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QUOTATONE APPORTIONMENT METHODS

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PREFACE

The problem of how to make a "fair division" of resources among competing interests arises in many areas of application at IIASA. One of the tasks in the System and Decision Sciences Area is the systematic investigation of different criteria of fairness and the formulation of allocation procedures based thereon.

A particular problem of fair division having wide application in governmental decision-making is the *apportionment* problem. An application has recently arisen in the debate over how many seats in the European Parliament to allocate to the different member countries. Discussions swirled around particular numbers, over which agreement was difficult to achieve. A systematic approach that seeks to establish bases for agreement on the criteria or "principles" for fair division should stand a better chance of acceptance in that it represents a scientific or system-analytic approach to the problem.

ABSTRACT

It has recently been pointed out that there exists more than one house monotone apportionment method satisfying quota.

This paper gives a simple characterization of all such methods as an immediate consequence of the Quota method's existence. Further, a manner of exposition is formulated which unites several key house monotone apportionment methods, thus clearly showing their connections.

<u>Quotatone Apportionment Methods</u>

1. INTRODUCTION

Let $\underset{\sim}{p} = (p_1,\ldots,p_s)$ be the populations of s states, where each $p_i > 0$, and $h \geqq 0$ the number of seats in the house. The problem is to find, for any $\underset{\sim}{p}$ and all h, an *apportionment for* h, that is, an s-tuple of non-negative integers $\underset{\sim}{a} = (a_1,\ldots,a_s)$ whose sum is h. A *solution* of the apportionment problem is therefore a function $\underset{\sim}{f}$ which to every $\underset{\sim}{p}$ and all h associates a unique apportionment for h, $a_i = f_i(\underset{\sim}{p},h) \geqq 0$ where $\sum_i a_i = h$. If $\underset{\sim}{f}$ is a solution and h a house size then $\underset{\sim}{f}^h$ is the function $\underset{\sim}{f}$ restricted to the domain $(\underset{\sim}{p},h')$, where $0 \leqq h' \leqq h$. $\underset{\sim}{f}^h$ is called a *solution up to* h and $\underset{\sim}{f}$ is called an *extension* of $\underset{\sim}{f}^h$. A specific apportionment "method" may give several different solutions, for "ties" may occur when using it, for example when two states have identical populations. For this reason it is useful to define an *apportionment method* $\underset{\sim}{M}$ as a non-empty set of solutions. A method $\underset{\sim}{M}$ is the unique one satisfying given properties if any other collection of solutions with these properties is a collection of $\underset{\sim}{M}$-solutions.

Let $p = \sum_i p_i$ be the total population. The *exact quota* of state j is $q_j(\underset{\sim}{p},h) = p_j h/p$, its *lower quota* is $\lfloor q_j(p,h) \rfloor$ (the largest integer less than or equal to q_j), and its *upper quota* is $\lceil q_j(\underset{\sim}{p},h) \rceil$ (the least integer greater than or equal to q_j). An apportionment method is said to *satisfy lower quota* if, for each of its solutions $\underset{\sim}{f}$, $f_i(\underset{\sim}{p},h) \geqq \lfloor q_i(\underset{\sim}{p},h) \rfloor$, to *satisfy upper quota* if $f_i(\underset{\sim}{p},h) \leqq \lceil q_i(\underset{\sim}{p},h) \rceil$, and to *satisfy quota* if it satisfies both lower and upper quota. A method is said to be *house monotone* if, for each of its solutions $\underset{\sim}{f}$, $\underset{\sim}{f}(\underset{\sim}{p},h+1) \geqq \underset{\sim}{f}(\underset{\sim}{p},h)$. A method $\underset{\sim}{M}$ is *quotatone* if $\underset{\sim}{M}$ is house monotone and satisfies quota.

The existence of a quotatone method was first established in [1,4]. This method, called the Quota Method, was also shown to be the *unique* such method satisfying a certain property of mathematical *consistency* (subject to satisfying quota).

On the other hand, not every quota, house monotone apportionment solution is a Quota method solution: indeed, it suffices to find just one example in which some Quota solution may be "twiddled" slightly (e.g. by interchanging the order in which some two states receive successive seats) while still satisfying house monotonicity and quota. Of course, such solutions will have a certain arbitrariness about them, and in particular will not be "consistent," thus violating an intrinsic idea of what is meant by a "method."

Nevertheless, it is interesting to ask *how far* some arbitrary, quotatone solution may deviate from a Quota method solution. Still [6] has given a characterization of all such solutions; here we shall give a simpler characterization that relates the class of quota, house monotone solutions to the Quota method and, at a further remove, to the Jefferson method, $\underset{\sim}{J}$.

2. THE DECK OF CARDS

Given $\underset{\sim}{p}$ we define the *Jefferson deck*, $\underset{\sim}{D} = \{(i,a,p_i/a)\}$, as a sequence of "cards," each card bearing the *name* of a state i, a number of seats a, and the average district size p_i/a if a seats are apportioned to state i, stacked, in decreasing order, by the *values* p_i/a, $1 \leqq i \leqq s$, and $a \geqq 1$ integer.

In the sequel we drop any redundant mention of the populations $\underset{\sim}{p}$.

Any house monotone method may be described in terms of $\underset{\sim}{D}$ as follows. At house size h = 0, set $\underset{\sim}{a} = \underset{\sim}{0}$ and begin with the full deck $\underset{\sim}{D} = \underset{\sim}{D}(\underset{\sim}{0},0)$. Given any apportionment $\underset{\sim}{a}$ for h, an apportionment for h + 1 is found by withdrawing a card of form $(i,a_i+1, p_i/(a_i+1))$ from the remaining deck $\underset{\sim}{D}(\underset{\sim}{a},h)$ and giving $a_i + 1$ seats to state i.

To say an apportionment $\underset{\sim}{a}$ for h satisfies lower quota is equivalent to saying $a_i + 1 > p_ih/p$ or

(1) $p_i/(a_i+1) < p/h$,

while to say $\underset{\sim}{a}$ satisfies upper quota is equivalent to saying $a_i - 1 < p_i h/p$ or (for $h > 0$ and $a_i \geqq 1$)

(2) $p_i/(a_i-1) > p/h$.

In this paper $p_i/0$ will be interpreted as having value plus infinity. So, knowing the position of p/h relative to the Jefferson deck, $\underset{\sim}{D}$ determines the apportionments which satisfy quota at h.

3. THE JEFFERSON METHOD

The *Jefferson method* $\underset{\sim}{J}$ [3,4] may be described as follows:

(i) $f_i(0) = 0$, $1 \leqq i \leqq s$ and $\underset{\sim}{D}(\underset{\sim}{0},0) = \underset{\sim}{D}$.

(ii) If $f_i(h) = a_i$, $1 \leqq i \leqq s$, is an apportionment for h and $\underset{\sim}{D}(\underset{\sim}{a},h)$ the remaining deck, let k be the name of the state on the topmost card. Then remove that card and let $f_k(h+1) = a_k + 1$ and $f_i(h+1) = a_i$ for $i \neq k$.

Notice that the number of seats on the discarded card is precisely equal to $a_k + 1$. Huntington [5] described $\underset{\sim}{J}$ and certain other methods in essentially this manner.

$\underset{\sim}{J}$ is clearly house monotone. It also satisfies lower quota. For, suppose not: then there is some state j with $p_j/(a_j+1) \geqq p/h$ (see (1)). Thus $p_j/a_j > p/h$ and

$$\frac{p}{h} = \frac{\sum_i p_i}{\sum_i a_i} > \min_i \frac{p_i}{a_i} = \frac{p_\ell}{a_\ell} \quad ,$$

implying $p_j/(a_j+1) > p_\ell/a_\ell$. This is a contradiction, since then the card $(j, a_j+1, p_j/(a_j+1))$ would have been chosen before the card $(\ell, a_\ell, p_\ell/a_\ell)$.

It has been shown that house monotonicity and satisfying lower quota together with a "consistency" property (see Section 8) uniquely characterizes $\underset{\sim}{J}$ [3].

4. THE QUOTA METHOD

Let $U(\underset{\sim}{a},h)$ be the set of states which are eligible to receive an extra seat in a house of size $h+1$ without violating upper quota, $U(\underset{\sim}{a},h) = \{i : p_i/a_i > p/(h+1)\}$. These states can be ascertained by looking at the ordered deck of discarded cards.

The *Quota method* $\underset{\sim}{Q}$ [1,4] may be described as follows:

(i) $f_i(0) = 0$, $1 \leq i \leq s$ and $\underset{\sim}{D}(\underset{\sim}{0},0) = \underset{\sim}{D}$.

(ii) If $f_i(h) = a_i$, $1 \leq i \leq s$, is an apportionment for h and $\underset{\sim}{D}(\underset{\sim}{a},h)$ the remaining deck, let k be the name of the state on the topmost card that belongs to $U(\underset{\sim}{a},h)$. Then, remove that card and let $f_k(h+1) = a_k + 1$ and $f_i(h+1) = a_i$ for $i \neq k$.

It has been shown that house monotonicity and satisfying quota together with a weakened "consistency" property uniquely characterize $\underset{\sim}{Q}$ [4].

5. QUOTATONE METHODS

Let $\underset{\sim}{D}$ be the Jefferson deck. For any house $h \geq 0$ and apportionment $\underset{\sim}{a}$ for h let $\alpha(\underset{\sim}{a},h)$ be the *first* integer $\alpha \geq 1$ such that there are at least α cards in the remaining deck $\underset{\sim}{D}(\underset{\sim}{a},h)$ with value $\geq p/(h+\alpha)$, and let $L(\underset{\sim}{a},h)$ be the set of all state names appearing on the first $\alpha(\underset{\sim}{a},h)$ cards. If no such α exists then set $\alpha = \infty$ and let $L(\underset{\sim}{a},h)$ be the set of all states (not all α need be checked; see below).

The meaning of $\alpha = \alpha(\underset{\sim}{a},h)$ is the following: if $\alpha < \infty$, $\underset{\sim}{f}(\underset{\sim}{p},h) = \underset{\sim}{a}$ is some apportionment at h, and the $(h+1)^{st}$ seat is given to some state $k \notin L(\underset{\sim}{a},h)$, then there can be no monotone extension $\underset{\sim}{g}$ of $\underset{\sim}{f}^h$ such that $\underset{\sim}{g}$ satisfies lower quota at house

$h + \alpha$. The reason is that the allocation by $\underset{\sim}{g}$ of each seat from $h + 1$ to $h + \alpha$ corresponds to the removal of a card from the remaining deck, so (by choice of α) at $h + \alpha$ there is still at least one card remaining corresponding to some state $j \in L(\underset{\sim}{a},h)$ and having value $p_j/b \geq p/(h+\alpha)$. Now, since at $h + \alpha$ state j has $b' < b$ seats we have

$$p_j/(b'+1) \geq p_j/b \geq p/(h+\alpha) \quad ,$$

which shows by (1) that state j violates lower quota at $h + \alpha$. Therefore if $\underset{\sim}{f}$ is a quotatone apportionment solution, then $\underset{\sim}{f}$ satisfies

(i) $\underset{\sim}{f}(\underset{\sim}{p},0) = \underset{\sim}{0}$,

(ii) if $\underset{\sim}{f}(\underset{\sim}{p},\underset{\sim}{h}) = \underset{\sim}{a}$ and $f_k(\underset{\sim}{p},h+1) = a_k + 1$, then

$k \in L(\underset{\sim}{a},h) \cap U(\underset{\sim}{a},h)$.

The significance of $\alpha(\underset{\sim}{a},h)$ was to determine which states belonged to $L(\underset{\sim}{a},h)$. If there is no "first" α, then all states must belong to $L(\underset{\sim}{a},h)$. It is clearly unnecessary to inspect values of α larger than those which assure that every card of form $(i,a_i+1,p_i/(a_i+1))$ has value greater than or equal to $p/(h+\alpha)$. Define, then, β_i to be the least positive integer satisfying $p_i/(a_i+1) \geqq p/(h+\beta_i)$, that is $\beta_i = \lceil p(a_i+1)/p_i - h\rceil$ and $\beta = \max_i \beta_i$. Then $L(\underset{\sim}{a},h)$ may be defined as before with this modification: if there is no first integer $\beta > \alpha \geqq 1$ for which at least α cards in $\underset{\sim}{D}(\underset{\sim}{a},h)$ have value $\geq p/(h+\alpha)$, then let $L(\underset{\sim}{a},h)$ be the set of all states.

Let $\underset{\sim}{\overline{Q}}$ be the class of all solutions $\underset{\sim}{f}$ satisfying (i) and (ii).

<u>*Theorem 1.*</u> *$\underset{\sim}{\overline{Q}}$ is precisely the set of all quotatone solutions.*

Proof. We know that every quotatone solution is in $\bar{Q}$, hence $Q \subseteq \bar{Q}$. Further, every $f \in \bar{Q}$ satisfies upper quota, by definition. Suppose $f(p,h+1) = b$ violates lower quota at $h + 1$ for state k. Then $p_k/(b_k+1) \geq p/(h+1)$ by (1), and card $d_0 = (k, b_k+1, p_k/(b_k+1))$ is in $D(b,h+1)$, hence also in $D(f(h),h)$. Therefore $\alpha(f(h),h) = 1$, hence the $(h+1)^{st}$ card removed, d_{h+1}, also had value greater than or equal to $p/(h+1)$, and $d_{h+1} \neq d_0$. Hence $\alpha(f(h-1),h-1) \leq 2$ so d_h (the h^{th} card removed) had value greater than or equal to $p/(h+1)$. In general, $\alpha(f(h'),h') + h' \leq h+1$ for $h' \leqq h$, and in D there were $h + 2$ cards $d_0, d_1, \ldots d_{h+1}$ with values $\geq p/(h+1)$. But then no house monotone solution can satisfy quota at $h + 1$, contradicting the fact that $\bar{Q} \supseteq Q \neq \phi$.□

Thus every quotatone solution is a variant of a Quota method solution in the following sense: instead of giving the additional seat at each successive house to the *first* state satisfying upper quota, give it to some state satisfying upper quota among the *first* α states. The problem is to decide *which* of the α states to select: the selection of the first one (satisfying upper quota) turns out to be the only resolution that is "consistent" (see Section 8) subject to satisfying quota.

6. GENERALIZED LOWER QUOTA

It is necessary to generalize definitions, methods and theorems to the need for any admissible apportionment a to satisfy certain minimum *requirements* $r = (r_1, \ldots, r_s)$, where the integer $r_i \geqq 0$ is the minimum number of seats which must be accorded to state i by mandate. Letting $h^* = \sum_i r_i$, an *apportionment* for $h \geq h^*$ is an n-tuple of integers $a = (a_i, \ldots, a_s)$, with $a \geqq r$ and $\sum_i a_i = h$. A *solution* is a function $f(p,r,h)$ which to every p and r, and all $h \geqq h^* = \sum_i r_i$, associates a unique apportionment for h, $a_i = f_i(p,r,h) \geqq r_i$, where $\sum_i a_i = h$. The concepts *method, extension, solution up to* h are defined analogously to the pure $(r=0)$ case.

It is impossible, for certain values r, to ask for solutions satisfying quota. Thus, this definition must be modified. The motivation is this. Suppose the exact quota of state i at h is less than or equal to r_i: then it deserves no more than r_i seats, but is required to receive at least r_i seats. Therefore, we reason, it should receive exactly r_i seats, and we say its lower and upper quota should be exactly r_i. Eliminate these states whose apportionment is fixed, and subtract the corresponding r_i's from $h = h_0$ to obtain h_1 seats which must be distributed among the remaining states. Using this smaller house h_1, compute exact quotas, that is, compute the proportional share of h_1 that each of the remaining states deserves, and iterate.

Specifically, let $J_0 = J_0(h) = \{1,\ldots,s\}$, $h_0 = h(\geq h^*)$ and define $J_1 = J_1(h) = \{i \in J_0; p_i h_0/\sum_{J_0} p_j > r_i\}$ and $h_1 = h_0 - \sum_{i \notin J_1} r_i$. In general, $J_{\beta+1} = J_{\beta+1}(h) = \{i \in J_\beta; p_i h_\beta/\sum_{J_\beta} p_j > r_i\}$ and $h_{\beta+1} = h_0 - \sum_{i \notin J_{\beta+1}} r_i < h_\beta$, the process stopping with J_μ when $J_{\mu+1} = J_\mu$. Thus, $J_0(h) \supset J_1(h) \supset \ldots \supset J_\mu(h) = J(h)$ and $h = h_0 > h_1 > \ldots > h_\mu$, with $p_i h_\mu/\sum_{J_\mu} p_j > r_i$ for $i \in J_\mu = J$. We define the *(generalized) exact quota* $q_i(\underset{\sim}{p},\underset{\sim}{r},h)$ of state i to be

$$q_i(\underset{\sim}{p},\underset{\sim}{r},h) = r_i \quad \text{for} \quad i \notin J(h)$$

$$= p_i h_\mu/\textstyle\sum_{J(h)} p_j \quad \text{for } i \in J(h) \quad .$$

Thus, the *(generalized) lower quota* of state i is $\ell_i(h) = \lfloor q_i(\underset{\sim}{p},\underset{\sim}{r},h) \rfloor$ and the *(generalized) upper quota* is $u_i(h) = \lceil q_i(\underset{\sim}{p},\underset{\sim}{r},h) \rceil$. This means, in particular, that $\ell_i(h) = u_i(h) = r_i$ for $i \notin J(h)$. Note that this definition is slightly more natural than that given previously in [4] and simplifies the proof of Theorem 4 in that paper.

There is a more direct way of computing J(h). By definition,

$$(3) \qquad p_i h_\beta/\textstyle\sum_{J_\beta} p_j \leqq r_i \text{ for } i \in J_\beta \sim J_{\beta+1} \quad , \quad 0 \leqq \beta < \mu \ .$$

Therefore,

$$h_{\beta+1} = h_\beta - \sum_{J_\beta \sim J_{\beta+1}} r_i \leqq h_\beta - \frac{h_\beta \sum_{J_\beta \sim J_{\beta+1}} p_i}{\sum_{J_\beta} p_j} = \frac{h_\beta \sum_{J_{\beta+1}} p_j}{\sum_{J_\beta} p_j} ,$$

and so

$$\sum_{J_{\beta+1}} p_j / h_{\beta+1} \geqq \sum_{J_\beta} p_j / h_\beta$$

or

$$\sum_{J_{\beta+1}} p_j / (h - h^* + \sum_{J_{\beta+1}} r_j) \geqq \sum_{J_\beta} p_j / (h - h^* + \sum_{J_\beta} r_j) .$$

From this and (3) we deduce

$$(4) \quad p_i/r_i > \frac{\sum_{J(h)} p_j}{h - h^* + \sum_{J(h)} r_j} \geqq p_k/r_k \quad \text{for all } i \in J(h),\ k \notin J(h) .$$

But (4) uniquely determines $J(h)$ by the following procedure. Suppose, for simplicity, $p_1/r_1 \geq p_2/r_2 \geq \cdots \geq p_s/r_s$. Given h and h^*, consider $\lambda_1 = p_1/(h - h^* + r_1)$. If $\lambda_1 \geqq p_2/r_2$, stop, $J(h) = \{1\}$. Otherwise, consider $\lambda_2 = (p_1 + p_2)/(h - h^* + r_1 + r_2) > \lambda_1$. If $\lambda_2 \geqq p_3/r_3$, stop, $J(h) = \{1,2\}$. Otherwise, continue similarly.

Define $U(\underset{\sim}{a},h)$ to be the set of states eligible to receive an extra seat in a house of size $h+1$ without violating (generalized) upper quota, $U(\underset{\sim}{a},h) = \{i : a_i + 1 \leqq u_i(h+1)\}$. Then the (generalized) *Quota method* $\underset{\sim}{Q}(\underset{\sim}{r})$ [1,3] with requirements is exactly the same as $\underset{\sim}{Q}$ except that $f_i(\underset{\sim}{p},\underset{\sim}{r},h^*) = r_i$ for all i, and $\underset{\sim}{D}(\underset{\sim}{a},h^*)$ is the original deck $\underset{\sim}{D}$ from which has been eliminated all cards $(i,a,p_i/a)$ with $a \leqq r_i$ for all i.

Still [6] attacks this definition of generalized lower quota because it "admits apportionments" not satisfying pure lower quota "even though...no violation of pure lower quota is necessary to satisfy the minimum requirements." The following theorem shows his objection to be inapplicable.

Theorem 2. If there exists an apportionment at h satisfying pure lower quota then there exists a $\underset{\sim}{Q}$-apportionment which does so.

Proof. Suppose that there exists an apportionment at some $h' (\geqq h^* = \sum_i r_i)$ which satisfies pure lower quotas. Then, surely, $\sum_i \max\{\lfloor p_i h'/p \rfloor, r_i\} \leqq h'$, where $p = \sum_i p_i$.

Let $p = \sum_i p_i$. Suppose $\underset{\sim}{a}$ is a $\underset{\sim}{Q}$-apportionment which does not satisfy $a_i \geqq \lfloor p_i h'/p \rfloor$ for all i. Then there exists j such that $a_j < \lfloor p_j h'/p \rfloor$, whence $p_j/(a_j + 1) \geqq p/h'$, and therefore there must be some ℓ with $a_\ell > p_\ell h'/p$.

Let $L = \{k ; a_k > p_k h'/p\} \neq \phi$, and $R = \{i ; r_i \geqq p_i h'/p\}$. For any $i \notin L$ we have $a_i \leqq \lfloor p_i h'/p \rfloor$; in particular, $a_j < \lfloor p_j h'/p \rfloor$. Further, if we assume $L \subseteq R$ then

$$\sum_{i\varepsilon L} a_i + \sum_{i\notin L} a_i \leqq \sum_{i\varepsilon L} r_i + \sum_{i\notin L} a_i < \sum_{i\varepsilon L} r_i + \sum_{i\notin L} \lfloor p_i h'/p \rfloor \leqq h' \quad ,$$

contradicting the fact that $\underset{\sim}{a}$ is an apportionment for h'. Thus, there exists $\ell \varepsilon L \sim R$, that is, a state for which $a_\ell > p_\ell h'/p > r_\ell$, implying

$$p_\ell/a_\ell < p/h' \leqq p_j/(a_j + 1) \quad .$$

Let h_ℓ be the house at which state ℓ received its $a_\ell{}^{th}$ seat; and choose $\ell \varepsilon L \sim R$ such that h_ℓ is largest. Clearly $h_\ell < h'$, since state j is eligible to receive an extra seat at h'. Let

$$K = \{i ; \text{state } i \text{ received a seat at } h,\ h_\ell < h \leqq h_0\} \quad .$$

By choice of ℓ, $\ell \notin K$.

Suppose $k \varepsilon K$ and $a_k > p_k h'/p$; then $k \varepsilon L$ and $k \notin R$, but $h_k > h_\ell$, a contradiction. Therefore, $k \varepsilon K$ implies

(5) $f_k(h') = a_k \leqq p_k h'/p$

However, $f_k(h_\ell) < a_k$ for $k \in K$, and so

$$p_k/(f_k(h_\ell) + 1) \geqq p_k/a_k \geqq p/h' > p_\ell/a_\ell = p_\ell/f_\ell(h_\ell) ,$$

showing that $k \in K$ is ineligible at h_ℓ, that is,

(6) $f_k(h_\ell) = f_k(h_\ell - 1) \geqq p_k h_\ell/p$ for $k \in K$.

But, in the interval $h_\ell < h \leqq h'$ exactly $h' - h_\ell$ seats were awarded to states in K, so $\sum_K \{f_k(h') - f_k(h_\ell)\} = h' - h_\ell$. Subtracting (6) from (5), then summing over K,

$$h' - h_\ell = \sum_K \{f_k(h') - f_k(h_\ell)\} \leqq \sum_K (p_k/p)(h' - h_\ell) ,$$

implying, since $h' - h_\ell > 0$, that $\sum_K p_k/p \geqq 1$, a contradiction since $\ell \notin K$. This completes the proof.

Still's definition [6] of generalized lower quota $\underset{\sim}{\ell}(h)$ may be given as follows. $\underset{\sim}{\ell}(h^*) = \underset{\sim}{r}$. For $h > h^*$, let $\underset{\sim}{\ell}^1(h)$ be defined by $\ell_i^1(h) = \max\{\lfloor q_i(h)\rfloor , \ell_i(h-1)\}$, and $h^1 = \sum \ell_i^1(h)$. In general, if $h^\beta > h$ then let $\underset{\sim}{\ell}^{\beta+1}(h)$ be defined by $\ell_i^{\beta+1}(h) = \max \{\ell_i^\beta(h) - 1 , \ell_i(h-1)\}$. Otherwise, if $h^\beta \leqq h$, then $\underset{\sim}{\ell}(h) = \underset{\sim}{\ell}^\beta(h)$. Thus Still successively reduces the pure lower quota of *every state* that can be reduced without going below the previous generalized lower quota. This is *not* a proportionally motivated scheme; in fact, it tends to consistently favor large states versus small states, as the example of Table 1 illustrates. It must be realized that a definition of generalized lower quota imposes de facto a method of apportionment for "small" house sizes, so this non-proportional bias is important. The $\ell_i(h)$ for h = 20,21,22,23,24,25, and 27 sum to h; hence the lower quotas are in these cases the only admissible apportionments for h which belong to Still's class of methods. Consider, in particular, h = 27. The lower quotas force an apportionment with

$\ell_{22}/\ell_{21} = a_{22}/a_{21} = 6$ whereas $p_{22}/p_{21} = 1.9925$. This is the result of neglecting proportionality in defining lower quotas. In contrast, the proportional approach to lower quotas at $h = 27$ gives $\underset{\sim}{\ell}(27) = (1,1,\ldots,1,2,4)$, so that the ratio $\ell_{22}/\ell_{21} = 2$. This analysis also shows that the Quota method with minimum requirements does not belong to Still's class. However, Still's generalized lower quota concept has no intuitive appeal, so we believe his class is not the appropriate one to consider. Instead, we generalize the description of quota, house monotone methods given in Section 5 for our lower quota definition.

Table 1. Still's lower quotas.

State	$\underset{\sim}{p}$	$\underset{\sim}{r}$	$\underset{\sim}{\ell}(20)$	$\underset{\sim}{\ell}(21)$	$\underset{\sim}{\ell}(22)$	$\underset{\sim}{\ell}(23)$	$\underset{\sim}{\ell}(24)$	$\underset{\sim}{\ell}(25)$	$\underset{\sim}{\ell}(26)$	$\underset{\sim}{\ell}(27)$
1	192	1	1	1	1	1	1	1	1	1
2	193	1	1	1	1	1	1	1	1	1
⋮										
20	211	1	1	1	1	1	1	1	1	1
21	1995	0	0	0	0	0	0	0	0	1
22	3975	0	0	1	2	3	4	5	5	6
	10,000	20	20	21	22	23	24	25	25	27

7. GENERALIZED QUOTATONE METHODS

Given minimum requirements $(r_1,\ldots,r_s) = \underset{\sim}{r}$, let $h^* = \sum_i r_i$, and for any $h \geq h^*$ let $\ell_i(h)$, $u_i(h)$ be the generalized lower and upper quotas for state i, and $J(h)$ the set of "slack" states. Then an apportionment $\underset{\sim}{a}$ for h *satisfies (generalized) quota* if and only if for each i, $u_i(h) \geq a_i \geq \ell_i(h)$, and a solution $\underset{\sim}{f}$ *satisfies quota* if all its apportionments do. We note that, for $i \notin J(h)$, $a_i \geq \ell_i(h)$ is equivalent to

$$a_i + 1 > p_i(h - h^* + \sum_{J(h)} r_j) \Big/ \sum_{J(h)} p_j \quad ,$$

or

(7) $$p_i/(a_i+1) < \sum_{J(h)} p_j \Big/ ((h - h^*) + \sum_{J(h)} r_j) \quad .$$

For any given $h \geq 0$ and any apportionment $\underset{\sim}{a}$ for h consider the Jefferson deck $\underset{\sim}{D}(\underset{\sim}{a},h)$ remaining after all cards $(i,a_i',p_i/a_i')$, $0 \leq a_i' \leq a_i$, are removed. Define $\alpha = \alpha(\underset{\sim}{a},h)$ to be the *first* integer $\alpha \geq 1$ such that there are at least α distinct cards in $\underset{\sim}{D}(\underset{\sim}{a},h)$ having value $\geq \sum_{J(h+\alpha)} p_j/((h - h^*) + \sum_{J(h+\alpha)} r_j + \alpha)$, and let $L(\underset{\sim}{a},h)$ be the set of all state names appearing on the first α cards. If no such α exists then set $\alpha = \infty$ and let $L(\underset{\sim}{a},h)$ be the set of all states (see below).

Suppose that $\underset{\sim}{f}(\underset{\sim}{p},h) = \underset{\sim}{a}$ is some apportionment at h, and the $(h+1)^{st}$ seat is given to some state $k \notin L(\underset{\sim}{a},h)$. Then $\alpha < \infty$, and in constructing a house monotone extension of $\underset{\sim}{f}^h$ exactly α cards must be removed in going from $h+1$ to $h+\alpha$. By choice of α there remains at $h+\alpha$ some card $(j,b_j,p_j/b_j)$ with value

$$p_j/b_j \geq \sum_{J(h+\alpha)} p_j \Big/ ((h - h^*) + \sum_{J(h+\alpha)} r_j + \alpha) \quad ,$$

and since state j has fewer than b_j seats at $h+\alpha$, it follows from (7) that lower quota at $h+\alpha$ is violated. Hence if $\underset{\sim}{f}$ is a house monotone apportionment solution satisfying quota (for the given requirements $\underset{\sim}{r}$), then we must have

(i) $\underset{\sim}{f}(h^*) = \underset{\sim}{r}$;

and

(ii) if $\underset{\sim}{f}(h) = \underset{\sim}{a}$ and $f_k(h+1) = a_k + 1$,

then

$$k \in L(\underset{\sim}{a},h) \cap U(\underset{\sim}{a},h) \quad .$$

At this point we note that, in the definition of $\alpha = \alpha(\underset{\sim}{a},h)$, it is unneccesary to inspect values of α larger than those which assure that the cards $(i,a_i+1,p_i/(a_i+1))$ have value

$p_i/(a_i+1) \geq \sum_{J(h+1)} p_j \Big/ ((h-h^*) + \sum_{J(h+1)} r_j + \alpha)$ for each i, $1 \leq i \leq s$, since for any $\alpha \geq 1$, $J(h+1) \subseteq J(h+\alpha)$, and by (4)

$$\sum_{J(h+1)} p_i \Big/ ((h-h^*) + \sum_{J(h+1)} r_j + \alpha) \geq \sum_{J(h+\alpha)} p_i \Big/ ((h-h^*) + \sum_{J(h+\alpha)} r_j + \alpha) \quad .$$

Hence if we define

$$\beta = \max_i \left\lceil \{(a_i + 1) \sum_{J(h+1)} p_j\}/p_i - \sum_{J(h+1)} r_j - (h - h^*) \right\rceil \quad ,$$

then $L(\underset{\sim}{a},h)$ may be defined as above with the modification: if there is no first integer α in the range $\beta > \alpha \geq 1$ satisfying the condition, let $L(\underset{\sim}{a},h)$ be the set of all states.

Let $\bar{Q}(\underset{\sim}{r})$ be the class of all solutions $\underset{\sim}{f}$ satisfying (i) and (ii).

Theorem 3. $\bar{Q}(\underset{\sim}{r})$ is precisely the set of all quotatone solutions for the requirements $\underset{\sim}{r}$.

The proof parallels that of Theorem 1.

8. CONCLUDING REMARKS

Two fundamental properties of apportionment methods are dictated by common sense and firmly grounded in the history of the problem: house monotonicity and satisfying quota.

Following the idea of Still that the class of all methods having these two properties are in some sense describable, we have shown that in fact they may all be described by using the Jefferson deck and choosing a card "near the top" that satisfies upper quota. If minimum requirements are given, we have shown that the quota idea has a natural generalization, and that the class of all house monotone (generalized) quota methods is again describable in a natural way in terms of the Jefferson deck. By contrast, the approach to generalized quotas proposed by Still, and the methods corresponding to them, were shown to lead to unnatural results.

Nevertheless, since there is a multiplicity of apportionment methods which satisfy the above two properties, the problem remains: which among these methods should be used? Here a third principle comes into play, which has its basis in the pioneering work on apportionment methods by E.V. Huntington in the early part of this century [5] and touches on the idea of what is meant intuitively by "method". Briefly stated, if for some problem and $\underset{\sim}{M}$-solution $\underset{\sim}{f}$ one state, having population $\bar{p}$ and $\bar{a}$ seats at house h, gets the "next" (i.e. $(h+1)^{st}$) seat before another state having population p* and a* seats, then the first state has *priority* over the second state, written $(\bar{p},\bar{a}) \gtrsim (p^*,a^*)$. If in another problem we also have $(p^*,a^*) \gtrsim (\bar{p},\bar{a})$, then we say the states are *tied*, written $(p^*,a^*) \sim (\bar{p},\bar{a})$. The method $\underset{\sim}{M}$ is said to be *consistent* if it treats tied states equally with respect to receiving one more seat; that is, there must be an alternate $\underset{\sim}{M}$-solution $\underset{\sim}{f}'$ which is an extension of $\underset{\sim}{f}^h$ and gives the $(h+1)^{st}$ seat instead to the p*-state. The essence of the idea is that a "method" should not *change* priorities between a pair of states if the data of some other state populations are altered.

The five methods proposed by Huntington, as well as their generalizations [2], all have this property. Moreover it may be shown that every house monotone, consistent method is necessarily a Huntington method, that is, uses a "rank index" r(p,a) that tells which state (with population p and number of seats a) most deserves to receive one more seat. Specifically,

(i) $\underset{\sim}{f}(0) = 0$,

and

(ii) if $\underset{\sim}{f}(h) = \underset{\sim}{a}$ and k is some one state maximizing $r(p_k,a_k)$, then

$$f_k(h+1) = a_k + 1 \quad , \quad f_i(h+1) = a_i \quad \text{all } i \neq k \quad .$$

The desirable features of these methods are: first, they are eminently computable, and second, they are based on the natural idea of comparing the states *pairwise* to determine which is worst off, hence most deserving of an extra seat. On the other hand, none of the Huntington methods satisfies quota [4].

Given the precedent -- in the political context -- of the two principles, house monotonicity and satisfying quota, it is natural to ask whether there is some modification of the consistency concept that leads to a computationally simple method. Indeed there is. Let consistency be modified to apply between pairs of states only when both states *are eligible* (i.e. both are in $U(\underset{\sim}{a},h)$). Then the Quota method is the *unique* method that is quota, house monotone, and consistent in this weaker sense [4]. (Note that with the present definition of generalized upper quotas, the restriction in [4] to unbiased requirements is unnecessary.) Moreover it is clear from the preceding that the Quota method $\underset{\sim}{Q}$ is the computationally simplest and most natural within the class $\underset{\sim}{\overline{Q}}$. If the concept is weakened still further to apply only between pairs of states that are both eligible *and* among the first $\alpha(\underset{\sim}{a},h)$ states, that is in $L(\underset{\sim}{a},h) \cap U(\underset{\sim}{a},h)$, then we may expect that this property, together with house monotonicity and satisfying quota, determines precisely the class of methods defined as follows.

Let $r(p,a)$ be a rank index and let $\underset{\sim}{r}$ be a given set of minimum requirements, $h^* = \sum_i r_i$. Define the *quotatone* method $\underset{\sim}{M}$ *based on* $r(p,a)$ to be the set of all solutions $\underset{\sim}{f}$ obtained as follows. For any $\underset{\sim}{r}$,

(i) $\underset{\sim}{f}(h^*) = \underset{\sim}{r}$;

(ii) if $\underset{\sim}{f}(h) = \underset{\sim}{a}$ and k is some one state that maximizes $r(p_k,a_k)$ over all $k \in L(\underset{\sim}{a},h) \cap U(\underset{\sim}{a},h)$ then let

$$f_k(h+1) = a_k + 1 \quad , \qquad f_i(h+1) = a_i \quad \text{all } i \neq k .$$

Thus, for example, $\underset{\sim}{Q}$ is the quotatone method basen on $p/(a+1)$.

Among the class of all such methods $\underset{\sim}{Q}$ is the simplest and most natural, since it does not depend on the computation of $L(\underset{\sim}{a},h)$, which in general is complex. Furthermore, although computers make possible the calculation of quotatone apportionments for any rank-index, it is nevertheless of paramount importance that political men both understand and feel comfortable with any method that is used. It may be that the set $L(\underset{\sim}{a},h)$ is simply beyond political understanding.

There are several criteria which are clearly of primary importance in choosing an apportionment method: satisfying quota, house monotonicity, consistency, and "simplicity." The desiderata cannot be met simultaneously. The question is to find a satisfactory reconciliation. Consistency and house monotonicity determine Huntington methods, which are simple but do not satisfy quota. A slightly weakened consistency notion together with satisfying quota gives the Quota method, which has an intuitive simplicity. A considerably weakened consistency idea leads to quotatone methods based on some $r(p,a)$ which, it seems, have mathematical appeal but lack simplicity.

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