	5	6	7	8	9	TOTAL
K (1958)	9,911,000	33,181,000	35,763,000	15,114,000	24,749,000	11,769,000	16,177,000	6,349,000	19,141,000	172,154,000
% (1958)	0.0576	0.1927	0.2077	0.0878	0.1438	0.0684	0.0940	0.0369	0.1112	1.0000
K (2008)	17,927,349	53,159,821	68,434,148	25,822,107	62,159,432	21,199,129	35,493,951	19,076,175	61,336,572	364,608,685
% (2008)	0.0492	0.1458	0.1877	0.0708	0.1705	0.0581	0.0973	0.0523	0.1682	1.0000
r (∞)				0.01554						
% (∞)	0.0360	0.0897	0.1516	0.0631	0.1748	0.0490	0.1107	0.0717	0.2533	1.0000
2008: Approximate Projection										
C (0-14)	0.3554	0.3378	0.3678	0.3690	0.3546	0.3655	0.3742	0.3728	0.3560	---
C (15-64)	0.5889	0.6004	0.5778	0.5751	0.5879	0.5814	0.5725	0.5740	0.5836	---
C (65+)	0.0567	0.0618	0.0544	0.0559	0.0575	0.0532	0.0533	0.0532	0.0604	---
2008: Original Projection										
C (0-14)	0.3560	0.3367	0.3642	0.3664	0.3513	0.3621	0.3709	0.3740	0.3587	0.3581
C (15-64)	0.5873	0.5988	0.5802	0.5713	0.5840	0.5765	0.5696	0.5719	0.5865	0.5825
C (65+)	0.0567	0.0644	0.0557	0.0623	0.0647	0.0614	0.0595	0.0541	0.0548	0.0594

the kind defined in Figure 3 are compensated not by net migration but by biregional aggregation. Specifically, each multiregional subsystem is augmented by an additional "rest-of-the-world" region which serves as the destination of all migration out of the subsystem and as the source of all migration into the subsystem. Table 9 presents the results produced by the application of such a method of shrinking to the projection model of Table 1. The particular decomposition scheme adopted was that of Figure 3B.

According to Table 9, cohort-biregional shrinking is a more accurate method of shrinking than cohort-components shrinking, at least with regard to the particular data set examined in this paper. The former projects regional age compositions that are virtually identical to those projected by the original large-scale model. The total multiregional population and its regional distribution are somewhat less accurately approximated, but nevertheless are, in general, closer approximations than those advanced by the cohort-components method of Table 8. Finally, the cohort-biregional shrinking can be more readily transformed into a method for approximating the intrinsic rate of growth and related stable growth parameters of the multiregional population system.

The cohort-components and the cohort-biregional methods of simple shrinking appear to be the most desirable shrinking methods among those examined in this paper. Table 10 indicates that they are the most accurate in projecting the total multiregional population. With the possible exception of the less-efficient biregional aggregation method of shrinking, they also appear to be the most accurate in projecting the regional shares and age compositions of the multiregional population. The accuracy with which the cohort-biregional method projects regional age compositions is especially remarkable and is well illustrated in Table 11, which presents the alternative projections of the age composition of the Pacific Division by way of example.

5. Conclusion

An increasing number of social scientists currently find themselves in the somewhat frustrating position of being asked to provide accurate projections at very fine levels of detail with resources that are scarcely sufficient for carrying out such projections at much more aggregate levels of resolution. Prominent among them are demographers who are called upon to produce consistent projections of regional populations disaggregated by age, color, race, sex, and such indicators of class and welfare as employment category and income. Since the computational requirements of this task are staggering, the need for developing improved methods for "shrinking" population projection models by reducing their dimensionality is an urgent one. The projections summarized in this paper suggest the following general conclusions:

Table 9. Multiregional projections to stability:
United States total population, 1958,
Decomposition B with biregional aggregation.

Projections and Stable Growth Parameters	DIVISION OF RESIDENCE 1 New England	2 Middle Atlantic	3 East North Central	4 West North Central	5 South Atlantic	6 East South Central	7 West South Central	8 Mountain	9 Pacific	TOTAL
K (1958)	9,911,000	33,181,000	35,763,000	15,114,000	24,749,000	11,769,000	16,177,000	6,349,000	19,141,000	172,154,000
% (1958)	0.0576	0.1927	0.2077	0.0878	0.1438	0.0684	0.0940	0.0369	0.1112	1.0000
K (2008)	21,737,424	59,648,676	81,625,712	30,941,760	70,046,704	24,580,972	40,261,452	22,187,936	72,139,368	423,170,004
% (2008)	0.0514	0.1410	0.1929	0.0731	0.1655	0.0581	0.0951	0.0524	0.1705	1.0000
r (∞)	0.02186	0.02186	0.02160	0.02160	0.02186	0.02160	0.02160	0.02160	0.02160	----
% (∞)	0.0451	0.1060	0.1870	0.0696	0.1684	0.0523	0.0969	0.0590	0.2158	1.0000
2008: Approximate Projection										
C (0-14)	0.3560	0.3369	0.3647	0.3665	0.3517	0.3627	0.3706	0.3730	0.3586	---
C (15-64)	0.5874	0.5990	0.5803	0.5713	0.5843	0.5766	0.5697	0.5719	0.5864	----
C (65+)	0.0566	0.0641	0.0550	0.0623	0.0640	0.0607	0.0597	0.0551	0.0550	----
2008: Original Projection										
C (0-14)	0.3560	0.3367	0.3642	0.3664	0.3513	0.3621	0.3709	0.3740	0.3587	0.3581
C (15-64)	0.5873	0.5988	0.5802	0.5713	0.5840	0.5765	0.5696	0.5719	0.5865	0.5825
C (65+)	0.0567	0.0644	0.0557	0.0623	0.0647	0.0614	0.0595	0.0541	0.0548	0.0594

Table 10. Alternative projections of the total population and its regional distribution in the year 2008: United States total population, 1958.

Alternative*	DIVISIONAL SHARES OF TOTAL POPULATION (2008)									TOTAL	TOTAL POPULATION
	1	2	3	4	5	6	7	8	9		
Table 1	0.0513	0.1403	0.1914	0.0739	0.1619	0.0578	0.0959	0.0541	0.1734	1.0000	421,862,143
Table 4	0.0523	0.1406	0.1938	0.0727	0.1654	0.0580	0.0944	0.0522	0.1707	1.0000	428,537,436
Table 5	0.0474	0.1215	0.1787	0.0619	0.1613	0.0478	0.0856	0.0618	0.2340	1.0000	450,782,060
Table 6	0.0472	0.1255	0.1807	0.0656	0.1573	0.0531	0.0955	0.0624	0.2127	1.0000	441,451,694
Table 7	0.0486	0.1319	0.1886	0.0703	0.1589	0.0512	0.0902	0.0585	0.2018	1.0000	435,234,788
Table 8	0.0492	0.1458	0.1877	0.0708	0.1705	0.0581	0.0973	0.0523	0.1682	1.0000	(419,173,278)
Table 9	0.0514	0.1410	0.1929	0.0731	0.1655	0.0581	0.0951	0.0524	0.1705	1.0000	423,170,004

*Table 1--original unconsolidated model
Table 4--biregional aggregations
Table 5--single-region decompositions (with net migration)
Table 6--decomposition A
Table 7--decomposition B
Table 8--single-region (no migration) decompositions with components-of-change aggregation (cohort-components shrinking)
Table 9--decomposition B with bi-regional aggregation (cohort-biregional shrinking)

Table 11. Alternative projections of the pacific division's age composition in the year 2008: United States total population, 1958.

	ALTERNATIVE*					
Age	Table 1	Table 4	Table 5	Table 6	Table 7	Table 8	Table 9
0-4	0.1352	0.1403	0.1417	0.1389	0.1387	0.1335	0.1351
5-9	0.1182	0.1234	0.1218	0.1202	0.1202	0.1176	0.1182
10-14	0.1053	0.1094	0.1067	0.1058	0.1062	0.1049	0.1053
15-19	0.0955	0.0956	0.0982	0.0968	0.0970	0.0940	0.0954
20-24	0.0861	0.0838	0.0910	0.0889	0.0888	0.0846	0.0860
25-29	0.0768	0.0745	0.0818	0.0798	0.0797	0.0758	0.0768
30-34	0.0686	0.0664	0.0723	0.0710	0.0706	0.0672	0.0686
35-39	0.0603	0.0582	0.0619	0.0615	0.0610	0.0590	0.0603
40-44	0.0523	0.0504	0.0515	0.0521	0.0517	0.0518	0.0523
45-49	0.0462	0.0445	0.0439	0.0448	0.0449	0.0473	0.0463
50-54	0.0395	0.0386	0.0356	0.0371	0.0375	0.0406	0.0395
55-59	0.0339	0.0332	0.0298	0.0314	0.0317	0.0351	0.0339
60-64	0.0272	0.0269	0.0231	0.0247	0.0248	0.0282	0.0273
65-69	0.0190	0.0192	0.0150	0.0166	0.0166	0.0201	0.0191
70-74	0.0136	0.0137	0.0106	0.0118	0.0118	0.0157	0.0137
75-79	0.0102	0.0101	0.0077	0.0087	0.0087	0.0123	0.0102
80-84	0.0067	0.0067	0.0050	0.0056	0.0057	0.0085	0.0068
85+	0.0052	0.0052	0.0022	0.0043	0.0044	0.0039	0.0052
TOTAL	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
TOTAL POPULATION	73,166,573	73,141,824	105,479,992	93,899,880	87,833,784	(70,515,742)	72,139,368

*Same as in Table 10

1) Components-of-change models are unreliable generators of middle and long-run projections of population totals, but seem to be reasonably accurate in projecting regional shares of such totals.

2) Biregional aggregation is an effective and relatively efficient method for shrinking projection models of a small to modest scale and may be used in situations where only gross out-migration and in-migration data are available for each region.

3) Modelling internal migration as a net flow can introduce serious biases into the projection process. Such biases are inevitably introduced in treating immigration and emigration as a net flow, but in most countries they tend to be relatively small.

4) Effective decompositions are not unique and may be difficult to identify in large systems. Consequently algorithms such as those discussed in Tewarson (1973) need to be adapted and applied in searches for decompositions that are in some sense "optimal."

5) The simple cohort-components method of shrinking is a reasonably accurate procedure, is easy to apply, and has the distinct advantage of not requiring age-specific migration flow data for its implementation. It therefore is the obvious choice for shrinking large-scale projection models of population systems for which such data are either unavailable or too costly to obtain.

6) The simple cohort-biregional method of shrinking appears to be very accurate and seems to be an effective compromise between biregional aggregation and single-region decomposition, combining the best features of each. It is especially well-suited for shrinking large-scale projection models of population systems that are comprised of several weakly connected subsystems.

The two principal approaches for shrinking examined in this paper have been aggregation and decomposition. They have been combined to define two fundamental methods of shrinking, both of which reflect the proposition that strongly interconnected regions should be modelled as separate closed subsystems using the cohort-survival model. The two methods differ in the way that they connect these subsystems together. The cohort-components method uses a components-of-change model to establish such connections; the cohort-biregional method relies instead on a residual "rest-of-the-world" region. Each alternative differs with respect to data inputs and outputs, computational efficiencies, and projection accuracy. Yet little can be said about the trade-offs between these attributes in the abstract, because they depend so much on the specifics of each empirical situation. The particular connectivity structure of an observed multiregional population, the particular data availability with regard to age-specific migration flows, the particular purposes for which the projections are being generated, all are important considerations in a rational choice between the two alternatives. Yet such considerations will vary from one situation to another, and will combine in different ways to suggest the superiority of one alternative over the other. In consequence, each particular situation requires a specific evaluation.

This paper represents a first and therefore preliminary examination of shrinking large-scale population projection models. Consequently it only outlines the fundamental problem and identifies what appear to be fruitful means for dealing with it. Much more remains to be done. For example, it is likely that further research could establish conditions for perfect decomposition" akin to those already established for perfect aggregation (Rogers, 1969 and 1975). The relative computational efficiencies of the two alternative methods in shrinking certain prototype connectivity structures could be examined profitably. More complex hierarchical extensions of the simple shrinking methods could be investigated, such as the extension of the simple cohort-components method to include several multiregional (no-migration) cohort-survival submodels, and the disaggregation of the "rest-of-the-world" region in the simple cohort-biregional method. Efficient algorithms for approximating the intrinsic rate of growth and other related stable growth measure using shrinking methods appears to be another promising direction for research. The potential applicability of the cohort-components and cohort-biregional methods of shrinking as approaches to multiregional life table construction when age-specific migration flow data are unavailable needs to be explored. The assumption of no interregional differentials in fertility and mortality has been used before to shrink a large-scale population projection model and deserves to be reconsidered in the context of this paper (Rogers, 1968, Ch. 3). Finally, the possibility of shrinking data input requirements by means of "model" schedules also merits careful examination (United Nations, 1967, Rogers, 1975, Ch. 6).

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