15th Triennial World Congress, Barcelona, Spain

ELSEVIER
IFAC
PUBLICATIONS
www.elsevier.com/locate/ifac

OPTIMAL TRENDS IN MODELS OF ECONOMIC GROWTH

A.M.Tarasyev[*,***,1], **C.Watanabe**[**,***]

* *Institute of Mathematics and Mechanics UrB of RAS, S.Kovalevskaya str. 16, Ekaterinburg 620219, Russia e-mail: tam@imm.uran.ru*

** *Tokyo Institute of Technology, 2-12-1 Ookayma Meguro-ku, 152 Tokyo, Japan e-mail: chihiro@me.titech.ac.jp*

*** *International Institute for Applied Systems Analysis (IIASA), A-2361 Laxenburg, Austria*

Abstract: The objective of this work is to design control strategies which optimize composition of production, technology stock and their rates in a nonlinear model of economic growth. The optimal control problem of R&D investment is formulated for a discounted utility function which correlates the amount of sales and production diversity. The maximum principle of Pontryagin is applied for designing optimal nonlinear dynamics. Quasioptimal feedbacks of the rational type for balancing the dynamical system are constructed. Properties of techno-economic trajectories are examined for different tangent slopes generated by R&D intensities. These properties correspond to the trends of economic growth intrinsic to econometric data. *Copyright © 2002 IFAC*

Keywords: Economic growth, control synthesis, econometric trends.

1. INTRODUCTION

Optimal regulation of R&D investment is a reasonable statement of the problem in models of economic growth. This question naturally arises due to presence of growth and decline trends in interaction between production and technology. Investment to R&D, from the one hand, generates new sales in the market competition, but, from the other hand, leads to the redistribution of resources between production and technology stock and introduces the risky factor of innovation. The discounted utility function correlates the amount of sales and production diversity in the model. The amount of sales is determined by the production growth and the production diversity depends on the accumulated and current R&D investment. Qualitatively the utility function expresses preferences of investors in the simultaneous growth of production, technology stock and technology rate.

The problem of optimal R&D investment is to find optimal innovation policy which maximizes the utility function and optimizes composition of production, technology stock and their rates. Such statement is connected with the classical problems of economic growth and optimal allocation of resources (see (Arrow, 1985), (Leitmann and Lee, 1999)), and refers to the endogenous growth theory (Grossman and Helpman, 1991). Unlike models (Grossman and Helpman, 1991) which treat dynamics of the knowledge stock as a function of the price for technology output we deal with dynamics which connects growth of sales with R&D investment. This dynamics comes naturally from adjustment of marginal productivities to the real econometric time

[1]The first author was supported by the Russian Fund for Fundamental Research under grants 99-01-00146, 00-15-96057, and Fujitsu Research Institute (FRI) under IIASA-FRI contract 00-117.

series (Watanabe, 1992). For description of utility we use the discounted integral with the consumption index of the logarithmic type and equal elasticity of substitution of invented products (Grossman and Helpman, 1991), (Intriligator, 1971).

We apply the Pontryagin maximum principle (Pontryagin *et al.*, 1962) to find the optimal R&D investment level. We show that the Hamiltonian system of the maximum principle has the unique equilibrium of the saddle type and the optimal trajectories should converge to this equilibrium. In analysis of the Hamiltonian dynamics we refer to the theory of nonlinear differential equations (Hartman, 1964). We use also qualitative methods for construction of optimal feedbacks in control problems and differential games with discounted payoff integrals (Dolcetta, 1983), (Tarasyev, 1999).

We propose several explicit approximations of the rational type – suboptimal feedbacks. The obtained suboptimal feedbacks have reasonable interpretations in terms of econometric characteristics. We examine growth and decline properties of suboptimal feedbacks for different tangent slopes generated by possible R&D intensities.

2. THE SYSTEM MODEL

We consider a nonlinear growth model which describes dynamics of aggregated production $y = y(t)$ and technology (accumulated R&D investment) $T = T(t)$ depending on the control parameter - the current level of R&D investment $r = r(t)$

$$\begin{aligned} \frac{\dot{y}}{y} &= f_1 + f_2\left(\frac{T}{y}\right)^{\gamma} - g\frac{r}{y} \\ \dot{T} &= r \end{aligned} \tag{1}$$

One can treat dynamic process (1) as the balanced equations of spending resources between the productivity rate $\dot{y} \ / \ y$ and R&D intensity $r \ / \ y$. Function $f_1(t)$ presents the non–R&D contribution to the production growth rate $r \ / \ y$. The term $f_2(t)$ $(T/y)^{\gamma}$ shows the growth effect of the technology intensity $T \ / \ y$ on production rate $\dot{y} \ / \ y$. Coefficient γ, $0 \leq \gamma \leq 1$ is the elasticity parameter of the growth effect. The negative sign $-g(t)$, $g(t) > 0$ of the net contribution by R&D means that in the short-run spending into R&D prevails on the rate of return.

Change $\dot{T} = r$ in technology T due to time lag m and obsolescence effect σ in technology, is not precisely equal to the current R&D investment r_t and is connected mainly with the R&D investment in initial stage r_{t-m}

$$\dot{T} = r = \frac{1}{(1-\sigma)}(-\sigma T + r_{t-m}), \quad 0 \leq \sigma < 1 \tag{2}$$

3. UTILITY OF THE SYSTEM TRAJECTORIES

We formulate now the utility principle for evaluating the quality of economic trajectories $(y(\cdot), T(\cdot), r(\cdot))$. For this purpose we introduce the discounted integral which measures utility in the long-run term (see, for example, (Arrow, 1985), (Grossman and Helpman, 1991))

$$U_t = \int_t^{\infty} e^{-\rho(s-t)} \ln D(s) ds \tag{3}$$

Here natural logarithm of the consumption index $D(s)$ represents instantaneous utility of products (technologies) at time s, ρ is the discount rate, s is the running time, t is the fixed initial time. For the consumption index D we choose a specification that imposes a constant and equal elasticity of substitution

$$\varepsilon = 1/(1-\alpha) > 1, \quad 0 < \alpha < 1 \tag{4}$$

between every pair of products

$$D(s) = (\int_0^{n(s)} (y(s)/n(s))^{\alpha} dj)^{1/\alpha}$$

Assuming that quantity of invented products n depends on the accumulated R&D investment T and the technology rate r (see (Watanabe, 1992))

$$n = n(s) = bT^{\beta_1} r^{\beta_2}, \ T = T(s), \ r = r(s) \tag{5}$$

we arrive to the following expression for the utility function

$$\begin{aligned} U &= \int_t^{\infty} e^{-\rho s}(\ln y(s) + a_1 \ln T(s) + a_2 \ln r(s)) ds \\ a_i &= A\beta_i, \quad i = 1, 2, \quad A = (1-\alpha)/\alpha \end{aligned} \tag{6}$$

Here coefficients $\beta_i > 0$, $i = 1, 2$ are parameters of elasticity.

We may assume that there exists lower bounds y_l, T_l for production y and technology T

$$0 < y_l \leq y, \quad 0 < T_l \leq T \tag{7}$$

One can introduce upper and lower bounds r_l, r_u for R&D intensity $r \ / \ y$

$$0 < r_l \leq r/y \leq r_u < +\infty \tag{8}$$

4. OPTIMALITY PRINCIPLES

The problem is to find the optimal level r^0 of investment, the corresponding optimal production y^0 and the optimal accumulated R&D investment T^0 subject to dynamics (1), (8) which maximize the utility function (6). For its solution one can use the maximum principle of Pontryagin (see (Pontryagin *et al.*, 1962)).

Remark 1. It is proved in (Aseev *et al.*, 2001) that the optimal control problem (1), (8), (6) has solution $(y^0(\cdot), T^0(\cdot), r^0(\cdot))$

$$\max_r U(y, T, r) = U(y^0, T^0, r^0) \tag{9}$$

Let us compose the Hamiltonian of the problem (1), (6)

$$H(y,T,r,\psi_1,\psi_2) = \ln y + a_1 \ln T + a_2 \ln r + \psi_1(f_1 y + f_2 T^{\gamma} y^{(1-\gamma)} - gr) + \psi_2 r \qquad (10)$$

Let us note that boundaries r_l, r_u in restrictions on control parameter r is given not precisely and scarcely can be identified from the real econometric data. Therefore, we will be interested in such regimes of optimal control r^0 which are realized at points of global maximum of the Hamiltonian H (10) for technology rates $r > 0$

$$\partial H/\partial r = a_2/r - g\psi_1 + \psi_2 = 0 \qquad (11)$$

So the maximum value is attained at the optimal technology rate r^0

$$r^0 = a_2/(g\psi_1 - \psi_2) \qquad (12)$$

Combining dynamics of real y, T and adjoint variables ψ_1, ψ_2 with the maximum condition for the Hamiltonian (12) we obtain the following closed system of differential equations

$$\begin{aligned} \dot{y}/y &= f_1 + f_2(T/y)^{\gamma} - ga_2/((g\psi_1 - \psi_2)y) \\ \dot{T} &= a_2/(g\psi_1 - \psi_2) \qquad (13) \\ \dot{\psi}_1/\psi_1 &= \rho - 1/(\psi_1 y) - (1-\gamma) f_2 (T/y)^{\gamma} - f_1 \\ \dot{\psi}_2/\psi_2 &= \rho - a_1/(\psi_2 T) - \gamma f_2 (\psi_1/\psi_2)(y/T)^{1-\gamma} \end{aligned}$$

Let us introduce notations for costs of production y and technology T

$$z_1 = \psi_1 y, \quad z_2 = \psi_2 T, \quad z = z_1 + z_2 \qquad (14)$$

On the finite horizon $[t,\vartheta]$ the transversality conditions can be written in terms of costs

$$z(\vartheta) = 0, \quad z_i(\vartheta) = 0, \quad i = 1,2 \qquad (15)$$

Proposition 1. The cost $z = z(s)$ satisfies the following differential equation

$$\dot{z}(s) = \rho(z(s) - p^0), \quad p^0 = (a_1 + a_2 + 1)/\rho \quad (16)$$

Its solution which meets transversality conditions (15) can be presented by Cauchy formula

$$z(s) = p^0(1 - e^{-\rho(\vartheta - s)}) \qquad (17)$$

Remark 2. Solution z (17), its components z_i, and adjoint variables ψ_i, $i = 1,2$ are bounded

$$\begin{aligned} &0 \le z(s) \le p^0, \quad 0 \le z_i(s) \le p^0, \quad i = 1,2 \\ &0 \le \psi_1(s) \le p^0/y_l, \quad 0 \le \psi_2(s) \le p^0/T_l \quad (18) \end{aligned}$$

It means that for times $\vartheta \to +\infty$ there exists a sequence of components of optimal solutions $y^k(\cdot)$, $T^k(\cdot)$, $z^k(\cdot)$, $\psi_i^k(\cdot)$, $z_i^k(\cdot)$, $i = 1,2$ for the problems with finite horizons ϑ_k which converges to the optimal solution of the problem (1), (6) with the infinite horizon.

Remark 3. The uniform estimate is valid for the sequence $z^k(\cdot)$

$$\sup_{s \ge t} |e^{-\rho s} z^k(s) - e^{-\rho s} p^0| = e^{-\rho \vartheta_k} p^0 \qquad (19)$$

For the terminal times growing to infinity $\vartheta_k \to +\infty$ the sequence of costs $\{e^{-\rho s}\ z^k(s)\}$ converges uniformly to the optimal cost $e^{-\rho s}\ p^0$ and therefore the constant

$$z = p^0 \qquad (20)$$

is the limit function for costs $\{z^k(\cdot)\}$. The constant function (first integral) $z = p^0$ is the unique solution of differential equation (16) which meets the well known transversality condition (see (Arrow, 1985))

$$\lim_{s \to \infty} e^{-\rho s} z(s) = 0 \qquad (21)$$

Transversality condition (21) means that the total cost $z(s)$ should not grow faster than exponent $e^{\rho s}$.

Our further task is to analyze the optimal dynamics (13) with condition (20) for cost z and estimates (18) for costs z_i, $i = 1,2$.

5. EQUILIBRIUM SOLUTION

Let us pass to analysis of nonlinear system (13) with transversality conditions (20) and estimates (18). Proofs of the main results are contained in (Tarasyev and Watanabe, 2001).

Assume that function f_1, f_2, g in dynamics (1) are constants and can be obtained, for example, as average values of the real econometric time series.

We introduce new variables

$$x_1 = y/T, \ x_2 = \psi_1 y, \ x_3 = 1/T, \ x_4 = \psi_2 T \quad (22)$$

Taking into account the first integral the system of new variables (22) and the first integral (20) we reduce system (13) to the three dimensional system with the block structure

$$\begin{aligned} \dot{x}_1 &= f_1 x_1 + f_2 x_1^{(1-\gamma)} - \frac{a_2(x_1 + g)x_1}{((x_1+g)x_2 - p^0 x_1)} \\ \dot{x}_2 &= \rho x_2 + \gamma f_2 x_2 \frac{1}{x_1^{\gamma}} - 1 - \frac{a_2 g x_2}{((x_1+g)x_2 - p^0 x_1)} \\ \dot{x}_3 &= -\frac{a_2 x_1 x_3}{((x_1+g)x_2 - p^0 x_1)} \end{aligned} \qquad (23)$$

In our analysis we assume that the following inequalities hold

$$0 \le \gamma \le 1, \quad f_1 - \rho = \nu > 0 \qquad (24)$$

Proposition 2. Assume that the growth conditions (24) hold. Then system (23) has stationary points x^0 with the following properties

$$0 < r_1 < x_1^0 \le u_1, \quad 0 \le r_2 < x_2^0 \le p^0 \qquad (25)$$

$$\xi_1 x_2^0 - p^0 x_1^0 > 0, \quad \xi_1 = x_1^0 + g, \quad x_3^0 = 0 \qquad (26)$$

Here parameters r_1, u_1 are unique positive solutions of the following equations

$$\frac{g}{(r_1+g)} = \frac{(\rho r_1^\gamma + \gamma f_2)}{(f_1 r_1^\gamma + f_2)}, \quad \frac{p^0 g}{(u_1+g)} = \frac{a_2 u_1^\gamma}{(f_1 u_1^\gamma + f_2)}$$

$$r_1 \ge g \min\{(f_1-\rho)/\rho,\ (1-\gamma)/\gamma\}$$

Parameter r_2 is defined by relation

$$r_2 = p^0 \min\{1-\gamma,\ 1 - a_1/(f_1 p^0 + 1)\}$$

If the growth rate γ and the corresponding transition coefficient f_2 are sufficiently small

$$f_2 \gamma^2 \le (a_2/p^0) \min\{1, g(a_1+1)/a_2\} \qquad (27)$$

then point x^0 is unique.

6. OPTIMAL TRAJECTORIES

In order to describe properties of the optimal control r^0 (12) we analyze stability of stationary point x^0. More precisely, we indicate the saddle character of this equilibrium and show the existence of optimal trajectories which converge to it. To this end we calculate the Jacobi matrix $DF = \{\partial F_i / \partial x_j\}$, $i,j = 1,2,3$ of the right hand side of system (23).

Proposition 3. The Jacobi matrix DF has at least one eigenvalue with positive real part and hence the stationary point x^0 is unstable.

Let us introduce the following assumptions for parameter a_1 and growth rate γ

$$a_1 \le 1, \quad \gamma \le \gamma^0 = \min\{1/2,\ \rho/f_1\} \qquad (28)$$

Proposition 4. Assume that conditions (24), (27), (28) hold. Then the Jacobi matrix DF has real eigenvalues: one - positive, and two - negative. Hence the stationary point x^0 is a saddle point.

Remark 4. If discriminant of the Jacobi matrix D is negative then the positive eigenvalue μ_1 provides the greater growth rate for trajectories of system (23) than the growth rate ρ, $\mu_1 > \rho > 0$, and the negative eigenvalue μ_2 can be presented through the positive one $\mu_2 = -(\mu_1 - \rho) < 0$.

Remark 5. Eigenvectors h_1, h_2 corresponding to eigenvalues μ_1, μ_2 have positive coordinates

$$h_1 = (b, a+\mu_1, 0)/(b^2 + (a+\mu_1)^2)^{1/2}$$
$$h_2 = (a+\mu_1, c, 0)/(c^2 + (a+\mu_1)^2)^{1/2} \qquad (29)$$
$$a = |\partial F_1/\partial x_1|,\ b = \partial F_1/\partial x_2,\ c = |\partial F_2/\partial x_1|$$

Proposition 5. Under conditions (24), (27), (28) linearized system has the following properties.

1. Equilibrium x^0 is the unique saddle point.

2. For any pair x_1^*, x_3^* there exists the unique component x_2^* such that initial position x^* is located on the plane generated by eigenvectors h_2, h_3 corresponding to negative eigenvalues μ_2, μ_3. Trajectory $x^*(\cdot)$ of the linearized system starting at initial position x^* tends to equilibrium x^0.

3. If relation $0 \le x_2^* \le p^0$ takes place then trajectory $x^*(\cdot)$ meets the condition $0 \le x_2^*(t) \le p^0$, $t > t_0$.

4. The second component $x_2(\cdot)$ of other trajectories $x(\cdot)$ starting at points $x = (x_1^*, x_2, x_3^*)$, $x_2 \ne x_2^*$ tends to infinity with the exponential growth rate $\mu_1 > \rho$.

According to the Grobman-Hartman theorem (see (Hartman, 1964)) nonlinear system (23) as well as the linearized system admits a trajectory which converges to equilibrium x^0.

Proposition 6. Nonlinear system (23) inherits the convergence property of the linearized system. Then the solution of optimal control problem (1), (6) exists and coincides with equilibrium trajectory $x^0(\cdot)$.

Remark 6. The third component $x_3^0 = 1 \ / \ T^0$ converges to zero $x_3^0 = 0$ (26) with negative velocity (23). It means that optimal technology stock $T^0(\cdot)$ monotonically grows to infinity.

The first component $x_1^0 = y^0 \ / \ T^0$ converges to the positive equilibrium value x_1^0. It shows that optimal production $y^0 = y^0(t)$ also grows to infinity with the same growth rate as technology T^0. In particular, this growth property of production y^0 means that its derivative in dynamics (1) is strictly positive $\dot{y}^0 > 0$.

If the initial ratio x_1^* is greater than at equilibrium x_1^0, $x_1^0 \le x_1^*$, then the optimal ratio $x_1^0(t) = y^0(t) \ / \ T^0(t)$ is decreasing from the initial state x_1^* to equilibrium x_1^0. It indicates that optimal technology stock T^0 is growing faster than production y^0.

Remark 7. The optimal trajectory $x^0(\cdot)$ and corresponding optimal control $r^0(\cdot)$ can be approximated numerically with any degree of accuracy in the backward procedure starting from points in the neighborhood of equilibrium x^0 which are located on the eigenvector h_2 corresponding to the negative eigenvalue μ_2. This numerical approximation obtained by means of software RATE (see (Reshmin *et al.*, 2002)) is demonstrated on Fig. 1 which shows trends of econometric time series for production and technology in Japan and optimal synthetic scenarios in R&D intensity $r \ / \ y$. Real data is depicted by curves with markers and synthetic optimal trajectories are shown by solid lines.

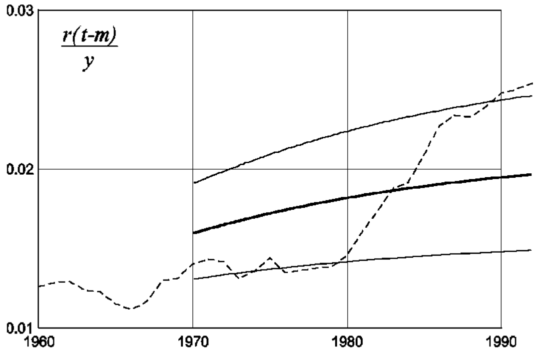

Fig. 1. Trends in optimal R&D intensity r / y.

7. CONTROL SYNTHESIS

It should be noted that the problem of searching the optimal trajectory $x^0(\cdot)$ which leads system (23) to the saddle point x^0 is very complicated due to the unstable properties of this equilibrium. We propose several analytic procedures for finding a suboptimal feedback which leads coordinates $x_1(\cdot)$, $x_3(\cdot)$ of the system to equilibrium x_1^0, x_3^0. To this end we consider the linear regime for the second coordinate $x_2(\cdot)$

$$x_2 = x_2^0 + \omega\Delta x_1,\ \Delta x_1 = x_1 - x_1^0,\ \omega \geq 0 \quad (30)$$

$$\dot{x}_1 = f_1 x_1 + f_2 x_1^{(1-\gamma)} - \frac{a_2(x_1+g)x_1}{(d+k(\omega)\Delta x_1 + \omega\Delta^2 x_1)}$$
$$\dot{x}_3 = -\frac{a_2 x_1 x_3}{(d+k(\omega)\Delta x_1 + \omega\Delta^2 x_1)} \quad (31)$$

Here parameters d, k are determined by relations

$$d = gx_2^0 - x_1^0\Delta x_2, \quad \Delta x_2 = p^0 - x_2^0$$
$$k(\omega) = k_1\omega + k_2,\ k_1 = \xi_1,\ k_2 = -\Delta x_2$$

and initial conditions x_1^*, x_3^* should satisfy conditions

$$x_1^0 \leq x_1^* < x_1^0 + \overline{x}_1(\omega), \quad x_3^* > 0 \quad (32)$$
$$\overline{x}_1(\omega) = 2d/(|k(\omega)| + (k^2(\omega) - 4\omega d)^{1/2})$$

We extract the expression for feedback $r = r(y,T)$ from system (31)

$$r = a_2 y/(d + k(\omega)\Delta x_1 + \omega\Delta^2 x_1) \quad (33)$$

Proposition 7. Assume that the slope coefficient ω of the second coordinate x_2 (30) satisfies conditions

$$0 \leq \omega \leq gp^0/\xi_1^2 = \omega_1 \quad (34)$$

Then the suboptimal rational feedback $r^*(\omega)$ (33) leads trajectories $x^*(\cdot)$ of system (31) from initial conditions x_1^*, x_3^* to equilibrium x_1^0, x_3^0.

Remark 8. The third component $x_3^* = 1$ / T^* converges to zero in the suboptimal regime (31). It means that technology stock T^* monotonically grows to infinity with the asymptotic growth rate $|\mu_3| > (f_1 - \rho) > 0$.

The first component $x_1^* = y^*$ / T^* converges to the positive equilibrium value x_1^0. It shows that suboptimal production y^* also grows to infinity with the same growth rate as technology T^*.

If the initial ratio x_1^* of production y^* to technology T^* is greater than the corresponding value at equilibrium x_1^0, then the suboptimal ratio $x_1^* = y^*$ / T^* is decreasing from the initial state x_1^* to equilibrium x_1^0. It indicates that in this case technology stock T^* is growing faster than production y^*.

Remark 9. In the expression for suboptimal control $r^*(\omega)$ (33) denominator tends to the positive constant value $(gx_2^0 - x_1^0\ \Delta x_2) > 0$, when $t \to \infty$, and numerator $a_2\ y^*$ is linear with respect to production y^*. It demonstrates that the value of suboptimal control $r^*(\omega)$ (33) is also growing to infinity with the same asymptotic growth rate as production y^* and technology T^*.

8. TRENDS OF R&D INTENSITIES

We examine the question about the evolutionary behavior of R&D intensities which is expressed by ratios r / y, r_{t-m} / y on suboptimal trajectories.

Proposition 8. There exists the interval of slopes

$$\omega_2 \leq \omega \leq \omega_1,\ \omega_1 = gp^0/\xi_1^2,\ \omega_2 = \Delta x_2/\xi_1 \quad (35)$$

such that suboptimal feedback $r^*(\omega)$ (33) leads trajectories $x^*(\cdot)$ from initial position x_1^*, x_3^* (32) to equilibrium x_1^0, x_3^0 with evolutionary decline of ratio $x_1 = y$ / T and growth of ratio r / y.

Let us consider the natural candidate for the slope of the suboptimal feedback (33) – the slope ω_0 of eigenvector h_2 (29) of the Jacobi matrix D which corresponds to the negative eigenvalue μ_2

$$\omega_0 = (a + \mu_2)/b \quad (36)$$

Proposition 9. The slope ω_0 (36) of eigenvector h_2 (29) corresponding to the negative eigenvalue μ_2 satisfies relations $0 \leq \omega_0 < \omega_1$, and, hence, the suboptimal feedback $r^*(\omega_0)$ (33) with slope ω_0 leads trajectories $x^*(\cdot)$ from initial position x_1^*, x_3^* to equilibrium x_1^0, x_3^0.

Proposition 10. There exists a threshold $\gamma^* > 0$ such that for parameters $0 \leq \gamma < \gamma^*$, $a_1 \geq 0$ the optimal slope ω_0 satisfies inequalities $0 \leq \omega_0 \leq \omega_2$

If inequalities $0 < \gamma < \gamma^*$ or $a_1 > 0$ take place then relations $0 < \omega_0 < \omega_2$ are valid.

It means that the suboptimal control $r^*(\omega_0)$ (33) with slope ω_0 provides the decline property of ratio r / y when x_1 tends to equilibrium x_1^0. Moreover, the optimal R&D intensity r^0 / y^0 has the same decline property, since the difference $(r^0$ / $y^0 - r$ / $y)$ between optimal and suboptimal R&D intensities is an infinitesimal value $o(\Delta x_1)$ of high order.

Remark 10. Assuming $\omega = 0$ in formula (33) one can obtain the suboptimal process with the constant value for the cost of production x_2^0

$$r = a_2 y / (\Delta x_1 (d - \Delta x_2)) \quad (37)$$

In the suboptimal process (37) ratio $r \;/\; y$ is growing while ratio $y \;/\; T$ is declining.

Setting the constant value for coordinate $x_1 = x_1^0$ in formula (37) one can derive the suboptimal process $r = a_2 \; y \;/\; d$ with the fixed second coordinate $x_2 = x_2^0$ and the constant ratio $r \;/\; y = a_2 \;/\; d$.

Suboptimal feedbacks lead trajectories $x^*(\cdot)$ of system (31) from initial conditions x_1^*, x_3^* to equilibrium x_1^0, x_3^0.

Let us examine the behavior of intensity $r_{t-m} \;/\; y$ in the suboptimal regime (33). Ratio $r_{t-m} \;/\; y$ is growing while ratio $x_1 = y \;/\; T$ is declining under condition

$$\omega \geq (\Delta x_2 - \sigma d^2 / (a_2 (1 - \sigma)(x_1^0)^2)) / \xi_1 = \omega_3 \quad (38)$$

which provides the positive sign of derivative $d(r_{t-m}/y) \;/\; dx_1$.

Remark 11. Summarizing previous results one can derive the following properties of the suboptimal control $r^*(\omega)$ (33):

1. if $0 \leq \omega < \max \{0, \omega_3\}$ then both ratios $r_{t-m} \;/\; y$ and $r \;/\; y$ are declining;

2. if $\max \{0, \omega_3\} \leq \omega < \omega_2$ then ratio $r_{t-m} \;/\; y$ is growing and ratio $r \;/\; y$ is declining.

3. if $\omega_2 \leq \omega \leq \omega_1$ then both ratios $r_{t-m} \;/\; y$ and $r \;/\; y$ are growing;

while x_1 is declining to equilibrium x_1^0.

Remark 12. If slope ω satisfies inequality

$$\omega < \Delta x_2 / \xi_1 + \gamma f_2 (x_1^0)^{-(1+\gamma)} d^2 / (a_2 g) = \omega_4 \quad (39)$$

then production rate $\dot{y}/y$ is growing while parameter x_1 is declining to equilibrium x_1^0.

REFERENCES

Arrow, K.J. (1985). *Production and Capital. Collected Papers*. The Belknap Press of Harvard University Press. Cambridge, Massachusetts, London.

Aseev, S.M., A.V. Kryazhimskii and A.M. Tarasyev (2001). The Pontryagin maximum principle and transversality conditions for an optimal control problem with infinite time interval. *Proceedings of the Steklov Institute of Mathematics* **233**, 64–80.

Dolcetta, I.C. (1983). On a discrete approximation of the Hamilton-Jacobi equation of dynamic programming. *Applied Mathematics and Optimization* **10**(4), 367–377.

Grossman, G.M. and E. Helpman (1991). *Innovation and Growth in the Global Economy*. The MIT Press. Cambridge, Massachusetts, London.

Hartman, Ph. (1964). *Ordinary Differential Equations*. J. Wiley & Sons. N.Y., London, Sydney.

Intriligator, M. (1971). *Mathematical Optimization and Economic Theory*. Prentice-Hall. New York.

Leitmann, G. and C.S. Lee (1999). On one aspect of science policy based on an uncertain model. *Annals of Operations Research* **88**, 199–214.

Pontryagin, L.S., V.G. Boltyanskii, R.V. Gamkrelidze and E.F. Mishchenko (1962). *The Mathematical Theory of Optimal Processes*. Interscience. NY.

Reshmin, S.A., A.M. Tarasyev and Watanabe C. (2002). Optimal trajectories of the innovation process and their matching with econometric data. *Journal of Optimization Theory and Applications*.

Tarasyev, A.M. (1999). Control synthesis in grid schemes for Hamilton-Jacobi equations. *Annals of Operations Research* **88**, 337–359.

Tarasyev, A.M. and C. Watanabe (2001). Optimal dynamics of innovation in models of economic growth. *Journal of Optimization Theory and Applications* **108**(1), 175–203.

Watanabe, C. (1992). Trends in the substitution of production factors to technology - empirical analysis of the inducing impact of the energy crisis on Japanese industrial technology. *Research Policy* **21**, 481–505.