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X AND Y OPERATORS FOR GENERAL LINEAR TRANSPORT PROCESSES

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1. Introduction

A central problem of atmospheric physics is the determination of radiation fields under various circumstances. Of special importance are the intensities of solar radiation transmitted through the earth's atmosphere and the amount of radiation reflected out the top. It can be shown that, under the hypothesis that the atmosphere may be regarded as a plane-parallel slab, most of the quantities of physical interest may be calculated in terms of the reflected, transmitted, and internal intensities.

The basic equation describing the intensities is the so-called "transport" equation, which is a linear two-point boundary value problem. Beginning with the work of Ambartsumian [1] and continued by Chandrasekhar [4], Sobolev [7], Bellman and Kalaba, et al. [2], new "imbedding" type initial value equations have been developed for calculating the basic quantities. Of special note in this regard is the contribution of Chandrasekhar who showed that, under special circumstances, the basic operator-Riccati equation associated with the computation could be replaced by two vector functions, now called the Chandrasekhar X-Y functions. This observation not only shed new light on the structure of the physical process, but also resulted in a significant reduction in the computing burden necessary to obtain the relevant quantities.

The purpose of this report is to detail the most general situation in which the X-Y-type of reduction may be expected to occur and to give the appropriate equations. A generalization of the usual algebraic formula relating the X-Y functions to the reflection function is also given, together with a treatment of the case where the atmosphere may be semi-infinite in extent. These results may prove useful to several IIASA studies, notably the climatology work of the Energy project and the associated work in the Ecology group.

2. Problem Statement

We consider the plane parallel slab $\Pi(a,r)$, $r > a$, having boundaries $z = a$ and $z = r$. The distribution of radiation in the direction of increasing and decreasing z is represented by $I^{\pm}(z)$, respectively. These quantities take into account frequency, degree of polarization, direction, and so forth. Thus, $I^{\pm}(z)$ take on values in a reproducing cone K of non-negative functions in a suitable separable Banach space B.

To each sub-slab $\Pi(z,z')$, $(z,z') \subset (a,r)$, there are associated reflection operators $R^{\pm}(z,z')$ and transmission operators $Q^{\pm}(z,z')$, which assume values from the Banach algebra $\mathscr{B}$ of bounded linear operators acting in B. The signs of $\pm$ refer to illumination of the sub-slab from the left and right, respectively (see Figure 1).

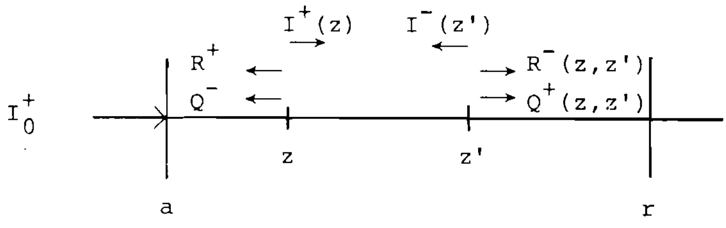

Figure 1. Plane-Parallel Slab.

In the medium, we assume $||Q^\pm + R^\pm|| \leq 1$ (no fission) and $Q^\pm(z,z') \longrightarrow I$, $R^\pm(z,z') \longrightarrow 0$ for $z' \longrightarrow z + 0$. We also assume the existence of the limits

$$T^\pm(z) \equiv \lim_{z' \to z + 0} \frac{I - Q^\pm(z,z')}{z - z'} \quad , \tag{1}$$

$$Z^\pm(z) \equiv \lim_{z' \to z + 0} \frac{R^\pm(z,z')}{z' - z} \quad .$$

In general, $T^\pm$, $Z^\pm$ are non-negative operators. For an homogeneous medium, $T^\pm$, $Z^\pm$ are independent of z, while for a locally isotropic medium $T^+ = T^-$ and $Z^+ = Z^-$.

On the medium $\Pi(a,r)$, let the flow I_0^+ be incident from the left. Then consideration of the regimes on the boundaries of the sub-slab $\Pi(z,z')$ shows that $I^\pm(z)$ satisfy the equations [5]:

$$\pm \frac{dI^\pm}{dz} + -A^\pm I^\pm + Z^\pm\left(I^+(z) + I^-(z)\right) \quad , \tag{2}$$

$$I^+(z) = I_0^+ \quad , \qquad I^-(r) = 0 \quad ,$$

where $A^\pm(z) = T^\pm(z) + Z^\pm(z)$.

In concrete transfer problems, the operators $A^{\pm}(z)$, $Z^{\pm}(z)$ are known and we are interested in methods for determining $R^{\pm}$ and $Q^{\pm}$.

3. Reflection, Transmission, and X-Y Operators

Consideration of Figure 1 shows that for $z' = r$, we have

$$I^{-}(z) = R(z)I^{+}(z) \quad , \qquad R(z) \equiv R^{+}(z,r) \quad . \tag{3}$$

Substitution of (3) into (2) leads to the Cauchy problem for the operator R:

$$\frac{-dR}{dz} = Z^{-}(z) - T^{-}(z)R - RT^{+}(z) + RZ^{+}(z)R \quad , \tag{4}$$

$$R(r) = 0 \quad .$$

Knowledge of R(z) allows us to simultaneously solve a family of different problems with different values of a. We determine $I^{+}(z)$ from the Cauchy problem

$$\frac{dI^{+}}{dz} = (Z^{+}R - T^{+})I^{+} \quad , \qquad I^{+}(a) = I_0^{+} \quad , \tag{5}$$

while $I^{-}(z)$ is determined from (3).

Since the pioneering work of Chandrasekhar [4] and Ambartsumian [1], it is well known that, in some cases, the solutions to the operator Riccati equation (4) may be expressed by an algebraic combination of lower-dimensional operators, the so-called X and Y operators. Our main result shows when this may be expected.

Theorem 1. Assume the medium is homogeneous, i.e. $T^{\pm}$, $Z^{\pm}$ are independent of z. Further, assume

i) dim range $Z^{-} = p < \infty$

ii) dim range $Z^{+} = q < \infty$

and that $Z^{\pm}$ are factored as $Z^{-} = MN$, $Z^{+} = UV$, where dim range $N = p =$ dim domain M, dim range $V = q =$ dim domain V. Then R admits the algebraic representation

$$T^{-}R(z) + RT^{+}(z) = Z^{-} + X_1(z)X_2(z) - Y_1(z)Y_2(z) \quad ,$$

where Y_1, Y_2, X_1, X_2 satisfy the equations

$$\frac{dY_1(z)}{dz} = \left(T^{-} - X_1(z)V\right)Y_1 \quad , \qquad Y_1(r) = -M \quad ,$$

$$\frac{dY_2(z)}{dz} = Y_2\left(T^{+} - UX_2(z)\right) \quad , \qquad Y_2(r) = N \quad ,$$

$$\frac{dX_1(z)}{dz} = -L_1L_2U \quad , \qquad X_1(r) = 0 \quad ,$$

$$\frac{dX_2(z)}{dz} = -VL_1L_2 \quad , \qquad X_2(r) = 0 \quad .$$

Proof: We follow the proof of [3] which was given for a special case of Eq.(4). Differentiate Eq.(4) with respect to z. This yields the following homogeneous equation for the operator $\frac{dR}{dz}$:

$$\frac{d}{dz}\left(\frac{dR}{dz}\right) = (T^{-} - RZ^{+})\frac{dR}{dz} + \frac{dR}{dz}(T^{+} - Z^{+}R) \quad ,$$

$$\left.\frac{dR}{dz}\right|_{z=r} = -Z^{-} = -MN \quad .$$

Making the definitions $X_1(z) = RU$, $X_2(z) = VR$, and using the representation

$$\frac{dR}{dz} = -\alpha MN\beta \quad ,$$

where

$$\frac{d\alpha}{dz} = (T^{-} - X_1 V)\alpha \quad , \qquad \alpha(r) = I \quad ,$$

$$\frac{d\beta}{dz} = \beta(T^{+} - UX_2) \quad , \qquad \beta(r) = I \quad .$$

The theorem follows with $Y_1 = \alpha M$, $Y_2 = N\beta$.

Remarks:

i) For an isotropically scattering medium, $Z^{+} = Z^{-}$ and $T^{+} = T^{-}$, with $T^{\pm}$ being self-adjoint. Thus, $Y_1 = Y_2{}^{*}$, $X_1 = X_2{}^{*}$, and the usual situation of a single X and a single Y operator is recovered;

ii) For slabs with a reflecting surface at z = r, the Riccati equation (4) has a non-zero initial condition at z = r, say R(r) = F. If F is independent of z, the foregoing arguments carry through, replacing assumption i) of the Theorem by i') dim range $-(Z^{-} - T^{-}F - FT^{+} + FZ^{+}F) < p \quad \infty$. For a specific application of this case to an atmosphere bounded by a Lambert law reflector, see [6].

iii) the finiteness of p and q is not essential. All that is required is that Z^{+} and Z^{-} project into lower dimensional subspaces of B. However, for computational considerations, the finite case is the most appropriate.

4. Semi-Infinite Media

We now treat the case of a semi-infinite media. In order to derive an equation for the operators $X_1(-\infty)$, $X_2(-\infty)$, we utilize the following lemma:

Lemma 1. Let P,A,Q be bounded linear operators of B to B. Then

$$\sigma(PAQ) = (Q^*\otimes P)\sigma(A) \quad , \tag{6}$$

where σ: $L(B,B) \longrightarrow \mathbb{C}^{(\dim B)^2}$ is the operator of "stacking" the "columns" of an element of L(B,B), and $\otimes$ is the usual tensor product of two operators.

Proof. Using the separability of B, the proof follows by a coordinate-wise comparison of the left and right sides of (6).

The result which generalizes the Chandrasekhar H-equation for the semi-infinite medium is

Theorem 2. Let $X_1(-\infty) = H_1$, $X_2(-\infty) = H_2$. Then H_1 and H_2 satisfy the equations

$$\sigma(H_1) = \left(U^*\otimes I\right)\left(I\otimes T^- + (T^+)^*\otimes I\right)^{-1}\sigma\left(Z^- + H_1H_2\right) \quad ,$$

$$\sigma(H_2) = \left(I\otimes V\right)\left(I\otimes T^- + (T^+)^*\otimes I\right)^{-1}\sigma\left(Z^- + H_1H_2\right) \quad .$$

Proof. From the Riccati equation (4), we have

$$T^-R(-\infty) + R(-\infty)T^+ = Z^- + H_1H_2 \quad .$$

Applying σ to both sides of this equation and using the results

$$\sigma(H_1) = \sigma(RU) = (U^*\otimes I)\sigma(R) \quad ,$$

$$\sigma(H_2) = \sigma(VR) = (I \otimes V)\sigma(R) \quad ,$$

the theorem easily follows.

<u>Remarks</u>: (i) Theorem 2 assumes that $\lambda_i + \mu_j \neq 0$, where $\{\lambda_i\}$ are the characteristic roots of T^- and $\{\mu_j\}$ are the roots of $(T^+)^*$; (ii) in both Theorems 1 and 2, considerable simplification occurs if Z^- and Z^+ are self-adjoint, while $T^+ = T^{-*}$ since in this case $X_1 = X_2^*$, $Y_1 = Y_2^*$, and $H_1 = H_2^*$. This is the situation which prevails in the classical plane-parallel, isotropic scattering, homogeneous case.

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