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ON HUNTINGTON METHODS OF APPORTIONMENT

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PREFACE

The problem of how to make a *"fair division"* of resources among competing interests arises in many areas of application at IIASA. One of the tasks in the System and Decision Sciences Area is the systematic investigation of different criteria of fairness and the formulation of allocation procedures satisfying them.

A particular problem of fair division having wide application in governmental decision-making is the *apportionment* problem. An application has recently arisen in the debate over how many seats in the European Parliament to allocate to the different member countries. Discussions swirled around particular numbers, over which agreement was difficult to achieve. A systematic approach that seeks to formulate principles or criteria of fair division should stand a better chance of acceptance in that it represents a scientific or system analytic approach to the problem.

ABSTRACT

A (generalized) *Huntington method* for apportioning representatives among states, or seats among parties, is one which distributes seats one by one by using a rank index that determines how deserving a state, or party, is to receive the next available seat. A characterization of these methods is given by two basic properties: consistency and house monotonicity.

The arguments used to establish this result are combinatorial in nature and use classical theorems concerning partial orders and their representation by a real-valued function.

On Huntington Methods of Apportionment

INTRODUCTION AND BRIEF HISTORY

The *apportionment problem* is the problem of determining how to divide a given integer number of representatives or delegates proportionally among given constituencies according to their respective sizes. The problem arises in deciding how to distribute a given number of delegates in a legislature among the component *states* of a country and also in determining how to divide a given number of candidates among the various political *parties* receiving votes in an election. In the latter guise this is the *proportional representation problem.*

In the United States the apportionment problem has a long and interesting history stemming from the Constitutional mandate, "Representatives and direct taxes shall be apportioned among the several States ... according to their respective numbers" (Article I, Section 2). This stipulation led to an early consideration of various methods by which apportionments might be computed. Jefferson, Hamilton, and Webster all actually proposed methods, and many important political figures in United States history concerned themselves with the apportionment problem at regular ten-year intervals following each census, thus testifying both to its political importance and its mathematical nontriviality. (For an historical account of the problem in the United States see [4,14].) In Europe, the question of apportionment *methods* does not seem to have been debated until the second half of the nineteenth century, and then in the context of proportional representation (see, e.g., [12]).

Formally, the apportionment problem may be stated as follows. Let $\underset{\sim}{p} = (p_1, p_2, \ldots, p_s)$ be the populations of s *states*, where each $p_i > 0$ is integer, and let $h \geqq 0$ be the number of seats in the *house* to be distributed. The problem is to find, for any $\underset{\sim}{p}$ and all house sizes $h \geqq 0$, an *apportionment* for h: an s-tuple of non-negative integers $\underset{\sim}{a} = (a_1, \ldots, a_s)$ whose sum is h. A *solution* of the apportionment problem is a function $\underset{\sim}{f}$ which to every $\underset{\sim}{p}$ and h associates a unique apportionment for h, $a_i = f_i(\underset{\sim}{p}, h) \geqq 0$ where $\sum_i a_i = h$. A specific apportionment method may give several different solutions, for "ties" may occur when using it -- for example when two states have identical populations and must share an odd number of seats. It is useful, for this reason, to define an *apportionment method* $\underset{\sim}{M}$ as a non-empty set of solutions. Two different apportionment solutions $\underset{\sim}{f}$ and $\underset{\sim}{g}$ of a method $\underset{\sim}{M}$ may be identical up to some house h and then branch, depending on how a particular tie is resolved. The restriction of $\underset{\sim}{f}$ to the domain $(\underset{\sim}{p}, h')$, $0 \leqq h' \leqq h$, will be called a *solution up to* h, $\underset{\sim}{f}^h$, and $\underset{\sim}{f}$ will be called an *extension* of $\underset{\sim}{f}^h$.

As early as 1792 Thomas Jefferson [10], then Secretary of State, pointed to the advantages of using a method of apportionment after each census, as opposed to relying on *ad hoc* procedures which are susceptible of endless political argument and manipulation. Moreover, he proposed a general and important method known today as Jefferson's method ($\underset{\sim}{J}$) [4]. This method, later rediscovered by the Belgian mathematician Victor d'Hondt, has been widely used for the proportional representation problem in Europe [12]. The United States apportionments based on the censuses of

1790 through 1830 were Jefferson's.

In 1792 Alexander Hamilton, then Secretary of the Treasury, proposed the following method [7]. Given the populations $(p_1, p_2, \ldots, p_s)$ and h, first compute the *exact quota* for each state i, $p_i h/(\Sigma_j p_j) = q_i$, and consider the fractional remainders $d_i = q_i - \lfloor q_i \rfloor$ (where $\lfloor x \rfloor$ represents the largest integer less than or equal to x) arranged in descending order, say $d_{i_1} \geqq d_{i_2} \geqq \cdots \geqq d_{i_s}$. Then *Hamilton's method* is to first give each state i $\lfloor q_i \rfloor$ seats, and if d_i is among the first $d = \sum_i d_i$ terms of the above list then it is given one more, or $\lfloor q_i \rfloor + 1$ seats. This method was proposed again after the 1850 census by Representative Samuel F. Vinton of Ohio, and was used (subject to politically motivated amendments) for the censuses of 1850 through 1900 under the name "Vinton's Method of 1850."

A serious difficulty with this method came to light in 1881 when C.W. Seaton, the Chief Clerk of the United States Census Office, discovered that, whereas the Hamilton method, in apportioning 299 seats among the states, gave Alabama 8, it gave her only 7 in a house of 300 seats. This phenomenon (which is no isolated quirk of the Hamilton method but in fact occurs frequently) was dubbed the *Alabama paradox*, and was immediately recognized as a critical flaw in the Hamilton method.

Beginning early in this century attention was therefore focused on developing methods that do *not* admit the Alabama paradox, that is, methods that are *house monotone* in the sense that $\underset{\sim}{f}(\underset{\sim}{p}, h+1) \geqq \underset{\sim}{f}(\underset{\sim}{p}, h)$ for every $\underset{\sim}{p}$ and h. W.F. Willcox [17] generalized an earlier proposal of Webster [16] to obtain a house monotone method, known alternately as the method of major fractions or *Webster's method* ($\underset{\sim}{W}$), which was used in 1911. This method was proposed independently by Sainte-Lagüe in 1910 [13] and has been used in proportional representation systems in Europe. Beginning at about this time E.V. Huntington [9], Professor of Mathematics at Harvard, undertook a formal investigation of house monotone methods.

From a computational point of view Huntington's approach may be summarized as follows. Let r(p,a) be any real-valued function of two variables, called a *rank index*. Then a house monotone apportionment method $\underset{\sim}{M}$ is obtained by taking all apportionment solutions $\underset{\sim}{f}$ defined recursively as follows:

(i) $f_i(\underset{\sim}{p},0) = 0 \quad , \qquad 1 \leq i \leq s \quad ,$

(ii) if $a_i = f_i(p_i,h)$ is an $\underset{\sim}{M}$-apportionment for h, and k is some one state for which $r(p_k,a_k) \geqq r(p_i,a_i)$ for $1 \leqq i \leqq s$, then

$$f_k(\underset{\sim}{p},h+1) = a_k + 1, \text{ and } f_i(\underset{\sim}{p},h+1) = a_i \text{ for } i \neq k.$$

The method obtained in this way will be called the *Huntington method based on* r(p,a), and as a class such methods will be called *Huntington methods* (see [4]). It is obvious that all Huntington methods are house monotone. But Huntington himself only considered five particular choices of ranking function -- these are listed in Table 1. As an example of a Huntington method Table 2 gives the Webster allocations ($r(p,a) = p/(a+\frac{1}{2})$) for a house ranging from 5 to 17 seats. That the five methods discussed by Huntington are, in fact, all different is shown in Table 3 by the apportionments obtained for a house of 36 seats for the same six-state example as that of Table 2.

Table 1. The five methods of Huntington.

Method	Rank Index	Test of Inequality ($p_i/a_i \geq p_j/a_j$)
Smallest Divisors (SD)	p/a	$a_j - a_i(p_j/p_i)$
Harmonic Mean (HM)	$p/\{2a(a+1)/(2a+1)\}$	$p_i/a_i - p_j/a_j$
Equal Proportions (EP)	$p/\{a(a+1)\}^{1/2}$	$p_i a_j/p_j a_i - 1$
Webster (W) (also known as Major Fractions and Sainte-Laguë)	$p/(a+1/2)$	$a_j/p_j - a_i/p_i$
Jefferson (J) (also known as Greatest Divisors or d'Hondt)	$p/(a+1)$	$a_j(p_i/p_j) - a_i$

Table 2. Sample Webster apportionments.

	A	B	C	D	E	F
State Population	27,744	25,178	19,947	14,614	9,225	3,292
House Size						
⋮						
5	2	1	1	1	0	0
6	2	1	1	1	1	0
7	2	2	1	1	1	0
8	2	2	2	1	1	0
9	3	2	2	1	1	0
10	3	3	2	1	1	0
11	3	3	2	2	1	0
12	3	3	3	2	1	0
13	4	3	3	2	1	0
14	4	4	3	2	1	0
15	4	4	3	2	1	1
16	5	4	3	2	1	1
17	5	4	3	2	2	1
⋮						
36	10	9	8	5	3	1

Table 3.

Party	Votes Received	Exact Quota	SD	HM	EP	W	J
			Apportionment for 36				
A	27,744	9.988	10	10	10	10	11
B	25,178	9.064	9	9	9	9	9
C	19,947	7.181	7	7	7	8	7
D	14,614	5.261	5	5	6	5	5
E	9,225	3.321	3	4	3	3	3
F	3,292	1.185	2	1	1	1	1
	100,000	36.000	36	36	36	36	36

Huntington derived these five particular methods from certain binary-comparison "tests of inequality." Given an apportionment $\underset{\sim}{a} = (a_1, a_2, \ldots, a_s)$ for h and populations $\underset{\sim}{p} = (p_1, p_2, \ldots, p_s)$, consider any pair of states i, j and the numbers p_i/a_i and p_j/a_j, which represent the average district sizes in states i and j respectively. Huntington then argued: "Now in a perfect apportionment, these two numbers would be exactly equal ... hence, in any practical case, ... if [the] inequality can be reduced by a transfer of a representative from one state to the other then ... the transfer should be made.... The question then comes down to this: what shall be meant by the inequality between these two numbers?" [9, p. 86]. Huntington then goes on to consider the *absolute difference*, $|p_i/a_i - p_j/a_j|$, versus the *relative difference*,

$$\frac{|p_i/a_i - p_j/a_j|}{\min\{p_i/a_i, p_j/a_j\}} .$$

Assume that i and j are chosen so that $p_i/a_i \geqq p_j/a_j$; then the relative difference is $p_i a_j / p_j a_i - 1$. Suppose the relative difference is chosen as the "right" measure of inequality. Then it is easily shown that $\underset{\sim}{a} = (a_1, \ldots, a_s)$ is an apportionment such that *no* transfer can be made between two states that reduces the amount of inequality if and only if, for all i and j,

$$p_j/\{a_j(a_j-1)\}^{\frac{1}{2}} \geqq p_i/\{a_i(a_i+1)\}^{\frac{1}{2}} ,$$

which holds if and only if $(a_1, a_2, \ldots, a_s)$ is obtained as a Huntington method solution with $r(p,a) = p/\{a(a+1)\}^{\frac{1}{2}}$, that is, EP [4]. Similarly, the test $p_i/a_i - p_j/a_j$ leads to the Harmonic Mean method. On the other hand, one could just as well begin by comparing the numbers a_j/p_j and a_i/p_i, or a_j and $a_i(p_j/p_i)$ or $a_j(p_i/p_j)$ and a_i, whose differences lead to W, SD, and J respectively, and whose relative differences all result in EP [9]. It is interesting to note in this context that Huntington's approach to J was quite different from Jefferson's; moreover Huntington was apparently not aware of Jefferson's proposal.

Huntington's goal was to show that EP is the best of the five methods, because it is based on what he felt was the most natural measure of difference -- namely, the relative difference. In this he was supported by two select committees which reported to the President of the National Academy of Sciences, one in 1929 [5] and one in 1948 [11]. These reports both argued for EP because, of the "now known" methods which are "unambiguous" and house monotone, EP satisfies a test that seems to be preferable to others and yields apportionments that are "neutral ... with respect to emphasis on larger and smaller states" [5]. The existence of house monotone methods based on rank indices other than Huntington's five had apparently escaped observation.

THE TWO BASIC PROPERTIES

By his tests of inequality Huntington restricted the field to five particular methods, but did not convincingly single out any one method as unequivocally "best." Here we ask, what are the essential properties that distinguish the class of Huntington methods from all others? The answer is surprisingly simple.

The first basic property of Huntington methods -- house monotonicity -- has already been mentioned: it was, indeed, the fundamental motivation for these methods. But the Huntington methods are not the only house monotone methods -- for example the Quota Method is a house monotone method that is not a Huntington method [1,4].

A further consideration of house monotonicity reveals a second basic property that we call in this context *consistency*. If $\underset{\sim}{M}$ is any house monotone method, and $\underset{\sim}{f}$ is a solution of $\underset{\sim}{M}$, then for any given populations $\underset{\sim}{p}$ the operation of $\underset{\sim}{f}$ can be fully described by specifying, for each h, which state gets the "next" (i.e., $(h+1)^{st}$) seat. For in going from $\underset{\sim}{f}(\underset{\sim}{p},h)$ to $\underset{\sim}{f}(\underset{\sim}{p},h+1)$, exactly one state must get one more seat while all the others stay the same. Why does some state i, having population p_i and current apportionment $a_i = f_i(\underset{\sim}{p},h)$, get the $(h+1)^{st}$ seat instead of some other state j with population p_j and apportionment $a_j = f_j(\underset{\sim}{p},h)$? Evidently because state i "deserves" it more than j. In comparing the relative claims to an extra seat between any *two* states i and j, the only relevant data should be their *populations* p_i and p_j, and their *current* numbers of seats a_i and a_j. That is, $\underset{\sim}{M}$ defines a partial relation $\underset{\sim}{\geq}$ on the set X of pairs of integers (p,a), $p > 0$, $a \geqq 0$, as follows:

(1) $(p,a) \succsim (q,b)$ if and only if there is some $\underset{\sim}{p}$, h and some i, j such that $p_i = p$, $p_j = q$, $f_i(\underset{\sim}{p},h) = a$, $f_j(\underset{\sim}{p},h) = b$ and $f_i(\underset{\sim}{p},h+1) = a + 1$, $f_j(\underset{\sim}{p},h+1) = b$.

In this case we say (p,a) has *weak priority* over (q,b). It should be noted that if $(p,a) \succsim (q,b)$ by some $\underset{\sim}{M}$ then this implies there is a problem with populations $\underset{\sim}{p} = (\ldots,p,\ldots,q,\ldots)$ and some h at which $\underset{\sim}{M}$ gives a seats to the state with population p and b seats to the state with population q. If $(p,a) \succsim (q,b)$ and not $(q,b) \succsim (p,a)$ we write $(p,a) \succ (q,b)$, whereas if $(p,a) \succsim (q,b)$ and $(q,b) \succsim (p,a)$ we write $(p,a) \sim (q,b)$ and say (p,a) and (q,b) are *tied*.

It is natural, from the context of apportionment itself, to require that the relation $\succsim$ satisfy:

(2) if (p,a) and (q,b) are tied, then $\underset{\sim}{M}$ should be "indifferent" between them; that is, whenever for some $\underset{\sim}{p}$ and h, $f_i(\underset{\sim}{p},h) = a$, $f_j(\underset{\sim}{p},h) = b$, $p_i = p$ and $p_j = q$, if $\underset{\sim}{f}$ gives the $(h+1)^{st}$ seat to state i then there should be an alternate solution $g \in M$ that is identical with $\underset{\sim}{f}$ up to h (i.e. $\underset{\sim}{g}^h = \underset{\sim}{f}^h$), but that gives the $(h+1)^{st}$ seat instead to state j.

Any method $\underset{\sim}{M}$ having property (2) will be said to be *consistent*. Basically, consistency means that if $(p,a) \sim (q,b)$, then any two states with populations p and q and apportionments a and b are equally deserving *in terms of the operation of the method* $\underset{\sim}{M}$.

Of course, there are other very natural properties that we might wish the priority relation to satisfy, e.g., transitivity. Remarkably enough, however, it turns out that something sufficiently close to transitivity -- namely, acyclicity -- is implied by the two conditions of house monotonicity and consistency. Indeed, these two properties precisely characterize the class of Huntington methods.

THE CHARACTERIZATION

Theorem. An apportionment method $\underset{\sim}{M}$ *is house monotone and consistent if and only if it is a Huntington method.*

The proof of this theorem needs two key lemmas concerning the relation $\gtrsim$. The first, which contains the meat of the argument, is established in the next section.

Lemma 1. Let $\gtrsim$ be the priority relation of a house monotone and consistent apportionment method $\underset{\sim}{M}$. If $(p_1,a_1) \gtrsim \ldots \gtrsim (p_k,a_k)$ then *not* $(p_k,a_k) \succ (p_1,a_1)$.

Recall that if π is any binary relation on some set S, then the *transitive closure* of π, π^t, is defined by $(x,y) \in \pi^t$ if and only if $(x,x^1),(x^1,x^2),\ldots,(x^m,y)$ are all in π for some sequence $x^1,x^2,\ldots,x^m \in S$. Clearly π^t is always transitive; and π^t is irreflexive and transitive if π is irreflexive and acyclic.

Let $\approx$ be the transitive closure of the relation $\sim$ in Lemma 1; then $\approx$ is symmetric and transitive. Define $(p,a) \approx (p,a)$ for all pairs $(p,a) \in X$, so that $\approx$ is an equivalence relation. Let $\bar{X} = X/\approx$ be the quotient set of X by $\approx$. Now define the binary relation μ on $\overline{X} \times \overline{X}$ by $(y,z) \in \mu$ if and only if $(p,a) \succ (q,b)$ for some $(p,a) \in y$ and some $(q,b) \in z$.

We claim that μ is acyclic. If not, then there is a sequence $y^1, y^2, \ldots, y^k \in \overline{X}$, $k \geqq 2$, such that $(y^1,y^2) \in \mu$, $(y^2,y^3) \in \mu, \ldots,$ $(y^k,y^1) \in \mu$. Hence there are equivalence class representatives $(p^i,a^i) \in y^i$ and $(q^i,b^i) \in y^i$ such that

(3) $\quad (p^1,a^1) \succ (q^2,b^2), (p^2,a^2) \succ (q^3,b^3), \ldots, (p^k,a^k) \succ (q^1,b^1)$.

$(p^i,a^i) \approx (q^i,b^i)$ for each i, $1 \leqq i \leqq k$, so that either $(p^i,a^i) = (q^i,b^i)$ or else there is a chain in X such that $(p^i,a^i) = (p^i_1,a^i_1) \sim \cdots \sim (p^i_n,a^i_n) = (q^i,b^i)$. From these and (3) we immediately derive a chain that contradicts Lemma 1. Hence μ is acyclic, and in particular asymmetric and irreflexive.

Let μ^t be the transitive closure of μ; μ^t is then a strict partial order on $\overline{X}$. We now need

Lemma 2. If π is a strict partial order on a countable set S, then there exists a real-valued, order-preserving function ϕ: $S \to \mathbb{R}$; that is, $(x,y) \in \pi$ if and only if $\phi(x) > \phi(y)$.

Proof. First we show that π is contained in a complete order π^* on S. Let $x^1, x^2, \ldots$ be a correspondence of S with the positive integers, and let Z be the set of all ordered pairs (x^i,x^j), where $i < j$ and $x^i, x^j \in S$. Z is also countable. Let $z^1, z^2, \ldots$ be a correspondence between Z and the positive integers. Let $(x^i,x^j) = z^\alpha$ be the *first* in this sequence such that neither $(x^i,x^j) \in \pi$ nor $(x^j,x^i) \in \pi$. (If there is no such z^α, π itself is complete.) Since π is transitive, $\pi \cup \{z^\alpha\}$ is acyclic, hence $\pi^1 = (\pi \cup \{z^\alpha\})^t$ is a partial order containing π. Beginning with π^1, construct π^2, and so forth. Then $\bigcup_{i=1}^{\infty} \pi^i = \pi^*$ is a complete order containing π.

By induction on k define, for each $k \geqq 1$, a function ϕ^k: $\{x^1, x^2, \ldots, x^k\} \to \mathbb{R}$ such that $\phi^k(x^i) > \phi^k(x^j)$ if and only if $(x^i, x^j) \in \pi^*$, and such that $\phi^{k+1}(x^i) = \phi^k(x^i)$ when $1 \leq i \leq k$. The union of all these ϕ^k's is the desired $\phi : S \to \mathbb{R}$. □

That π is contained in a complete order is a special case of a result known as Szpilrajn's Theorem (which Szpilrajn attributes to Banach, Kuratowski and Tarski [15]). The existence of a real representation of a complete order on a countable set is a special case of a result of Debreu [6].

The proof of the theorem is now completed as follows. Since the set X of all pairs (p,a) is countable, $\overline{X} = X/\approx$ is also countable. Let $\phi : \overline{X} \to \mathbb{R}$ be an order-preserving function relative to μ^t as guaranteed by Lemma 2. From ϕ we then define $r : X \to \mathbb{R}$ such that $r(p,a) = \phi(y)$ if and only if y is the equivalence class of $\approx$ containing (p,a).

We claim that $\underset{\sim}{M}$, the house monotone, consistent method of Lemma 1, *is* the Huntington method based on r(p,a). Indeed let $\underset{\sim}{f} \in \underset{\sim}{M}$. For any $\underset{\sim}{p}$ suppose there is a first h such that $\underset{\sim}{f}(\underset{\sim}{p},h) = (a_1, a_2, \ldots, a_s)$ is a Huntington apportionment for h (based on r) but $\underset{\sim}{f}(\underset{\sim}{p},h+1)$ is not. Then there must be distinct states i and j such that $r(p_i,a_i) > r(p_j,a_j)$ but $\underset{\sim}{f}$ gives state j the $(h+1)^{st}$ seat. Thus $(p_j,a_j) \gtrsim (p_i,a_i)$; but by the definition of r, we would have $r(p_j,a_j) \geq r(p_i,a_i)$, a contradiction. Thus every $\underset{\sim}{M}$-solution is also a Huntington solution. The converse is established similarly.

This completes the proof of the theorem and leaves only the proof of Lemma 1. □

ESTABLISHING ACYCLICITY

In effect, given a sequence $(p_1,a_1) \succsim (p_2,a_2) \succsim \ldots \succsim (p_k,a_k)$ what we would like to do is construct a solution $\underset{\sim}{f}$ such that for $1 \leq i \leq k$, $f_i((p_1,\ldots,p_k),h) = a_i$ where $h = \Sigma_i a_i$. This turns out to be technically quite involved. A particular stumbling block is that $\succsim$ is only a *partial* relation, so that not every two pairs (p,a) and (q,b) are comparable.

Let us say that a sequence $S = ((p_1,a_1),(p_2,a_2),\ldots,(p_k,a_k))$ is *constructible*, written $C((p_1,a_1),(p_2,a_2),\ldots,(p_k,a_k))$, if there exists $\underset{\sim}{f} \in \underset{\sim}{M}$ such that, for some $\underset{\sim}{q} = (q_1,\ldots,q_s)$, $s \geq k$, satisfying $q_{i_1} = p_1, q_{i_2} = p_2, \ldots, q_{i_k} = p_k$ and some h, we have $f_{i_j}(\underset{\sim}{q},h) = a_j$ for $1 \leq j \leq k$. It follows that if S is constructible, then it is constructible for the population vector $(p_1,p_2,\ldots,p_k)$ since consistency permits one to imitate the solution for $\underset{\sim}{q}$ restricted to these populations. Also, if it is known that $(p,a) \succsim (q,b)$ by some $\underset{\sim}{M}$ then, of course, we have $C((p,a),(q,b))$.

Suppose that $C((p,a),(q,b))$, and let $\underset{\sim}{f} \in \underset{\sim}{M}$, $\underset{\sim}{p}$ and h be such that $p_i = p$, $f_i(\underset{\sim}{p},h) = a$ and $p_j = q$, $f_j(\underset{\sim}{p},h) = b$. Consider the sequence of pairs $(f_i(\underset{\sim}{p},0),f_j(\underset{\sim}{p},0)),(f_i,(\underset{\sim}{p},1),(f_j(\underset{\sim}{p},1)),\ldots,(f_i(\underset{\sim}{p},h),f_j(\underset{\sim}{p},h))$ -- that is, the record of how states i and j went from zero seats each to an apportionment of a and b respectively. After eliminating *redundant* elements from this sequence we obtain the *history* H(a,b) *for* p , q. Evidently, since $\underset{\sim}{f}$ is house monotone, any element $(x_1,x_2) \in H(a,b)$ satisfies $0 \leq x_1 \leq a$, $0 \leq x_2 \leq b$, and if $(x_1,x_2) \neq (a,b)$ the successor of (x_1,x_2) is either (x_1+1,x_2) or (x_1,x_2+1). Note that if (x_1+1,x_2) follows (x_1,x_2) then $(p,x_1) \succsim (q,x_2)$ and if (x_1,x_2+1) follows (x_1,x_2) then $(q,x_2) \succsim (p,x_1)$. We represent H(a,b) by a tableau of form:

p	0 ...	x_1	...	a	:	H(a,b) .
q	0 ...	x_2	...	b		
	0 ...	x_1+x_2	...	a+b		

Any sequence $S = ((p_{i_1}, a_{i_1}), \ldots, (p_{i_k}, a_{i_k}))$ such that $(p_{i_1}, a_{i_1}) \gtrsim (p_{i_2}, a_{i_2}) \gtrsim \ldots \gtrsim (p_{i_k}, a_{i_k}) \gtrsim (p_{i_1}, a_{i_1})$ is called a *cycle*, and S is a *strict* cycle if at least one of the relations $\gtrsim$ is satisfied as $\succ$.

The proof of Lemma 1 now proceeds by several sublemmas. Throughout, the relation $\gtrsim$ is that given to us by the method $\underset{\sim}{M}$.

Lemma 1a. No strict cycle is constructible.

Proof. Suppose that $(p_1,a_1) \gtrsim (p_2,a_2) \gtrsim \ldots \gtrsim (p_k,a_k) \gtrsim (p_1,a_1)$ is a strict cycle and constructible. Then for some $\underset{\sim}{f} \in \underset{\sim}{M}$ we would have $a_i = f_i(p_1,\ldots,p_k,h)$ where $h = \Sigma_i a_i$. Let i be such that $f_i((p_1,\ldots,p_k),h+1) = a_i + 1$. Then $(p_i,a_i) \gtrsim (p_{i-1},a_{i-1})$ whereas by assumption $(p_{i-1},a_{i-1}) \gtrsim (p_i,a_i)$, hence $(p_i,a_i) \sim (p_{i-1},a_{i-1})$ (if i = 1 let i - 1 always mean k). Therefore, by consistency there exists an extension $\underset{\sim}{g}$ of $\underset{\sim}{f}^h$ such that $\underset{\sim}{g}$ gives the $(h+1)^{st}$ seat to state i - 1. Continuing in this manner we establish that $(p_i,a_i) \sim (p_{i-1},a_{i-1})$, $1 \leq i \leq k$. But this contradicts the assumption that for some i, $(p_{i-1},a_{i-1}) \succ (p_i,a_i)$. Hence S is not constructible.

Lemma 1b. If $(p,a) \gtrsim (q,b)$ then C((q,b),(q,b)).

Proof. Since $(p,a) \gtrsim (q,b)$ we must have C((p,a),(q,b)).

Let H(a,b) be a particular history for p , q. Define $\underset{\sim}{p}$ = (p,q,q). Consider the *largest* house $h \leqq a + b + b = h^0$ for which there exists an $\underset{\sim}{M}$-apportionment $\underset{\sim}{x} = (x_1,x_2,x_3)$ satisfying

(4) $x_1 \leqq a$, $x_2 \leqq b$, $x_3 \leqq b$,

(5) $(x_1,x_2) \in H(a,b)$ and $(x_1,x_3) \in H(a,b)$.

p	0	...	x_1	...
q	0	...	x_2	...
q	0	...	x_3	...
	0	...	$h = x_1+x_2+x_3$	...

.

Without loss of generality take $x_3 \geqq x_2$.

Case 1. $x_3 > x_2$. Then $H(a,b)$ has form

p	0 ...	x_1	...	x_1	...	a
q	0 ...	x_2	...	x_3	...	b

: $H(a,b)$.

In particular $(x_1,x_2+1) \in H(a,b)$ so

(6) $(q,x_2) \gtrsim (p,x_1)$.

If also $(q,x_2) \gtrsim (q,x_3)$, then there exists an apportionment for h+1

p	0	...	x_1	x_1	...
q	0	...	x_2	x_2+1	...
q	0	...	x_3	x_3	...
			h	h+1	

and since $x_2 < x_3 \leqq b$, (4) and (5) are satisfied for the larger house h+1, a contradiction.

Otherwise $(q,x_3) \succ (q,x_2)$, so by Lemma 1a

(7) $(q,x_3) \succ (p,x_1)$.

Case 1a. If also $x_3 < b$, then $(x_1,x_3+1) \in H(a_1,a_2)$ and

p	0 ...	x_1	x_1
q	0 ...	x_2	x_2
q	0 ...	x_3	x_3+1
		h	h+1

is an $\underset{\sim}{M}$-apportionment for h+1 satisfying (4) and (5), a contradiction.

Hence $x_3 = b$. We cannot also have $x_1 < a$, because then the history H(a,b) would imply that $(x_1+1,b) \in H(a,b)$, so that $(p,x_1) \gtrsim (q,x_3) = (q,b)$ contrary to (7).

Case 1b. $x_3 = b$ and $x_1 = a$. Then we have $(q,x_2) \gtrsim (p,a)$ by (6) and $(p,a) \gtrsim (q,b)$ by the hypothesis of the lemma, so

p	0 ...	a	a
q	0 ...	x_2	x_2+1
q	0 ...	b·	b
		h	h+1

is an apportionment for h+1 satisfying (4) and (5), a contradiction.

Case 2. $x_3 = x_2$. If $x_1 < a$ and $x_2 = x_3 < b$, then the successor of (x_1,x_2) in the history H(a,b) determines whether state 1 or state 2 gets the $(h+1)^{st}$ seat and in either case (4) and (5) are satisfied, a contradiction. If $x_1 < a$ and $x_2 = x_3 = b$ then $(p,x_1) \gtrsim (q,b)$ by the history and (x_1+1,b,b) is an $\underset{\sim}{M}$-apportionment for h+1 satisfying (4) and (5), again a contradiction. Finally, if $x_1 = a$ and $x_2 = x_3 < b$ then (a,x_2+1,x_2) is an $\underset{\sim}{M}$-apportionment for h+1 satisfying (4) and (5), which is a contradiction once again.

It follows that we must have $h = a+b+b$, proving the lemma. □

Lemma 1c. If $C((q,b),(q,b))$ then for any b', $0 \leq b' \leq b$, there exists a sequence $b' = b_0 \leq b_1 \leq \ldots \leq b_k = b$ such that $C((q,b_{i-1}),(q,b_i))$ for $1 \leqq i \leqq k$ and $(q,b_0) \gtrsim (q,b_1) \gtrsim \ldots \gtrsim (q,b_k)$.

Proof. The proof is by induction on $b - b'$, the result for $b = b'$ being trivial. Let $H(b,b)$ be any history for q, q. Then there exists a pair $(x,y) \in H(b,b)$ such that $x = b'$ or $y = b'$. Choose any such pair (x,y) with $x + y$ maximum. Say without loss of generality that $x = b'$. Then by choice of (x,y), $y > b'$ and $(q,b') \gtrsim (q,y)$. Set $b_1 = y$. If $b_1 = b$ we are done; otherwise $b_1 < b$ and we argue as with b' to find a $b_2 > b_1$ such that $(q,b_1) \gtrsim (q,b_2)$ and so forth. This completes the proof of Lemma 1c. □

For any sequence S of pairs $(p_{i_1},a_{i_1}),(p_{i_2},a_{i_2}),\ldots,(p_{i_k},a_{i_k})$ define b_S to be the maximum of the integers a_{i_j}, $1 \leqq j \leqq k$, and define n_S to be the number of a_{i_j} such that $a_{i_j} = b_S$. We say that sequence S *precedes* T, written $S << T$ if either $b_S < b_T$ or $b_S = b_T$ and $n_S < n_T$.

Clearly any sequence other than a trivial one of form $S = ((p,0))$ has a predecessor.

Suppose, contrary to Lemma 1, that $(p_1,a_1) \gtrsim \ldots \gtrsim (p_k,a_k) \gtrsim (p_1,a_1)$ is a strict cycle S. By Lemma 1a S is not constructible, hence in particular $b_S > 0$. We may therefore assume inductively that S is the "first" strict cycle; i.e. that $T << S$ for no strict cycle T. Also, we may assume (by relabelling if necessary) that $a_2 = b_S$. We shall now derive a contradiction.

For each i, $1 \leq i \leq k-1$, the fact that $(p_i,a_i) \gtrsim (p_{i+1},a_{i+1})$ implies $C((p_i,a_i),(p_{i+1},a_{i+1}))$; hence for each such i choose a history $H(a_i,a_{i+1})$ for p_i, p_{i+1}. Letting $\underset{\sim}{p} = (p_1,p_2,\ldots,p_k)$ and

$h^o = \Sigma_i a_i$, consider the *largest* $h' \leqq h^o$ for which there exists $\underset{\sim}{f} \in \underset{\sim}{M}$ satisfying

$$(8) \qquad (f_i(\underset{\sim}{p},h), f_{i+1}(\underset{\sim}{p},h)) \in H(a_i, a_{i+1}) \text{ for all } i, \quad 1 \leqq i \leqq k-1 \text{ and all } h \leqq h' \; .$$

Clearly h' exists and $h' \geqq 0$. Moreover, if $h' = h^o$ then S is constructible, a contradiction. Therefore, $h' < h^o$. Let $x_i = f_i(\underset{\sim}{p},h')$ for each i; in particular, by (8) we have $x_i \leqq a_i$. Let V be the set of all pairs (p_i,x_i), $1 \leqq i \leqq k$. Any two elements of V are comparable relative to $\succsim$ because V has actually been constructed; for the same reason there are no strict cycles in V. Second, define $E = \{(p_i,x_i) \in V : x_i = a_i\}$.

We now construct a *partial* relation R on V as follows: for $1 \leqq i \leqq k-1$ if $(x_i,x_{i+1}) \lneqq (a_i,a_{i+1})$ and the successor of (x_i,x_{i+1}) in $H(a_i,a_{i+1})$ is (x_i+1,x_{i+1}) then $(p_i,x_i) \; R \; (p_{i+1},x_{i+1})$ whereas if the successor is $(x_i,x_{i+1}+1)$ then $(p_{i+1},x_{i+1}) \; R \; (p_i,x_i)$. These are all the relations in R. The significance of R is the following: if $(p_i,x_i) \in V-E$ is undominated relative to R then the successive pairs from the sequence $(x_1,x_2,\ldots,x_i+1,x_{i+1},\ldots,x_k)$ are again members of the histories $H(a_1,a_2),\ldots,H(a_{k-1},a_k)$.

Notice that vRw implies $v \succsim w$ for any $v,w \in V$. Further, comparable pairs under R form a forest (in fact, a forest in which no vertex has degree greater than two) on the vertex set V. Since, by definition, we never have $(p_i,x_i) \; R \; (p_j,x_j)$ for any $(p_i,x_i) \in E$ it follows that the set of R-undominated elements in $V-E$ is nonempty. Let $V^R = \{v \in V-E\text{: not } w R v \text{ for any } w \in V\} \neq \emptyset$.

Finally, let $v^* = (p_\ell,x_\ell)$ be a maximum element of V^R relative to $\succsim$. Notice that v^* cannot also be maximum in V relative to $\succsim$, for if it were then by consistency there would exist an $\underset{\sim}{M}$-

apportionment for h'+1 giving the $(h'+1)^{st}$ seat to state ℓ. Moreover, this would agree with the given histories, contradicting our assumption on h'.

We claim

(9) $\qquad (p_1,x_1),(p_2,x_2) \in E$.

For any $w_0 \in V - E$ there is a chain $w_n R w_{n-1} \ldots R w_0$ in $V-E$ such that $w_n \in V^R$, hence $v^* \gtrsim w_n \gtrsim \ldots \gtrsim w_0$. In particular, E cannot be empty, else v* would be maximum in V. Suppose (9) is false, and let $(p_i,x_i) = (p_i,a_i)$ be any element of E. First, if $i \neq 1$, let j be the largest index *less* than i such that $(p_j,x_j) \notin E$. (Such a j always exists by the assumption that (9) is false.) Then $(x_j,a_{j+1}) \in H(a_j,a_{j+1})$, hence $(p_j,x_j) \gtrsim (p_{j+1},a_{j+1})$. Moreover, by assumption on S, $(p_{j+1},a_{j+1}) \gtrsim (p_{j+2},a_{j+2}) \gtrsim \ldots \gtrsim (p_i,a_i)$. But $(p_j,x_j),(p_{j+1},a_{j+1}), \ldots ,(p_i,a_i)$ has been constructed, so by Lemma 1a it cannot be a strict cycle. Hence $(p_j,x_j) \succ (p_i,a_i)$. Second, if i = 1 then $(p_2,a_2) \notin E$ implies $(a_1,x_2) \in H(a_1,a_2)$ so $(p_2,x_2) \gtrsim (p_1,a_1)$. Since in the above argument (p_i,x_i) was arbitrary in E, it follows that for all $(p_i,a_i) \in E$ there exists $(p_j, x_j) \notin E$ such that $(p_j,x_j) \succ (p_i,a_i)$. But then v* would be maximum in V, a contradiction. Thus (9) is established.

Since $v^* = (p_\ell,x_\ell)$ cannot be maximum in V, but $v^* \succ w_0$ for all $w_0 \in V-E$, there must exist $w \in E$, say $w = (p_j,a_j)$, such that $w \succ v^*$. Suppose that $j > \ell$. Observe that since $(p_{\ell-1},a_{\ell-1}) \gtrsim (p_\ell,a_\ell)$, Lemma 1b tells us that $C(p_\ell,a_\ell;p_\ell,a_\ell)$. Hence, by Lemma 1c, there exists a sequence $x_\ell = a_\ell^1 \leq a_\ell^2 \leq \ldots \leq a_\ell^n = a_\ell$ such that $(p_\ell,a_\ell^1) \gtrsim (p_\ell,a_\ell^2) \gtrsim \ldots \gtrsim (p_\ell,a_\ell^n)$. Then $(p_j,a_j) \succ (p_\ell,a_\ell^1) \gtrsim (p_\ell,a_\ell^2) \gtrsim \ldots \gtrsim (p_\ell,a_\ell) \gtrsim (p_{\ell+1},a_{\ell+1}) \gtrsim \ldots \gtrsim (p_j,a_j)$ is a strict cycle T that does *not* include the pair (p_2,a_2), hence does not contain as many values b_s as did the original cycle S. Since a_2 was chosen to be the maximum value of a_i in S, T << S, contradicting the inductive hypothesis that no strict cycle precedes S.

Suppose, then, that $j < \ell$. Let t be the largest index less than j (if such exists) such that $x_t < a_t$. Then $(x_t, a_{t+1}) \in H(a_t, a_{t+1})$ since $x_{t+1} = a_{t+1}$ and so $(p_t, x_t) \gtrsim (p_{t+1}, a_{t+1}) \gtrsim \ldots \gtrsim (p_j, a_j) \succ (p_\ell, x_\ell)$. Since this sequence has been constructed we have $(p_t, x_t) \succ (p_\ell, x_\ell)$. But this implies $v^* = (p_\ell, x_\ell)$ is not a maximum in V - E, a contradiction. So $x_i = a_i$ for $1 \leqq i \leqq j$, and $(p_1, a_1) \gtrsim (p_2, a_2) \gtrsim \ldots \gtrsim (p_j, a_j) \succ (p_\ell, x_\ell)$. But this sequence has been constructed so $(p_1, a_1) \succ (p_\ell, x_\ell)$. Thus, as before, $(p_1, a_1) \succ (p_\ell, x_\ell) = (p_\ell, a_\ell^1) \gtrsim \ldots \gtrsim (p_\ell, a_\ell^n) = (p_\ell, a_\ell) \gtrsim (p_{\ell+1}, a_{\ell+1}) \gtrsim \ldots \gtrsim (p_k, a_k) \gtrsim (p_1, a_1)$ is a strict cycle T (not necessarily constructed) with T << S. This contradiction concludes the proof of Lemma 1 and hence of the theorem. □

FURTHER AXIOMATIC CHARACTERIZATIONS

This paper has shown how the five methods discussed by Huntington find their place in an axiomatic setting which uniquely characterizes the class of "generalized" Huntington methods by two basic properties: house monotonicity and consistency.

Particular Huntington methods may be uniquely characterized by various additional axioms. A method $\underset{\sim}{M}$ is said to be the *unique* one satisfying given properties if any other set $\underset{\sim}{M}'$ of solutions having these properties is a set of $\underset{\sim}{M}$-solutions, i.e. $\underset{\sim}{M}' \subseteq \underset{\sim}{M}$. One of the fundamental types of axioms *not* considered by Huntington is that an apportionment should not differ from the exact quotas by one whole integer or more. A method is said to *satisfy quota* if any apportionment $(a_1, a_2, \ldots, a_s)$ for $(p_1, p_2, \ldots, p_s)$ at house h has the property that $\lfloor q_i \rfloor \leqq a_i \leqq \lceil q_i \rceil$ where q_i is the exact quota of state i. A method is said to *satisfy upper quota* if $a_i \leqq \lceil q_i \rceil$ for all apportionments a_i and to *satisfy lower quota* if $a_i \geqq \lfloor q_i \rfloor$. It may then be shown that $\underset{\sim}{J}$ (Jefferson) is the unique house monotone, consistent method satisfying lower quota [3]. Also, $\underset{\sim}{SD}$ (Smallest Divisors) is the unique house monotone, consistent method satisfying upper quota [3]. Since $\underset{\sim}{SD}$ and $\underset{\sim}{J}$ are not the same method (e.g. see Table 2) it follows, in particular, that there is no house monotone, consistent method satisfying quota.

In view of the desirability of house monotonicity and satisfying quota as properties of an apportionment method, it is natural to ask whether there exists any method that obeys both properties. There is; moreover, if consistency is weakened to "consistency satisfying quota," then there exists a unique method, the Quota method, satisfying the three properties [1,4].

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