NORMATIVE MODELLING IN DEMO-ECONOMICS

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Abstract

This paper reviews recent attempts in demography and economics to design comprehensive dynamic demo-economic policy models. The policy models are formally stated as optimal control problems. Two groups of models are distinguished: Planning-oriented models, which originated in demography and which are designed to aid policymakers to solve practical problems; and theoretically-oriented models, which have been developed as part of the economic growth theory and which are intended to gain theoretical insights into demo-economic systems.

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1. Introduction

Social concern with population processes arises when the demographic acts of individuals affect the welfare of others and combine in ways that produce a sharp divergence between the sum of individual (private) preferences and the social well-being. In such instances, population processes properly become the subject of public debate and the object of public policy.

Population policy has a special feature that makes it a difficult research topic. Although a central element of any demographic policy is the size and the distribution of population, neither the goals nor the means of such a policy are purely demographic in nature. A population trend is viewed as being good or bad in the light of its presumed social and economic consequences. That is, a population policy is an instrument to achieve non-demographic goals. Davis (1971, p.7) described a population policy as a policy that tries to eliminate the demographic causes of the problems to be solved.

The importance of social and economic considerations in the formulation of population policies was stressed by the World Population Conference in Bucharest in 1974. The Conference strongly endorsed the view that demographic matters considered in isolation from economic and social factors, have little significance (Tabah, 1975, p.380). This is particularly true in the case of migration.

It is impossible to determine the goals and means of population distribution policies without considering general economic and social policies. Typically, human settlement programs are part of a regional economic policy, or of a land use policy, or of a physical planning program (for an illustrative review of population distribution policies in several developed countries, see Willekens, 1976a,pp.31-55).

The interdependence between economics and demography is also reflected in policy modelling efforts. Models of demo-economic growth and policy are receiving considerable attention in economics and in demography.

In most of modern economic growth theory, population is entered as an exogenous variable affecting economic growth through the labor supply, but itself being unaffected by changes in economic conditions. A few economists have endogenized population in their models by relating it to per capita income or a similar economic index. Although the treatment of population as an endogenous variable in economic growth models is of recent date, the notion itself has a long history. Classical economists such as Adam Smith, Thomas Malthus, J.S. Mill, and Ricardo all viewed population as being intimately dependent on the state of the economy.

Demographers, too, have only recently attempted to extend purely demographic models of population growth to include economic factors. And, again, the notion is not new. More than fifty years ago, Lotka (1925) already was stressing the importance of an interdisciplinary approach to the study of population. He did not consider population in isolation, but viewed it as part of a larger ecosystem. The demographic growth model he proposed, one which has become the basis of modern mathematical demography, was derived as part of a larger ecological study.

This paper reviews several recent policy models in which demographic and economic variables are endogenous. The models investigated are those with explicit policy objectives. To place the models in perspective, however, we first propose a classification of existing demo-economic models. Then, we introduce the basic mathematical apparatus common to all models treated in the remainder of the paper.

1.1 Demo-economic models: A classification

Because the number of demo-economic models is large and growing rapidly, a classification may be useful. Realizing that any classification is to some degree arbitrary, and that classifying items is done more for pragmatic than for scientific reasons, we adopt the following three-way classification scheme (for another classification, see McNicoll, 1975):

a. Models of demo-economic growth
b. Demo-economic simulation models
c. Models of demo-economic policy

a. Models of demo-economic growth:

The main purpose of these models is to describe or to explain demographic and/or economic growth by considering both demographic and economic variables in an interdependent way. Their fundamental feature is the simultaneous endogenous treatment of demographic and economic growth. We shall consider three types of such models:

i. Demometric growth models
ii. Neoclassical growth models
iii. Dualistic growth models

The first category contains empirically-oriented models that frequently are developed as part of an inductive investigation of demo-economic growth. They are closely related to econometric growth models. Greenwood's (1975) simultaneous equations model of urban growth and migration is an example of this class of models.

The second and third categories contain theoretically-oriented models developed as aids in the deductive analysis of demo-economic growth. Their purpose is not to predict real situations, but rather to gain theoretical insights into real processes.

Neoclassical growth models have received most attention in economic growth theory (see e.g. Burmeister and Dobell, 1970). The term "neoclassical" is used to describe supply-oriented models. The contrast is with Keynesian models, in which production and the use of resources are determined by aggregate demand rather than supply (McNicoll, 1975, p.649).

Models of the dual economy are best known in the economic development literature. Although not generally thought of as demo-economic growth models, they are included in this classification because of their use of migration as an adjustment mechanism assuring labor market equilibrium. Models of the dual economy have been developed by Lewis (1954), Ranis and Fei (1961) and Jorgenson (1961) and extended by Kelley, Williamson and Cheetham (1972) and Todaro (1969), among others.

b. Demo-economic simulation models

The distinction between simulation models and descriptive and explanatory growth models is fuzzy. Any demo-economic simulation model contains a growth model as its central and vital element. The quality of a simulation model varies directly with the quality of its growth model. The fundamental difference arises as a consequence of the aims of the two kinds of models. While the first group of models are intended to describe or to explain, simulation models are meant to demonstrate and compare impacts of alternative policies or of alternative trajectories of exogenous variables. Simulation models, therefore, are impact evaluation tools. In this sense, policy simulation models may be descriptive or explanatory models that are adapted to investigate the sensitivity of the system to be studied to changes in predefined instrument and exogenous variables.

Demo-economic simulation models usually are designed to explore the economic implications of alternative population trajectories and trajectories of demographic parameters, and very rarely do they try to describe the evolution of a complete demo-economic system under changing conditions. One of the earliest examples of models of this type is the Coale-Hoover (1958) model for India. It focused on the impact of fertility reduction and the consequent changes in the size and age distribution of the population on economic development, in casu employment (labor), investments (capital) and per capita income. This model has produced a number of progeny (Demeny, 1965; Enke et al., 1968).

Only a few attempts have been made to simulate in a truly integrated manner, the interdependencies in a demo-economic system. The first models of this sort are macro-economic models with population as an endogenous variable. Their principal purpose is not to represent the full complexity of the real situation, but rather to identify important insights about the demo-economic process. This class of macro-simulation models of demo-economic systems is illustrated by the Yap (1975) model for Brazil, designed to simulate the interaction between rural-urban migration and economic development, and by Kelley and Williamson's (1975) model of Meiji, Japan.

Finally, large-scale data-based models have been developed to simulate the evolution of demo-economic systems. The "Bachue" model of the International Labour Office, for example, considers a multisectoral economy and a disaggregated demographic system in a study of alternative employment generation strategies. Economic development depends on demographic change, and fertility, mortality and migration patterns are determined by the economy. For a critical review of some of the models, see Arthur and McNicoll (1976).

c. Models of demo-economic policy:

Demo-economic policy models strive to prescribe comprehensive demographic and economic policies. Policy objectives and policy instruments are stated explicitly. Objectives may be expressed in terms of a set of targets to be reached, in terms of an overall welfare index to be maximized, or as a combination of both. The policy models are dynamic in the sense that both instruments and objectives belong to different time periods. Formally, the dynamic policy problem is that of choosing time paths for certain variables called instrument or control variables from a given set of feasible time paths, so as to maximize a given objective or to achieve given targets (compare this formal statement with that of Intrilligator, 1971, p. 292). When presented in this form, the dynamic policy problem becomes an optimal control or dynamic optimization problem. Therefore, a convenient analytical framework for the study of quantative dynamic demo-economic policy is the theory of optimal control. In other words, the population policy problem may formally be stated as a problem of optimal control (see Arthur and McNicoll, 1972, p. 2; Willekens, 1976b, p.86). For the analogy in economic policy, see for example Chow (1973, 1975) and Pindyck (1973).

Within the formal framework of optimal control, two groups of demo-economic policy models may be distinguished:

i. Planning-oriented policy models.
ii. Theoretically-oriented policy models.

The models in the latter category are set up to gain theoretical insights into the characteristics of an optimal demo-economic system. The aggregation level of these policy models is usually high and the underlying growth model is generally of the neo-classical type. The first category contains models designed to aid policy-makers to solve practical problems. They are usually more disaggregated and imbed an empirically-oriented or demometric growth model. While most (i)-models have been developed by authors more directly interested in the planning of the growth and the distribution of the population, most (ii)-models originated in the economic growth theory.

In the remaining sections of the paper, we investigate some features of demo-economic policy models of the optimal control type. The first part is devoted to planning-oriented models. The second part reviews the theoretically-oriented models. First, however, we need to introduce some of the optimal control vocabulary.

1.2 The formal dynamic policy problem

The basic ingredients of an optimal control problem are (i) a state equation describing the dynamics or "laws of motion" of the system, (ii) a set of constraints on the state and control variables, (iii) a set of boundary conditions and (iv) a performance index or objective function (see for example Bryson and Ho, 1969).

(i) State equation

Let the vector $\{\underset{\sim}{x}(t)\}$ denote the state of the system at time t. The state vector may refer to the population distribution by age or region, or to economic stock variables such as capital. The control vector $\{\underset{\sim}{u}(t)\}$ contains the instruments or policy variables which may be controlled by the policy maker. The dynamics of the system are described by a set of differential or difference equations, the so-called state equations:

$$\{\dot{\underset{\sim}{x}}(t)\} = f[\{\underset{\sim}{x}(t)\}, \{\underset{\sim}{u}(t)\}, t] \quad (1)$$

or

$$\{\underset{\sim}{x}(t + 1)\} = h[\{\underset{\sim}{x}(t)\}, \{\underset{\sim}{u}(t)\}, t]. \quad (2)$$

In this paper, the state equations usually describe population growth and capital accumulation. In other words, the state variables are population and capital.

(ii) Constraints

The dynamics of the state and control variables may be constrained for economic, political or other reasons. Let the set of admissible state and control variables be defined by the vector-valued function $\{\underset{\sim}{g}(.)\}$:

$$\{\underset{\sim}{g}(\{\underset{\sim}{x}(t)\}, \{\underset{\sim}{u}(t)\}, t)\} \geq \{\underset{\sim}{0}\}. \tag{3}$$

(iii) Boundary conditions

The initial state is given:

$$\{\underset{\sim}{x}(0)\} = \{\underset{\sim}{x}_0\}. \tag{4}$$

Sometimes the values of the state variables must satisfy certain conditions at the planning horizon T. These are described by the vector valued function

$$\{\underset{\sim}{m}(\{\underset{\sim}{x}(T)\})\} = \{\underset{\sim}{0}\} \tag{5}$$

(iv) Performance index

The general formulation of the performance index to be optimized is

$$J = \int_0^T U(\{\underset{\sim}{x}(t)\}, \{\underset{\sim}{u}(t)\}, t)\ dt \tag{6}$$

for the continuous model, and

$$J = K(\{\underset{\sim}{x}(T)\}) + \sum_{i=0}^{T-1} L(\{\underset{\sim}{x}(t)\}, \{\underset{\sim}{u}(t)\}, t) \tag{7}$$

for the discrete model.

The dynamic policy or optimal control problem is then formulated as the determination of the control sequence $\{\underset{\sim}{u}^*(t)\}$ for $t = 0, \ldots T - 1$, and the corresponding trajectory of the state vector $\{\underset{\sim}{x}^*(t)\}$ for t = 0,...T, such that the systems dynamics (1) or (2), the constraint (3), and the boundary conditions (4) and (5) are satisfied and such that the performance index (6) or (7) is optimized. The sequence $\{\underset{\sim}{u}^*(t)\}$ is the optimal control and $\{\underset{\sim}{x}^*(t)\}$ is the optimal trajectory. In other words, the optimal control problem is to steer a dynamic system so as to optimize a performance index, subject to constraints. This formulation is a very general one and encompasses most dynamic population policy problems.

2. Planning-Oriented Policy Models

The models discussed in this section all have a common feature: they may be considered as logical extensions of demographic growth models to the policy domain. To demonstrate this, we will gradually build up policy models of greater degrees of complexity, starting with growth models that have been studied in mathematical demography.

Malthus can probably be credited with formulating the first model of population growth: "Population, when unchecked, increases at a geometrical ratio" (Malthus, 1798, p.13). Denoting the population size by N and the rate of population growth by n, Malthus' model may be represented by a first-order differential equation

$$\frac{dN(t)}{dt} = nN(t), \tag{8}$$

with the solution

$$N(t) = N_0 e^{nt} \quad . \tag{9}$$

The discrete form of Malthus' model is the difference equation

$$N(t + 1) = (\bar{n} + 1)\, N(t) = gN(t) \tag{10}$$

with the solution

$$N(t) = N_0 g^t \quad . \tag{11}$$

More recently, the aggregate model in (10) has been disaggregated to treat population growth by age (Leslie, 1945; Keyfitz, 1968), by region (Rogers, 1968) and by age and region (Rogers, 1975). The disaggregated model takes the form of a set of linear, first-order, homogenous difference equations, that has the simple expression:

$$\{\underset{\sim}{K}(t+1)\} = \underset{\sim}{G}\{\underset{\sim}{K}(t)\}, \tag{12}$$

where the elements of the vector $\{\underset{\sim}{K}(t)\}$ denote the number of people at time t by age group and/or by region, and $\underset{\sim}{G}$ is the associated population growth matrix.

The solution of (12) for an unchanging $\underset{\sim}{G}$ is, of course

$$\{K(t)\} = \underset{\sim}{G}^t\{\underset{\sim}{K}(0)\}. \tag{13}$$

In terms of the standard optimal control problem presented in the previous section, the growth model (12) constitutes the homogenous part of the state equation (1). The population distribution vector $\{\underset{\sim}{K}(t)\}$ is the state vector and the growth matrix $\underset{\sim}{G}$ is the transition matrix. The model describes the dynamics of an age- and/or region-specific population system that is undisturbed by exogenous forces (such as external migration) and that is free of any policy interventions.

The demographic growth model may be converted into a complete policy model in a number of steps. It is the purpose of this section to build up such a policy model and to provide a framework for comparing existing and potential planning-oriented population policy models.

The first step is to transform the growth model (12) into a complete state equation (1) by adding a sequence of vectors, describing control actions in time (and space). The simplest model (Rogers, 1966, 1968, 1971) is a purely demographic model, i.e. both the state and the control vector are in terms of demographic variables, such as fertility and migration. This model and its variants will be reviewed in the first part of this section. The second part extends this policy model to include economic control and state variables, and considers constraints and objective functions explicitly.

2.1 A matrix model of population control

Recall that the growth of a demographic system may be represented by the matrix equation (12). To investigate the effects of a birth or migration control policy on an interregional population system, one may introduce a constant intervention (contol) vector $\{\bar{m}\}$ which is added to the population in each time period as follows:

$$\{K(t + 1)\} = G\{K(t)\} + \{\bar{m}\}. \tag{14}$$

The vector $\{\bar{m}\}$ may have both positive and negative components. A positive $\bar{m}_i$ indicates the number of people that must be added to a region's population during each unit interval of time; a negative $\bar{m}_i$ denotes the population that has to be periodically withdrawn from region i. In analyses of alternative birth control policies, a negative $\bar{m}_i$ may be interpreted as the number of births that must be prevented from occurring during each unit interval of time.

Beginning with an initial population at some point in time, say, t = 0, we may trace through the effects of a particular policy control measure over time by repeatedly applying (14):

$$\{K(1)\} = G\{K(0)\} + \{\bar{m}\}$$

$$\{K(2)\} = G\{K(1)\} + \{\bar{m}\}$$

$$= G^2\{K(0)\} + G\{\bar{m}\} + \{\bar{m}\}$$

$$\vdots$$

$$\{K(t)\} = G^t\{K(0)\} + [\sum_{i=0}^{T-1} G^{T-1-i}] \{\bar{m}\} \tag{15}$$

and

$$\{K(t)\} - G^t\{K(0)\} = [\sum_{i=0}^{T-1} G^{t-1-i}] \{\bar{m}\}. \tag{16}$$

Premultiplying both sides of (16) by $\underset{\sim}{G}$, and subtracting the result from (16), gives, for a particular class of $\underset{\sim}{G}$ matrices,

$$(\underset{\sim}{I} - \underset{\sim}{G})\ [\{\underset{\sim}{K}(t)\} - \underset{\sim}{G}^t\{\underset{\sim}{K}(0)\}] = (\underset{\sim}{I} - \underset{\sim}{G}^t)\ \{\underset{\sim}{\bar{m}}\}\ . \qquad (17)$$

Therefore,

$$\{\underset{\sim}{K}(t)\} = \underset{\sim}{G}^t\{\underset{\sim}{K}(0)\} + (\underset{\sim}{I} - \underset{\sim}{G})^{-1}\ (\underset{\sim}{I} - \underset{\sim}{G}^t)\ \{\underset{\sim}{\bar{m}}\}\ . \qquad (18)$$

Note that (18) is the solution to equation (14).

Equation (18) may easily be transformed into a policy model under certain conditions. First, the goals of a population distribution policy must be expressed in terms of population targets at a planning horizon. Second, the matrix $(\underset{\sim}{I} - \underset{\sim}{G}^t)$ is nonsingular. The latter is the necessary and sufficient condition for <u>controllability</u> of the system described by (14) (Willekens, 1976b, Chapter 2). Assuming now a vector of target populations at the planning horizon T, $\{\underset{\sim}{\bar{K}}(T)\}$, the intervention vector which will assure that the targets are reached is easily computed:

$$\{\underset{\sim}{\bar{m}}\} = (\underset{\sim}{I} - \underset{\sim}{G}^T)^{-1}\ \left[(\underset{\sim}{I} - \underset{\sim}{G})\ [\{\underset{\sim}{\bar{K}}(T)\} - \underset{\sim}{G}^T\{\underset{\sim}{K}(0)\}]\right]. \qquad (19)$$

In applying the population control model (14) to empirical data, it is important to ensure that the interpretation of the intervention vector makes sense. For example, in a "pure" internal migration policy, the total population of the system remains constant. An inmigrant with respect to one region is an outmigrant with respect to another. The sum of inmigrants must equal the outmigrants, i.e.

$$\{\underset{\sim}{1}\}'\{\underset{\sim}{\bar{m}}\} = 0\ . \qquad (20)$$

As a consequence, the policy-maker cannot specify a target population for all regions.

The procedure to compute the control vector in the case of a pure migration model is described by Rogers (1971, p.106) as follows: the migration rates are taken out of the growth matrix and the migration flows are introduced via the control vector. The new growth matrix is $\underset{\sim}{S}$, say. After computing $\{\bar{\underset{\sim}{m}}\}$ by (19) with the growth matrix $\underset{\sim}{S}$ and a target vector $\{\bar{\underset{\sim}{K}}(T)\}$ some elements of $\{\bar{\underset{\sim}{m}}\}$ are adjusted such that the constraint (20) holds, and a revised target vector is calculated. The constraints placed on the control variables makes the system (14) uncontrollable, i.e. any target population cannot be reached. The problem, therefore, is to find a vector $\{\bar{\underset{\sim}{m}}\}$ which, given the equation (14) and the constraint equation (20), brings the population distribution at the horizon T as close to the target population $\{\bar{\underset{\sim}{K}}(T)\}$ as possible. This policy problem may be expressed as the following optimal control problem:

$$\text{min.} \quad \{\underset{\sim}{K}(T)\} - \{\bar{\underset{\sim}{K}}(T)\} \tag{21}$$

$$\text{s.t.} \quad \{\underset{\sim}{K}(t+1)\} = \underset{\sim}{S}\{\underset{\sim}{K}(t)\} + \{\bar{\underset{\sim}{m}}\}$$

$$\{\underset{\sim}{1}\}'\{\bar{\underset{\sim}{m}}\} = 0.$$

An interesting extension of the above policy model follows from the relaxation of the assumption of fixed policy vector. If the degree or level of a population policy may decline over time, then the vector $\{\bar{\underset{\sim}{m}}\}$ is added only at the beginning and $w\{\bar{\underset{\sim}{m}}\}$ is added during the next time period $(0<w<1)$, and i.e.

$$\{\underset{\sim}{m}(t+1)\} = w\{\underset{\sim}{m}(t)\} = w^{t+1}\{\bar{\underset{\sim}{m}}\}, \tag{22}$$

with w being a scalar. The control at a certain time period t is a constant fraction of the control in the previous time period. In other words, the value of the control vector is decreasing exponentially in time. The impact of an initial policy $\{\bar{\underset{\sim}{m}}\}$ on the population growth path is therefore

$$\{\underset{\sim}{K}(1)\} = \underset{\sim}{G}\{\underset{\sim}{K}(0)\} + \{\bar{\underset{\sim}{m}}\}$$

$$\{\underset{\sim}{K}(2)\} = \underset{\sim}{G}\{\underset{\sim}{K}(1)\} + w\{\bar{\underset{\sim}{m}}\} = \underset{\sim}{G}^2\{\underset{\sim}{K}(0)\} + \underset{\sim}{G}\{\bar{\underset{\sim}{m}}\} + w\{\bar{\underset{\sim}{m}}\}$$

$$\vdots$$

$$\{\underset{\sim}{K}(t)\} = \underset{\sim}{G}^t\{\underset{\sim}{K}(0)\} + \left[\sum_{i=0}^{T-1} w^{T-1-i}\underset{\sim}{G}^i\right]\{\bar{\underset{\sim}{m}}\} \tag{23}$$

and

$$\{\underset{\sim}{K}(t)\} - \underset{\sim}{G}^t\{\underset{\sim}{K}(0)\} = [\sum_{i=0}^{T-1} w^{T-1-i} \underset{\sim}{G}^i]\{\bar{\underset{\sim}{m}}\}. \tag{24}$$

Premultiplying both sides by $(w\underset{\sim}{I} - \underset{\sim}{G})$ yields

$$(w\underset{\sim}{I} - \underset{\sim}{G})[\{\underset{\sim}{K}(t)\} - \underset{\sim}{G}^t\{\underset{\sim}{K}(0)\}] = (w^t \underset{\sim}{I} - \underset{\sim}{G}^t)\{\bar{\underset{\sim}{m}}\},$$

whence

$$\{\underset{\sim}{K}(t)\} = \underset{\sim}{G}^t\{\underset{\sim}{K}(0)\} + (w\underset{\sim}{I} - \underset{\sim}{G})^{-1} (w^t \underset{\sim}{I} - \underset{\sim}{G}^t)\{\bar{\underset{\sim}{m}}\}. \tag{25}$$

Assuming a target vector $\{\bar{\underset{\sim}{K}}(T)\}$, the initial control vector that assures the achievements of the targets under this policy regime is

$$\{\bar{\underset{\sim}{m}}\} = [w^t \underset{\sim}{I} - \underset{\sim}{G}^t]^{-1} \left[(w\underset{\sim}{I} - \underset{\sim}{G})[\{\bar{\underset{\sim}{K}}(T)\} - \underset{\sim}{G}^t\{\underset{\sim}{K}(0)\}]\right]. \tag{26}$$

The policy model (14) may be extended a step further (Willekens, 1976b, pp. 69-71). Instead of assuming a constant relative decline in the value of the control vector, suppose that the control vector at each time period is a linear combination of the control vector at the previous time period, i.e.

$$\{\underset{\sim}{m}(t)\} = \underset{\sim}{W}\{\underset{\sim}{m}(t-1)\}, \tag{27}$$

and

$$\{\underset{\sim}{m}(t)\} = \underset{\sim}{W}^t\{\underset{\sim}{m}(0)\} = \underset{\sim}{W}^t\{\bar{\underset{\sim}{m}}\}, \tag{28}$$

where $\underset{\sim}{W}$ is assumed to be nonsingular. Introducing (28) into (14) and solving yields

$$\{\underset{\sim}{K}(t)\} = \underset{\sim}{G}^t\{\underset{\sim}{K}(0)\} + \left[\sum_{i=0}^{T-1} \underset{\sim}{G}^i \underset{\sim}{W}^{T-1-i}\right]\{\bar{\underset{\sim}{m}}\}. \tag{29}$$

The control vector at the initial time period yielding a target population distribution $\{\bar{K}(T)\}$ at horizon T may be computed easily.

Note that equations (18) and (25) are particular cases of (29). In the case of a constant control vector, the matrix W is an identity matrix. In the case of an exponentially declining control vector, on the other hand, W is a scalar matrix, i.e. $W = wI$.

The policy models discussed in this section have an interesting common feature. Since the matrices G and W are assumed to be time-invariant, the matrix sum in (29) only depends on the planning horizon T. Let

$$\sum_{i=0}^{T-1} G^{T-1-i} W^{i} = A(T),$$

then equation (29) may be written for t = T as

$$\{K(T)\} = G^{T}\{K(0)\} + A(T)\ \{\bar{m}\} \tag{30}$$

where $\{K(T)\}$ is the population distribution at time T,
$\{K(0)\}$ is the initial population distribution, and
$\{\bar{m}\}$ is the control vector in the initial time period.

Hence, although a population policy is implemented in each time period, the population trajectory is completely determined, once the control vector in the base year is fixed. The dynamic, multi-period population policy problem reduces therefore to a single-period problem. Policy models, where the control at t is a fixed linear combination of the control vector in the initial period, have been called <u>initial period control models</u> (Willekens, 1976, p.69). In the next section, we will drop the constraint on the control vector and introduce the possibility of intervening in population redistribution by applying economic policy instruments.

2.2 Elaboration of the matrix model of population control

The expansion of the above matrix model and its variants to a complete dynamic policy problem would involve (Willekens, 1976, Chapter 3);

i. Introducing economic control variables and the specification of the impact of these variables on the population distribution.

ii. Dropping the stringent constraints on the control vector, i.e., the extension of the initial period control problem to a truly dynamic control problem.

iii. Allowing for other constraints on both the state and the control variables, and for formulations of the policy objectives other than in terms of targets.

a. Introduction of non-demographic control variables

It was stressed in the introduction that a fundamental feature of population policy is that it does not occur in a vacuum. The ultimate goals of demographic intervention are non-demographic in nature, and the instruments are socio-economic in character. Policy models therefore must reflect this connection. The first link between population policy and socio-economic policy lies in the instruments. Policy-makers usually do not directly alter the volume of migration in order to mold a population distribution into a desired pattern. Rather, the intervention is indirect, through economic variables such as regional income, employment, housing, accessibility, government expenditures, and so on. Therefore, $\{\bar{\underset{\sim}{u}}\}$ is a vector of socio-economic control variables, and the impacts of the instruments on the population distribution in the next time period is given by vector function

$$\{\bar{\underset{\sim}{m}}\} = \{h(\{\bar{\underset{\sim}{u}}\})\}. \tag{31}$$

For the sake of simplicity, we will assume a linear relationship,

$$\{\bar{\underset{\sim}{m}}\} = \underset{\sim}{B}\{\bar{\underset{\sim}{u}}\} \text{ , say,} \tag{32}$$

where $\underset{\sim}{B}$ is a time-invariant coefficient matrix of dimension N x K, with N being the dimension of $\{\bar{\underset{\sim}{m}}\}$ and hence the dimension of the state vector (e.g. number of regions), and where K is the dimension of $\{\bar{\underset{\sim}{u}}\}$ or the number of instruments. An element b_{ij} denotes the impact of the j-th control variable on the i-th element of $\{\bar{\underset{\sim}{m}}\}$.

The ratio $-b_{ij}/b_{ik}$ is the amount by which the j-th instrument may be cut down without changing the level of the i-th element of $\{\bar{m}\}$, if the value of the k-th instrument is increased with one unit. It is, therefore, the marginal rate of substitution between the two instruments (Fromm and Taubman, 1968, p. 109). Introducing (32) into (14) gives

$$\{K(t+1) = G\{K(t)\} + B\{\bar{u}\}. \qquad (33)$$

Equation (33) relates the population distribution in a certain time period to the population distribution in the previous time period and to socio-economic policies. Since $\{\bar{u}\}$ may contain lagged policy variables, the direct effects of earlier policies may be included. If $\{\bar{u}\}$ has no lagged instruments, B coincides with what is known in economics as the matrix of impact multipliers or the matrix multiplier. The matrix multiplier plays a pivotal role in the study of the controllability of dynamic systems (Willekens, 1976b, Chapter 2, Aoki, 1976).

At this point, two remarks are in order:

i. If the population policy is a purely demographic policy, then (33) reduces to the basic matrix model of population control or its variants. In the basic model $\{\bar{m}\} = \{\bar{u}\}$ and the matrix multiplier is a diagonal matrix. In the intervention model with exponentially declining policies, B reduces to a scalar matrix. However, this matrix is no longer time-invariant.

ii. The policy problem represented by (33) is still an initial period control problem. It is closely related to the static policy model developed by Tinbergen (1963), in which $\{K(t+1)\} = \{y\}$ is the vector of target variables and $\{K(t)\} = \{z\}$ is the vector of uncontrollable exogenous variables:

$$\{y\} = G\{z\} + B\{\bar{u}\}. \qquad (34)$$

The Tinbergen policy model is therefore a special case of (33) in which there is only one time period. A solution to (34) exists if the rank of B is equal to the number of targets. The solution is unique if B is nonsingular, or, in the words of Tinbergen, if the number of instruments is equal to the number of targets. Then,

$$\{\bar{\underset{\sim}{u}}\} = \underset{\sim}{B}^{-1} [\{\underset{\sim}{y}\} - \underset{\sim}{G}\{\underset{\sim}{z}\}]. \tag{35}$$

b. The multi-period control problem

The policy models considered thus far are not really dynamic. Although there is a control vector for each time period, the trajectory of the controls is fixed such that the only freedom the policy-maker has is in choosing the instruments of the initial time period. Once the initial controls are chosen, future values of the controls, and hence of the state variables, follow automatically. In this section, the assumption of dependency of controls is dropped. The state equation (33) becomes

$$\{\underset{\sim}{K}(t+1)\} = \underset{\sim}{G}\{K(t)\} + \underset{\sim}{B}\{\underset{\sim}{u}(t)\} . \tag{36}$$

the solution to this truly dynamic policy model is

$$\{\underset{\sim}{K}(t)\} = \underset{\sim}{G}^{t}\{\underset{\sim}{K}(0)\} + \sum_{i=0}^{t-1} \underset{\sim}{G}^{(t-1-i)} \underset{\sim}{B}\{\underset{\sim}{u}(i)\}. \tag{37}$$

From the model (36) and its solution (37), two multi-period policy problems may be derived:

i. Horizon-oriented policy. The horizon-oriented policy problem may be formulated as follows: given the initial condition $\{\underset{\sim}{K}(0)\}$ and the assumption of time-invariance of the coefficient matrices, which sequence of control vectors $\{\underset{\sim}{u}(i)\}$ ensures that a target vector at a predefined horizon T will be reached?

ii. Trajectory-oriented policy. In a trajectory-oriented policy, the question is whether there exists a sequence of control vectors $\{\underset{\sim}{u}(i)\}$ such that, for a given initial condition and for time-invariant coefficient matrices, any sequence of target vectors $\{\bar{\underset{\sim}{K}}(t)\}$ can be realized.

In mathematical systems theory, the first policy problem is known as state controllability, (Wolovich, 1974). The second problem will be denoted as complete state controllability.

Both policy problems will now be treated in formal terms.

i. Horizon-oriented policy:

Equation (37) may be written for t = T as

$$\{\underset{\sim}{K}(T)\} - \underset{\sim}{G}^{T}\{\underset{\sim}{K}(0)\} = [\underset{\sim}{B} \vdots \underset{\sim}{G}\underset{\sim}{B} \vdots \cdots \vdots \underset{\sim}{G}^{T-1}\underset{\sim}{B}] \begin{bmatrix} \{\underset{\sim}{u}(T-1)\} \\ \vdots \\ \{\underset{\sim}{u}(1)\} \\ \{\underset{\sim}{u}(0)\} \end{bmatrix} \tag{38}$$

$$= \underset{\sim}{D}\{\hat{\underset{\sim}{u}}\} \quad , \quad \text{say} . \tag{39}$$

The system (38) is state controllable if the N x KT matrix $\underset{\sim}{D}$ is of rank N, where N is the dimension of the target vector $\{\bar{\underset{\sim}{K}}(T)\}$ (Wolovich, 1974, p. 65). If $\underset{\sim}{D}$ is nonsingular, there is a unique control sequence, which is given by

$$\{\hat{\underset{\sim}{u}}\} = \underset{\sim}{D}^{-1}[\{\bar{\underset{\sim}{K}}(T)\} - \underset{\sim}{G}^{T}\{\underset{\sim}{K}(0)\}] . \tag{40}$$

In the dynamic policy model (36) and (39), it is the combined magnitude of the number of instruments and the planning horizon that determines the state controllability. Any target vector may be reached by only one instrument (K = 1), provided that the planning horizon is not less than N and certain other conditions hold (Preston, 1974, p.70; Willekens, 1976b, p. 55). Also, any set of targets may be reached in only one time period, if the policy-maker can manipulate at least N instruments[1].

ii. Trajectory-oriented policy:

The policy problem discussed in the previous section dealt with the existence of a sequence of control vectors, necessary for

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[1]This is exactly the controllability condition derived by Tinbergen (1963) for a static policy model. For t=1, $\underset{\sim}{D}$ coincides with $\underset{\sim}{B}$.

the achievement of the desired target vector at a predefined planning horizon. In practice, policy-makers would be interested not only in achieving desired target values, but also in keeping them on some desired trajectory once achieved, or achieving the targets along a desired path. It is not uncommon in politics that short-term objectives conflict with long-term goals. Long-term goals may become unattainable because of short-term policies. Consequently, not only is the state at the planning horizon of interest, but also the trajectory. It is, therefore, relevant to consider the policy problem in which targets are formulated at each time period.

Writing (38) for each time period gives

$$\begin{bmatrix} \{\underset{\sim}{K}(T)\} - \underset{\sim}{G}^{T}\{\underset{\sim}{K}(0)\} \\ \{\underset{\sim}{K}(T-1)\} - \underset{\sim}{G}^{T-1}\{\underset{\sim}{K}(0)\} \\ \vdots \\ \{\underset{\sim}{K}(1)\} - \underset{\sim}{G}\{\underset{\sim}{K}(0)\} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{B} & \underset{\sim}{G}\underset{\sim}{B} & \underset{\sim}{G}^2\underset{\sim}{B} & \cdots & \underset{\sim}{G}^{T-1}\underset{\sim}{B} \\ \underset{\sim}{0} & \underset{\sim}{B} & \underset{\sim}{G}\underset{\sim}{B} & \cdots & \underset{\sim}{G}^{T-2}\underset{\sim}{B} \\ \vdots & & \vdots & \ddots & \vdots \\ \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} & \cdots & \underset{\sim}{B} \end{bmatrix} \begin{bmatrix} \{\underset{\sim}{u}(T-1)\} \\ \{\underset{\sim}{u}(T-2)\} \\ \vdots \\ \{\underset{\sim}{u}(0)\} \end{bmatrix} \quad (41)$$

$$\{\hat{\underset{\sim}{d}}\} = \underset{\sim}{F}\{\hat{\underset{\sim}{u}}\}, \text{ say.} \quad (42)$$

The system (41) is controllable if the NT x KT matrix $\underset{\sim}{F}$ is of rank NT (for $T<N$). If $\underset{\sim}{F}$ is nonsingular, then there exists a unique control sequence:

$$\{\hat{\underset{\sim}{u}}\} = \underset{\sim}{F}^{-1}\{\hat{\underset{\sim}{d}}\}. \quad (43)$$

Note that horizon-oriented policy problems form a special case of the trajectory-oriented policy problem. If in (41), $\{\underset{\sim}{K}(t)\}$ is not predefined for $t = 1\ldots T-1$, then it reduces to (38), i.e. the horizon-oriented policy problem. Therefore, complete state controllability implies state controllability.

The computation of the unique policy sequence is straightforward once the existence of such a policy is demonstrated. But what if $\underset{\sim}{F}$ (or $\underset{\sim}{D}$ or $\underset{\sim}{B}$) is singular? In this case, there may be an infinite number of control sequences which give the desired values of the state vectors. Or there may be no control sequence at all that reaches the targets. Two cases may be considered:

CASE 1 : NT < KT

If $\underset{\sim}{F}$ is rectangular and of rank NT, the number of controls exercised, KT, exceeds the number of targets specified, NT. Consequently, there are an infinite number of solutions to (42) and, therefore, an infinite number of control sequences that lead to the predefined targets. All the solutions to (42) may be expressed using the notion of a generalized inverse. If $\underset{\sim}{F}^{(1)}$ is a generalized inverse of $\underset{\sim}{F}$, satisfying

$$\underset{\sim}{F}\,\underset{\sim}{F}^{(1)}\,\underset{\sim}{F} = \underset{\sim}{F},$$

and if $\{\underset{\sim}{c}\}$ is an arbitrary vector of dimension KT, then the general solution to (42) is (Rogers, 1971, p.258)

$$\{\hat{\underset{\sim}{u}}\} = \underset{\sim}{F}^{(1)}\{\hat{\underset{\sim}{d}}\} + [\underset{\sim}{I} - \underset{\sim}{F}^{(1)}\underset{\sim}{F}]\ \{\underset{\sim}{c}\}. \qquad (44)$$

Out of the infinite number of feasible control sequences, the policy-maker must choose a single one. In order to get a unique solution to (42), the policy-maker may force the number of instruments exercised, KT, to be equal to the number of targets specified NT, by deleting some instrument variables at certain time periods. Instead of deleting instruments, he may constrain the values that the instrument variables can take on. A wide variety of possible constraints exists, but we consider only two categories:

i. Introduction of linear dependency among several instruments.

By making some instruments linearly dependent, the freedom of policy action is reduced in a way such that only one control sequence is available to achieve the targets. An illustration of this type of constraint has been given in the previous section.

ii. Introduction of acceptable values of the instruments.

In many cases, the policy-maker has a good idea of what levels of control variables are acceptable politically and economically. Minimizing some measure of deviation between the realized and the most acceptable values assures a unique sequence of instrument vectors. For example, the solution of the following tracking problem may yield a unique control sequence:

$$\min W = \sum_{t=0}^{T-1} [\{\underset{\sim}{u}(t)\} - \{\underset{\sim}{\bar{\bar{u}}}(t)\}]' \underset{\sim}{R} [\{\underset{\sim}{u}(t)\} - \{\underset{\sim}{\bar{\bar{u}}}(t)\}] \quad (45)$$

$$\text{s.t.} \quad \{\underset{\sim}{K}(t + 1)\} = \underset{\sim}{G} \{\underset{\sim}{K}(t)\} + \underset{\sim}{B} \{u(t)\}.$$

In equation (45), the objective is to minimize the sum of the squared deviations between $\{\underset{\sim}{u}(t)\}$ and an acceptable or desired control vector at time t, $\{\underset{\sim}{\bar{\bar{u}}}(t)\}$. It is a simple linear quadratic control problem.

The use of a quadratic objective function with linear constraints is common in economic policy analysis*. It is based on two assumptions. The first is that the policy-maker's preferences are quadratic. The second assumption is that each of the targets depends linearly on all of the instruments, the coefficients of these linear relations being fixed and known.

<u>CASE 2 : NT > KT</u>

If the number of targets specified exceeds the number of instruments exercised, the system (42) is inconsistant, and not all of the target values can be reached. This poses an additional decision problem for the policy-maker. Does he give up some targets in order to reach others, or does he want to approximate all of the targets as closely as possible? In the latter case, we again have

*Theil's quadratic programming model for statis and dynamic policy analysis (Theil, 1964, pp.34-35 and Chapter 4; Friedman, 1975, pp.158-160) is frequently used, as is the linear-quadratic control model (Sengupta, 1970; Pindyck, 1973a, 1973b; Vishwakarma, 1974; Garbade, 1975; Chow, 1972, 1975 Chapter 9). The linear quadratic control model is particularly successful in applied problems of quantitative economic stabilization policy.

a tracking problem, but now in the state variables instead of in the controls. A policy model analogous to (45) may be formulated as follows:

$$\min W = \sum_{t=1}^{T} [\{\underset{\sim}{K}(t)\} - \{\bar{\underset{\sim}{K}}(t)\}]' \underset{\sim}{Q} [\{\underset{\sim}{K}(t)\} - \{\bar{\underset{\sim}{K}}(t)\}] \quad (46)$$

$$\text{s.t.} \quad \{\underset{\sim}{K}(t+1)\} = \underset{\sim}{G} \{\underset{\sim}{K}(t)\} + \underset{\sim}{B} \{\underset{\sim}{u}(t)\}.$$

A combination of tracking problems (45) and (46) leads to the dual tracking problem. Desirable values are given for the trajectories of both the state and the control variables. Some extensions of the dual tracking problem are given by Willekens (1976b,pp.98-101).

c. The generalized dynamic policy problem.

In the policy problems considered thus far, it was assumed that the policy-maker's preference system could be expressed completely in terms of target values for the state variables, and that the achievement of these targets was constrained only by the "law of motion" or state equation describing the dynamic behavior of the system. The advantage of this formulation of the policy problem is that its solution can be investigated analytically. It has been shown that the existence of a control vector or of a sequence of control vectors ensuring the achievements of the targets is determined by the rank of the matrix multipliers $\underset{\sim}{B}$, $\underset{\sim}{D}$ or $\underset{\sim}{F}$. In other words, the ranks of the matrix multipliers determine the controllability of the dynamic demo-economic system. Once the existence of a feasible policy has been demonstrated, the computation of the control vector or sequence of control vectors, is straightforward. The design of an optimal policy is particularly simple if the matrix multiplier is nonsingular. In this case, only one feasible combination of controls exists: the optimal combination.

In the previous sections, no direct constraints were imposed on the state variables. The control variables were constrained in a very simple way; namely, through the introduction of linear dependency. In this section, we expand the possible constraints, thus reducing the set of feasible control vectors. In addition, more realistic policy objectives are discussed.

In practical policy-making, the values that the state and control vectors can take on are restricted by political, economic and social considerations. For example, it is politically unacceptable for the values of policy instruments to fluctuate heavily from one period to another. To remedy possible problems of instrument instability, Holbrook (1972, p.57) proposes to include the instruments in the policy-maker's preference function. Each element of the control vector also may be required to lie within a lower and upper boundary:

$$\underline{u}_i(t) \leq u_i(t) \leq \bar{u}_i(t). \tag{47}$$

Population policy is not cost free. Imposing controls implies the incurrence of costs. It is, therefore, natural to assume a budget constraint that limits the action space of the policy-maker. We distinguish between a budget constraint for each period:

$$\{\underset{\sim}{c}(t)\}' \{\underset{\sim}{u}(t)\} \leq C(t) \tag{48}$$

and a global budget constraint:

$$\sum_{t=0}^{T-1} \{\underset{\sim}{c}(t)\}' \{\underset{\sim}{u}(t)\} \leq C. \tag{49}$$

The cost vector $\{\underset{\sim}{c}(t)\}'$ contains the unit costs of each instrument.

Constraints (47) to (49) are related to the control vector. Frequently, the state vector itself, i.e. the population distribution, is constrained in addition to the control vector. For example, the policy-maker may want to put upper and lower limits on the population in each region in order to avoid the social costs of excessive density or of depopulation. Other constraints on the state vector may be formulated. The general formula expressing constraints on state and/or control vectors is given by equation (3). Usually, however, such constraints take the form of a set of linear inequalities. Together with the boundary conditions and the state equations these delineate the feasible set of controls, out of which an optimal control vector or control sequence may be chosen according to an objective function.

In the previous section, quadratic objective functions have been considered. Other illustrations of formal population policy problems with quadratic objectives and linear constraints are given by Evtushenko and MacKinnon (1976) and by Mehra (1975). If constraints and objectives are both linear, the policy model takes the format of a dynamic linear programming problem (Propoi and Willekens, 1977).

The most general formulation of a dynamic policy problem is presented by equations (1) to (7) of the first section of this paper. Neither constraints nor objectives need to take simple linear or quadratic forms. In general, however, simplifications are adopted to facilitate the computational task of finding the optimum. Solution algorithms for dynamic mathematical programming or optimal control problems are beyond the scope of this paper. Descriptions and numerical illustrations may be found in textbooks, such as those of Bryson and Ho (1969) and McReynolds (1970) and Noton (1972).

3. Theoretically-Oriented Policy Models

Theoretically-oriented policy models have been developed to gain insights into the characteristics of an optimal demo-economic system. Most originated in economics, particularly in the field of economic growth theory. Their main concern is the study of the existence, uniqueness, stability, and efficiency properties of equilibrium growth paths (McNicoll, 1975, p.651).

The basic format of these models is that of an optimal control problem, as described in equations (1) to (7). As in the previous section, we will begin our exposition with the simple economic growth model that underlies most theoretically-oriented policy models, the neoclassical growth model, and then gradually build up policy models of greater complexity.

In contrast to demographic growth, there is no unique indicator that measures economic growth. Gross product or output, value added, consumption and other such variables all have been used. Consider, for example the activity of production. The output of a production process is described by the production function. Usually, only two production factors are considered, capital and labor. The production function, therefore may be expressed as

$$Q(t) = F[K(t), L(t)]. \tag{50}$$

where $K(t)$ denotes the capital stock at time t, and $L(t)$ is the corresponding stock of labor.

In a well-defined production function, growth of total output is uniquely determined by the growth of the factors of production. Solow (1956) suggested simple hypotheses about the development of factor endowments that close the system and enable a study of the growth path generated by the model economy. Meade (1961) and Swan (1956) independently developed similar models leading to the same conclusions. Their model is known as the neoclassical or Solow-Swan growth model.

Assume a neoclassical production function, and a growth path of capital labor obeying the following assumptions:

a. The labor force $L(t)$ grows at a constant relative rate n, which is equal to the growth rate of the population and is given exogenously:

$$\frac{dL(t)}{dt} = nL(t), \quad L(0) = L_0 > 0. \tag{51}$$

The labor supply function is the solution of the differential equation in (51):

$$L(t) = L_0 e^{nt} . \tag{52}$$

b. A constant fraction s of the total output flow $Q(t)$ is saved and all the savings are invested in the capital stock $K(t)$. Assuming moreover that capital does not depreciate, the growth of the capital stock is given by the investment function:

$$\frac{dK(t)}{dt} = sQ(t) , \qquad K(0) = K_0 > 0 . \tag{53}$$

In the production function in (50), L(t) stands for the total employment, or labor demand. In (52), L(t) stands for labor supply. By equating the two, we are assuming that full employment is prepetually maintained. In addition, the exponential growth of labor at a predetermined rate n implies that labor is completely inelastic. The labor supply curve is a vertical line which shifts to the right over time as the quantity of labor increases. Wages have no impact on labor supply or on labor demand. The real wage rate, or equivalently the marginal productivity of labor, adjusts each time period so that all available labor is employed.

If there is unemployment, the wage rate should fall. Labor becomes cheaper and induces a substitution of labor for capital. This lowers the capital-labor ratio until full employment is restored.

Inserting (50) and (51) in (53) gives

$$\frac{dK(t)}{dt} = sF[K(t), L_0 e^{nt}] \quad . \tag{54}$$

This is the basic equation that determines the time path of capital accumulation that must be followed for full employment to be maintained. For each time t, the supply of labor and capital is inelastic. Labor is given by (52) and the capital stock is a result of previous accumulation. All labor and capital that exist at t will be fully employed. This is broughtabout by an adjustment of the marginal productivities. For each t, the output may be computed by the production function. How much of the output will be saved and reinvested is given by (54). This investment adds to the capital stock of the next period.

In the Solow-Swan model of economic growth, the possibility of factor substitution assures that full employment is maintained. The burden of adjustment falls on the marginal productivities of capital and labor, or equivalently, on the marginal capital-output ratio. To study the relationship between the time path of capital accumulation and population growth, we express the capital stock at time t as

$$K(t) = kL(t) \tag{55}$$

where k is the capital-labor ratio, defined as $k = K(t)/L(t)$. Differentiating both sides of (55) gives

$$\frac{dK(t)}{dt} = \frac{dk}{dt} L(t) + k \frac{dL(t)}{dt} . \tag{56}$$

Substitution of (52) in (56) gives:

$$\frac{dK(t)}{dt} = \frac{dk}{dt} L_0 e^{nt} + knL_0 e^{nt}. \tag{57}$$

Equating (57) to (54) yields:

$$\left[\frac{dk}{dt} + kn\right] L_0 e^{nt} = sF[K(t), L_0 e^{nt}]. \tag{58}$$

The production function is neoclassical and obeys the assumption of constant returns to scale. If capital and labor are multiplied by some constant, the output is multiplied by the same constant.

Therefore, we may divide both sides of (58) by $L_0 e^{nt}$:*

$$\frac{dk}{dt} + kn = sF[k,1] .$$

The time path of the capital-labor ratio is given by the differential equation:

$$\frac{dk}{dt} = sF(k,1) - nk. \tag{59}$$

The function $F(k,1)$ is the <u>per capita production function</u>. It is the total product curve that arises as varying amounts k of capital are employed with one unit of labor. In other words, it gives the output per worker as a function of capital per worker. It only depends on the capital-labor ratio or capital-output ratio because of constant returns to scale. Equation (59) shows that the growth rate of the capital-labor ratio $\frac{1}{k}\frac{dk}{dt}$ is equal to the growth rate of capital, or the rate of capital accumulation $sF(k,1)/k$,

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*If there is no constant returns to scale, we must consider (50) and (52) directly without this simplification.

minus the growth rate of labor, n. Note that $k/F(k,1)$ is the capital-output ratio, C say. Hence, we may write

$$\frac{1}{k}\frac{dk}{dt} = s/C - n \ . \qquad (60)$$

The quantity s/C is the <u>warranted rate of growth</u> (Harrod, 1970, p. 47). In Harrod's version of the Harrod-Domar model, it is the rate of growth of output for which the actual level of production coincides with the expected demand. The producer produces <u>neither more nor</u> less than the right amount.

If $\frac{dk}{dt} = 0$, the capital-labor ratio is constant, and the capital stock must be expanding at the same rate as the labor force, namely n, to maintain full employment. But this is exactly the formulation of the Harrod-Domar consistency condition. It is the condition that an economic system must satisfy in order for steady-state growth to be possible under a fixed capital-output ratio, a constant savings rate, and full employment. In Harrod's formulation, the condition is that the warranted rate of growth equals the <u>natural growth rate</u> n (the growth rate of the population):

$$\frac{s}{C} = n. \qquad (61)$$

This equilibrium situation is labeled by Robinson (1970, p. 133) the "<u>golden age</u>", to indicate a steady, smooth growth with full employment. In Solow's extension of the Harrod-Domar model, the capital output ratio is not fixed, but changes automatically in response to changes in factor supplies (measured by changes in s or n).

It can be shown that for any positive s and n, satisfying

$$0 < \frac{n}{s} < F'(0) \quad , \qquad (62)$$

there exists a unique positive capital-labor ratio k* such that $\frac{dk^*}{dt} = 0$ (Burmeister and Dobell, 1970, p. 25), i.e. such that equilibrium or steady state is feasible. How the capital-labor ratio changes as the system converges to its equilibrium position is portrayed in Figure 1. It is the phase diagram for the differential equation in (59).

For any point on the $\frac{dk}{dt}$ curve, there is full employment and hence short-run equilibrium. The position of the economy is described by k, and its growth by $\frac{dk}{dt}$. In the long-run equilibrium, $\frac{dk}{dt} = 0$, the capital-labor ratio is constant, and capital grows at the same rate as labor.

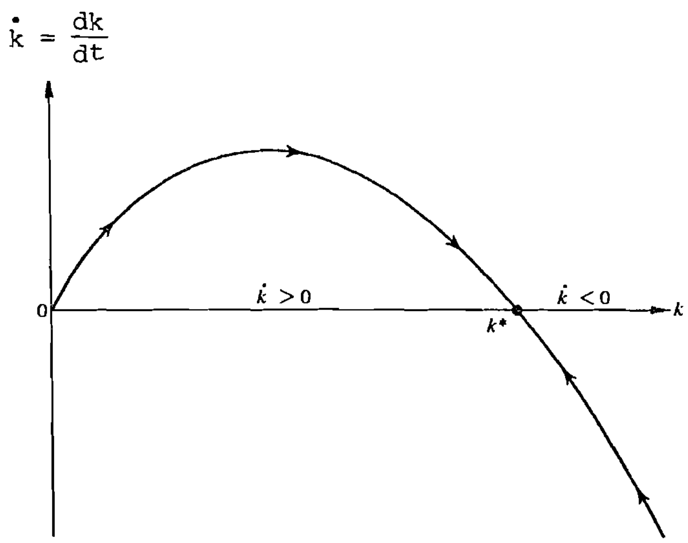

Figure 1. Phase diagram for Solow's fundamental differential equation $(\frac{dk}{dt} = sF(k,1) - nk)$.*

Source: Burmeister and Dobell, 1970, p. 26.

Equations (59) and (60) describe the growth rate of the economy in terms of the growth rates of factor supplies. It presents the "law of motion" of the economic system.

In these models, and in various extensions, labor grows at a constant rate. To convert the neoclassical growth model (59) into a complete demo-economic policy model, we may consider a number of additions. The first is to drop the assumption of exogenously defined labor increase by endogenizing the growth rate of labor;

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*Throughout the text, $\dot{k}$ and $\frac{dk}{dt}$ are equivalent.

the second is the introduction of policy-objectives and other constraints.

3.1 Economic growth with population endogenous.

Classical economists such as Adam Smith, viewed the size of population as being positively related to the wage level. High wages would affect birth and death rates. They would encourage early marriage, and hence higher birth rates. In addition, children would become more valuable as future workers and as a form of retirement insurance. This would induce parents to take greater care of their children, and would thereby diminish the infant death rate.

Ricardo considered a third factor of production, land, whose total supply is fixed. Constant returns to scale is assumed for the three factors: land, capital, and labor. Therefore, a production function, containing only capital and labor, exhibits decreasing returns. As did Smith, Ricardo linked population growth to the wage level. He assumed that there was a subsistence wage. If the actual wage received fell below the subsistence wage, women would adopt a net reproduction rate of less than unity. The conclusion of Ricardo's analysis was that population and the economy would approach a stationary state ($n = 0$), with wages at a subsistence level.*

Although Smith and Ricardo both devoted some attention to the economics of population growth, and indicated that population is endogenous to economic growth, Malthus was the first to succeed in systemizing a general theory of population. According to Malthus, birth rates are biologically determined, but death rates are affected by economic conditions.

The formal treatment of an endogenous population in economic growth models is of a more recent date. This section reviews some attempts to endogenize the demographic component. In addition, it investigates the impact of an endogenously changing labor force participation rate and of the explicit consideration of consumption.

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*A production function with decreasing returns to scale is, somewhat surprisingly, not a sufficient condition to ensure that a stationary population will be approached (Niehaus, 1963; Enke, 1963; Pitchford, 1974, p. 56-70).

a. The neoclassical model with population endogenous.

To illustrate how population growth may affect economic growth, consider the fundamental equation of the Solow-Swan model with the population growth rate a function of the wage rate $w(k)$, or per capita income or consumption (Solow, 1970, p.189):

$$\frac{dk}{dt} = sF(k,1) - kn[w(k)]. \tag{63}$$

Assume that, when the capital-labor ratio k is low, and hence the wage rate w(k) is low, the population is unable to maintain itself, and the growth rate of labor is negative. As wages rise, the population growth rate increases until the wage rate reaches such a level that the wealthy population decides to cut down its growth rate. Such a case might be represented by a growth rate equation for n[w(k)] or n(k), or by a phase diagram such as Figure 2.

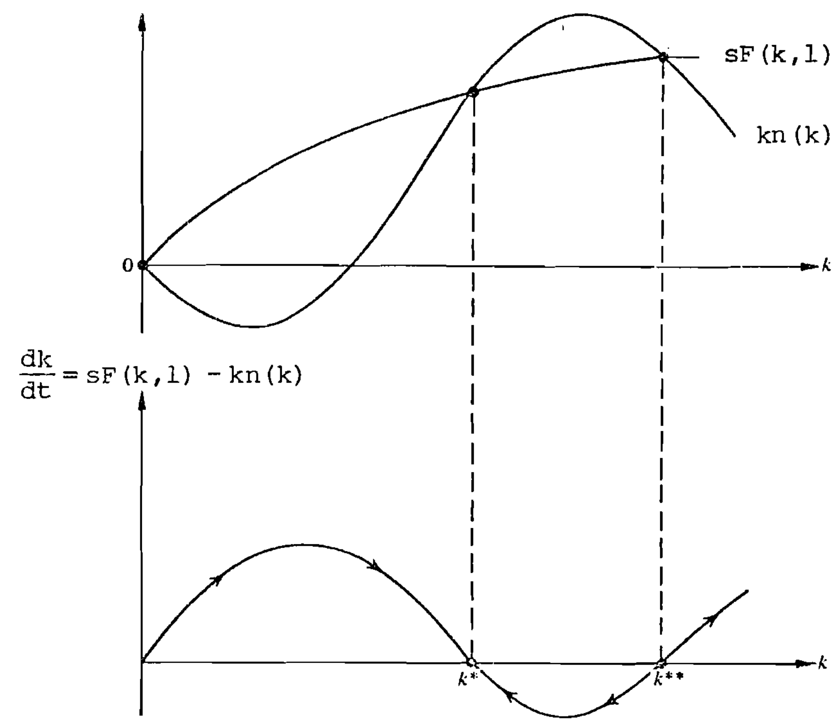

Figure 2. Phase diagram for Solow-Swan model with population growth endogenous (equation $\frac{dk}{dt} = sF(k,1) - kn(k)$).

Source: Burmeister and Dobell, 1970, p.37.

As Figure 2 shows, there is the possibility of multiple equilibria and hence also of unstable ones. Below the equilibrium point k** small perturbations of k force the capital-labor ratio to an equilibrium value k*. When the capital-labor ratio increases beyond the unstable equilibrium point k**, the economy is on a path with a perpetually rising capital-labor ratio, and hence per capita income.

Instead of focusing on the wage rate, one may make population growth depend on per capita income or consumption. In his original article, Solow (1970, p.188) treats these equivalent cases. In general, the population growth equation becomes

$$\frac{dL}{dt} = L(t)\ n(\xi) \tag{64}$$

where ξ denotes wages, per capita income, or per capita consumption.

In the Sato and Davis (1971, p.881) model, ξ denotes per capita income y or F(k,1). The economic dynamics is therefore given by

$$\frac{dk}{dt} = sF(k,1) - k\ n[F(k,1)]. \tag{65}$$

The function n(•) is monotonic ($n' \geq 0$). The logistic growth curve in Figure 3 reflects the fact that the death rate decreases with income and that the birth rate increases up to a certain level of y and declines thereafter.

The assumption that population growth depends on per capita income and that the labor force participation rate is constant has also been made by Lane (1975, p.58) and Pitchford (1974, p.167).

In recent years there has been a revived interest in population as an endogenous variable in economic growth models, particularly within the perspective of policy formulation. The rationale for making population an endogenous variable of dynamic economic growth models, has been given by Dasgupta. His approach is to treat capital accumulation and population growth as inter-dependent: "The economic welfare of a community is affected by policies that determine 1) the rate of capital accumulation; and 2) the rate of growth of population. At any moment of time the optimum size of population will depend on the size of the existing capital stock and the optimum

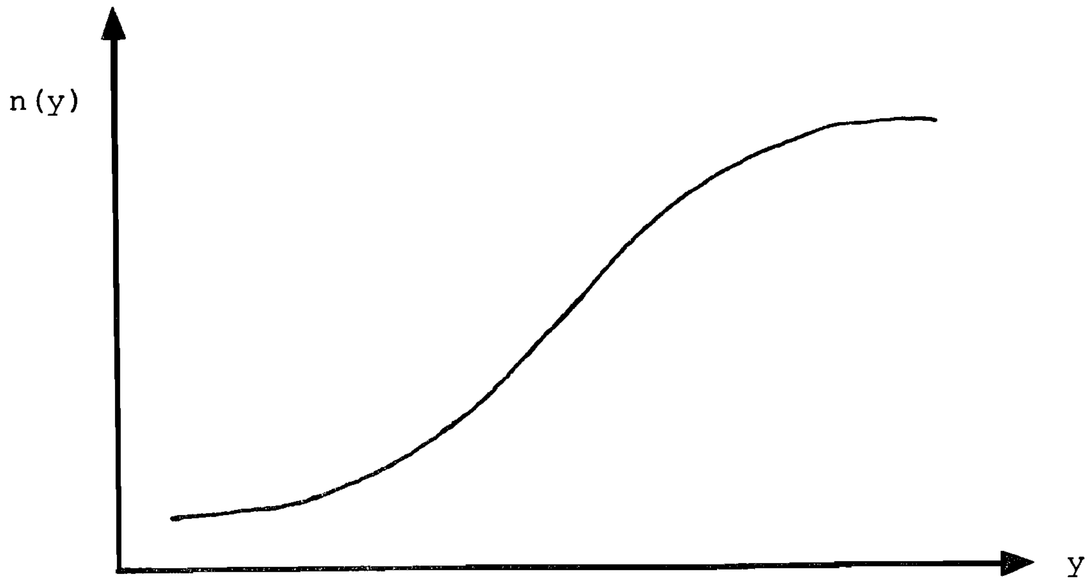

Figure 3. Relation between population growth and per capita income.

rate of savings will depend on the existing number of people. In this sense a population policy cannot be formulated without a concurrent savings policy. The two must be considered together". (Dasgupta, 1969, p.295).

b. Population growth and labor force participation.

An assumption which is frequently made by authors attempting to endogenize population growth in economic models is that labor force and population are interchangeable variables. The rationale for this is given by Pitchford (1974, p.55): "If the proportional rate of growth of population is constant and has been for a long time, it is not unreasonable to assume that a fixed ratio between the work force and population has been established". In terms of mathematical demography, it means that the population is assumed to be stable. In the stable population, the age composition is constant, and hence the population in the active age groups is a fixed proportion of the total population.

$$L(t) = pN(t) \qquad (66)$$

where $L(t)$ is the labor force,

$N(t)$ is the population at time t, and

p is the labor force participation rate.

If population grows at a constant rate n, and p is constant, the labor grows at the same rate n.

In reality, the labor force participation rate is not constant, but depends on the age composition of the population and on economic conditions. We may therefore ask the question how economic growth would be affected if the labor force participation rate varied with changes in the economic situation.

Consider the neoclassical growth model in (59):

$$\frac{dk}{dt} = sF(k,1) - nk \ . \tag{59}$$

Assume that the labor force participation rate depends on the prevailing wage w(k), which depends on the factor shares. The labor force at time t then is

$$L(t) = p[w(k)]\ N(t) = p[w(k)]\ N_0 e^{nt} \ . \tag{67}$$

Therefore,

$$\frac{dL(t)}{dt} = \frac{dp}{dw} \cdot \frac{dw}{dk} \cdot \frac{dk}{dt} N_0 e^{nt} + pN_0 ne^{nt}$$

or

$$\frac{dL(t)}{dt} \cdot \frac{1}{L(t)} = \frac{dp}{dw} \cdot \frac{dw}{dk} \cdot \frac{dk}{dt} \cdot \frac{1}{p} + n \ , \tag{68}$$

$$\frac{dk}{dt} = \frac{dK}{dt}\frac{1}{L} - \frac{dL}{dt}\frac{1}{L} k \ . \tag{69}$$

Substituting (59) and (68) in (69) gives

$$\frac{dk}{dt} = sF(k,1) - \left(\frac{dp}{dw}\ \frac{dw}{dk}\ \frac{dk}{dt}\ \frac{1}{p} + n\right) k$$

$$= sF(k,1) - \left[\left(\frac{dp}{dw} \cdot \frac{w}{p}\right)\left(\frac{dw}{dt} \cdot \frac{k}{w}\right)\left(\frac{dk}{dt} \cdot \frac{1}{k}\right) + n\right] k$$

$$= sF(k,1) - z_p\ z_w \frac{dk}{dt} - nk \tag{70}$$

where z_p represents the elasticity of the participation rate with respect to wages, and

z_w represents the elasticity of wages with respect to the capital-labor ratio.

The growth path of the capital-labor ratio reduces to

$$\frac{dk}{dt} = \frac{sF(k,1) - nk}{1 + z_p\ z_w} \quad . \tag{71}$$

Balanced growth equilibrium, i.e., where $\frac{dk}{dt} = 0$, occurs as in the Solow-Swan model. Convergence to the equilibrium point, however, takes more time. Since $\frac{dp}{dw}$ may be expected to be positive, and since $\frac{dw}{dk}$ is positive, z_p and z_w are positive, making the denominator of (71) greater than one, slowing the speed of convergence. This reflects the fact that rising wages associated with a rising capital-labor ratio, induce the entry of a larger proportion of the population into the labor force.

c. The neoclassical model with consumption.

In the Solow-Swan model, the savings rate s was held constant. Different values of s would lead to different allocations of capital in the golden age growth path, and therefore to different equilibrium capital-labor ratios, different wage rates, different personal income streams, and different consumption rates. A logical extension of the neoclassical model, therefore, is to introduce consumption explicitly. One may begin by expressing total savings as

$$S = sF(K,L) = F(K,L) - cN \tag{72}$$

where c is consumption per capita, and N is the total population. Assuming a labor force participation rate of unity, i.e. L = N, and constant returns to scale, (72) may be written as

$$S/L = sF(k,1) = F(k,1) - c. \tag{73}$$

Substituting (73) in the Solow-Swan growth model in (59) gives*

$$\frac{dk}{dt} = F(k,1) - c - nk\ . \tag{74}$$

There is a different golden-age growth path for every k. For a given consumption rate c, the steady-state capital-output ratio is

$$C = \frac{F(k,1)}{k} = \frac{c}{k} - n, \tag{75}$$

where c/k is consumption as a fraction of total income. On the other hand, for a given capital-output ratio, or equivalently, capital-labor ratio, the steady-state consumption rate is

$$c = F(k,1) - nk\ . \tag{76}$$

Equation (76) provides a direct link between growth theory and growth policy. Although the rate of population growth, n, is held constant, one can derive rules for capital accumulation which may be compared with those obtained with different rates of population growth. In addition, (76) or (74) show up as constraints in a number of demo-economic optimization models.

Optimization and policy considerations come in when both per capita consumption and the capital-labor ratio are allowed to vary. The relevant policy question is: what steady-state capital-labor ratio is able to sustain a maximum per capita consumption? The first order condition for maximizing per capita consumption is

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*Recall that this implies full employment and that all savings are invested.

$$\frac{\delta c}{\delta k} = 0 = \frac{\delta F(k,1)}{\delta k} - n$$

or

$$F'(k,1) = n, \tag{77}$$

where $\frac{\delta F(k,1)}{\delta k} = F'(k,1)$ is the marginal product of capital. This means that the interest rate equals the rate of labor force growth. The capital output-ratio maximizing c will be denoted k*.

Equation (77) is the "golden rule of capital accumulation". It has been discovered independently by Swan (1964) and Phelps (1961; see also 1970, p.198), and was already implicit in Ramsey's (1928) work. Among all the possible golden age paths of natural growth, that golden age is best which practices the golden rule: the investment made by each generation is such that the next generation has the highest possible per capita consumption. Under the golden rule, the relative share of output going to capital is the optimal savings ratio

$$s = \frac{k^*F'(k^*,1)}{F(k^*,1)} = \frac{k^*n}{F(k^*,1)} \quad . \tag{78}$$

(See also Burmeister and Dobell, 1970, pp.49-53). Therefore, the savings rate which maximizes per capita consumption in the long run is equal to the share of profit in national income.

The golden rule consumption per head is given by

$$c^* = F(k^*,1) - n\,k^*. \tag{79}$$

How does an increase of the population growth rate n affect c* (Phelps, 1966, pp. 178)? If $k^*>0$, an increase of n leads to an increase in $F'(k^*,1)$, by (77). But $F'(k^*,1)$ is decreasing in k, so k* will decrease, which implies a decline in per capita output $F(k^*,1)$. The net effect upon c* of increasing n may be found by differentiating (79) with respect to n. This yields:

$$\frac{dc^*}{dn} = F'(k^*,1)\cdot\frac{dk^*}{dn} - (k^* + n\frac{dk^*}{dn})$$

$$= [F'(k^*,1) - n]\,\frac{dk^*}{dn} - k^* \;,$$

which, after applying (77) gives:

$$\frac{dc^*}{dn} = - k^* . \tag{80}$$

Therefore, the golden rule per capita consumption declines as the population growth rate rises, and the lower the population growth rate, the better.

In Phelps's model population growth is given exogenously. Davis (1969) has extended the model to allow the population growth rate to vary with per capita income, i.e. $n = n[F(k,1)]$. When population grows endogenously, the golden rule savings rate is no longer equal to (78) but is modified to

$$s = \frac{k^*F'(k^*,1)}{F(k^*,1)} \left[1 - k^* \, n'[F(k^*,1)]\right] \tag{81}$$

The ratio $k^* \frac{F'(k^*,1)}{F(k^*,1)}$ is the relative share of output going to capital. Whether the modified golden rule savings rate is greater or less than the relative share of capital is determined by the sign of $n'(.)$. Correspondingly, the growth rate of the economy under endogenous labor supply is greater than or less than the marginal productivity of capital depending upon whether $n'(.)$ is negative or positive*. A positive $n'(.)$ implies a monotonic function describing a positive relationship between the population growth rate and per capita income.

d. Age-specificity in demo-economic growth.

Population growth and capital accumulation are represented by simple aggregate models. In the models of economic growth discussed so far, capital and labor are assumed to be homogenous.

A few growth models have focused on a disaggregation of population by age. Samuelson (1958, 1975) and Arthur and McNicoll (1977a and 1977b) have shown how the introduction of age groups affects the optimum rate of population growth. In the aggregated version of the growth model, given by (74), a small population growth rate is preferrable, since an increase in population growth calls for a greater investment to maintain the capital-labor ratio, or capital per head. Capital widening diverts resources from consumption and capital deepening, i.e., increases in capital per head. Samuelson

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*It is assumed that $[1 - k^*n'(.)] > 0$.

(1958) has shown that the introduction of aging with the possibility of transfers between age groups may alter the conclusion. We will return to age-specific demo-economic policy models in the next section.

3.2 Optimum demo-economic growth

Heretofore, we have investigated several models of demo-economic growth and their steady-state properties. Policy objectives were introduced to select a unique steady-state or golden age growth path. The focus was not on the objectives themselves, but on the characterization of a unique steady-state.

The transformation of these models into truly dynamic demo-economic policy models requires:

i. The introduction of an explicit population control variable.

ii. The introduction of explicit <u>intertemporal</u> policy objectives. This involves problems of definition of the welfare criterion and of the social rate of discount.

iii. An allowance for other constraints on both the state and control variables. This will not be studied here.

iv. A more realistic description of the population system by introducing age-specificity.

These requirements will be dealt with separately below.

a. The population control variable.

In the models considered thus far, population was treated exogenously or endogenously, but was not considered as a direct policy variable. The problem was to determine the optimal capital-labor ratio or the savings rate for a given population growth rate. Recall, for example, the Golden Rule of capital accumulation. Associated with each population growth rate n is a Golden Rule state. The Golden Rule consumption per head is given by (79) and depends on n, i.e. $c^* = c(n)$. For each n, the implied optimal savings rate is easily derived, since $c = (1 - s) F(k,1)$. This savings rate is optimal in the sense that it maximizes per capita consumption under the given regime of population growth.

Phelps (1966, pp.179-182) went a step further. He addressed the policy problem of finding the growth rate n which yields a Golden Rule state that is socially preferred. This step completely integrates economic (savings) policy and population policy. Which Golden Rule state is preferred depends of course on the objectives. According to Phelps, society not only wants to consume as much as possible, but also wants to grow. Social welfare is, therefore, a function of both the consumption per head and the population growth rate. Hence, the function to be maximized is $u(n,c^*)$. Assuming a constant mortality rate, the welfare function may be written in terms of the birth rate b. Writing $c^* = h(b)$, the policy problem becomes

$$\max_{b} u[b,h(b)] \quad (82)$$

The problem now is to find the optimal combination of fertility and consumption in a situation of balanced growth (i.e. when output, capital, and consumption all grow at the same rate as labor). It is a simple, but complete demo-economic policy problem. The optimality condition is

$$\frac{\delta u}{\delta b} = 0 = u_b + u_c \, h'(b^*) \ ,$$

or

$$-\frac{\delta c^*}{\delta b} = -h'(b^*) = u_b/u_c \ . \quad (83)$$

Equation (83) states that for the social welfare function in (82) to be at a maximum, the birth rate must be such that the marginal cost of the birth rate (per capita births), in terms of Golden Rule consumption per head, is equal to the marginal rate of substitution between the birth rate and the Golden Rule consumption per head. The birth rate b* which yields the Golden Rule state in which (82) is maximum is the Golden Rule of Procreation.

Phelps's Golden Rule of Procreation gives the optimum population growth rate under the assumption that population policy is costless. Changing the population growth rate, however, requires resources that could have been directed to productive investments. The portion of the income allocated to population control is denoted by McNicoll (1975, p.671) as demographic investment*. The

*McNicoll's definition of demographic investment differs from that of Sauvy (1976, p.64), who considers it to be that part of total investment which is required to maintain the standard of living or the capital-labor ratio. The latter perspective is identical to capital widening.

function describing the relationship between demographic investment per capita at time t, j(t) say, and the population growth rate n is n[j(t)]. In other words, j(t) denotes the per capita expenditures required to reach a population growth rate n. If j(t) = 0, n is equal to the natural rate of population growth. Total savings are now divided among investment in the capital stock and demographic investment. The basic technological relation (59) is then

$$\frac{dk}{dt} = sF(k,1) - k\, n[j(t)] - j(t). \tag{84}$$

In the simple case, j(t) is a constant fraction of per capita income, i.e. j(t) = g F(k,1), hence the demographic response function is n[F(k,1)] and per capita consumption is

$$c = (1 - s - g)\, F(k,1). \tag{85}$$

To find the steady-state or equilibrium level of demographic investment we recall Phelps's Golden Rule of Procreation. The population growth rate or the birth rate that maximizes the social welfare derived from both per capita consumption and growth rate is such that

$$\frac{\delta c^*}{\delta b} = h'(b^*) = - u_b/u_c. \tag{86}$$

The demographic investment associated with a birth rate b* is j*. Hence

$$\frac{\delta c^*}{\delta b} = h'[b(j^*)] = - u_b/u_c, \tag{87}$$

and

$$h'[b(j^*)]\, u_c = -u_b\ . \tag{88}$$

The optimal demographic investment is such that the loss in utility from reducing the population by one unit is exactly equal to the utility derived from the higher per capita consumption. Therefore, at the optimum, a given increment of investment has the same impact whether allocated to production (and consumption) or to population control.

Another approach to finding the optimum level of demographic investment has been taken by Sato and Davis (1971, p. 890). The authors assume that population grows endogenously, but can be influenced by direct policy intervention. The demographic response function is therefore $n[f(k,1),g]$, where g is the fraction of per capita income allocated to population control. This is an extension of the problem, discussed in the previous section, of determining an optimum savings rate when population grows endogenously. Maximization of per capita consumption (85) yields the "modified" Golden Rule of capital accumulation:

$$s^* = \frac{k^*F'(k^*,1)}{F(k^*,1)} (1 - g - k^* n') \tag{89}$$

and

$$\frac{\delta n}{\delta g} = \frac{F(k,1)}{k} = - \frac{1}{C} \tag{99}$$

where $n' = \dfrac{\delta n[F(k,1),g]}{\delta F(k,1)}$.

The introduction of direct population control reduces the optimum savings rate even further than before (compare (89) with (78). Population policy should be implemented until the marginal impact of public expenditures is equal to the average productivity of capital (output-capital ratio $1/C$). This implies that at the optimum, per capita income or output is equal to the product of the capital-labor ratio and the marginal impact of population control expenditures. Consequently, since in the steady state $n = s/C$, the equilibrium growth rate of population must be

$$n^* = - s \frac{\delta n}{\delta g} . \tag{100}$$

Since s/C is Harrod's warranted rate of growth, the quantity $-s \frac{\delta n}{\delta g}$ may be called the modified warranted rate of growth for endogenous population growth and direct policy intervention.

The population policy variable or decision rule considered by Phelps is the growth rate, (or birth rate if mortality is constant). Other authors have addressed the question of the optimum size or density of the population. For example, Dasgupta (1969) treats the problem first formulated by Wicksell; namely, what size (density) of population under given circumstances is the most advantageous? Posing the question of optimum population size implies the assumption that a zero growth rate is best since only if $n = 0$ can an optimum population be maintained (other conditions being equal). The second decision rule is more suitable for "classical" economic regimes in which the reality of finite resources or of some fixed production factors as land eventually leads to diminishing returns. The first decision rule on the other hand fits the "neoclassical" regime, with no resource constraints but with constant returns to the production factors*.

b. Policy objectives.

What is the optimum population size or the optimum growth rate of the population? According to Phelps, an optimum growth rate is one which maximizes (82). In the demo-economic policy literature, the policy objective usually involves a measure of per capita consumption. Two types of welfare indices are used frequently: i) social welfare is a direct function of per capita utility; ii) social welfare is a weighted function of per capita utility, the weight being the population size. We consider both indices in a static and a dynamic framework.

(1) <u>Static analysis</u>

i. Per capita utility

The total consumption stream available to the population is equal to the amount of the total production $F[K(t), L(t)]$ less the amount of investment $\dot{K}(t)$. The first criterion relates to per capita consumption. It is equal to $\left[F[K(t),L(t)] - \dot{K}(t)\right]/L(t)$. For example, in a static analysis, the welfare criterion is $u(c)$. The optimum population size is obtained when per capita consumption is at a maximum, i.e.

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*These definitions of classical and neoclassical economic regimes follow Arthur and McNicoll (1977, p.114).

$$\frac{\delta c}{\delta L} = \frac{\delta[F(K,L) - \dot{K}] / L}{\delta L} = \frac{1}{L}\left[\frac{\delta F(K,L)}{\delta L} - c\right] = 0 \qquad (101)$$

or $$c = \delta F(K,L) / \delta L.$$

Therefore, the optimum population size is reached when the contribution to production of a marginal person is equal to his consumption (which is the average consumption).

ii. Total utility.

This criterion supposes that social welfare is equal to the average individual utility weighted by population size. The use of such an approach has been strongly endorsed by Meade (1955). Meade performed a static analysis. The objective function U(C) is simply the product u(c) . L, where L denotes the population size. This is known as the Bentham-criterion (McNicoll, 1975, p.666)*. The optimum population size is given by the condition

$$\frac{\delta[u(c)L]}{\delta L} = 0 = u(c) + L\, u'(c)\, \frac{\delta c}{\delta L}, \qquad (102)$$

where $\delta c/\delta L$ is given by (101). Hence optimality requires that

$$u(c) = u'(c)\left[c - \frac{\delta F(K,L)}{\delta L}\right], \qquad (103)$$

which means that the utility of a marginal individual entering the population (and consuming at the average level) must equal the disutility he causes to the other members of the population.

(2) <u>Dynamic analysis.</u>

i. Discounted per capita utility:

$$\int_0^T e^{-\rho t}\, u[c(t)]\, dt \qquad (104)$$

where ρ is the rate of discount,

c(t) is the per capita consumption at time t, i.e.

$$c(t) = [1 - s(t)]\, F(k,1),$$

u(.) is a utility function, and T is the planning horizon.

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*According to Meade (1955, p.88), there exists a consumption level C_0 at which life is just enjoyable, i.e. $U(C_0) = 0$. The quantity C_0 is referred to as the "welfare subsistence level".

ii. Discounted total utility:

$$\int_0^T e^{-\rho t} u[c(t)] L(t) dt, \tag{105}$$

where L(t) is the population at time t.

Both of the above criteria are frequently used in dynamic policy models. Dasgupta (1969, p.297) compares alternative policies by assessing their impact on total welfare (105). He argues that it is a better measure to compare the ultimate value of having one more person in the world with the ultimate value of present people having a bit more to consume. Both population size and utility from per capita consumption enter the objective function directly. Sato and Davis (1971) compare the theoretical implications of both welfare indices on the optimum policy, under the assumption that population grows endogenously, i.e. that the economic dynamics are those given by (65). Maximization of (104) subject to (65) yields an optimum steady state savings rate equal to

$$s^* = \frac{k^*[F'(k^*,1)\ (1 - k^*n') - \rho]}{F(k^*,1)} . \tag{106}$$

The optimum per capita consumption is of course $c^* = F(k^*,1) - k^*n$. Note that for $\rho = 0$, (106) reduces to the Modified Golden Rule of capital accumulation (81).

Maximization of (105) subject to (65), on the other hand, yields quite different results. The steady state is given by the following relationship:

$$\frac{u}{u'}n' = \frac{(\rho - n)}{F'(k^*,1)}\ [\rho - F(k^*,1)\ (1 - k^*n')], \tag{107}$$

where u is the utility function. Note that when population grows exogenously, $n' = 0$ and (107) reduces to

$$F'(k^*,1) = \rho,$$

i.e. the marginal product of capital must be equal to the rate of discount or time preference*. The steady state capital-labor ratio is determined entirely by the discount rate. The form of the utility function has no effect at all**. But if population grows endogenously, the utility function does play a role.

The choice of the rate of discount has been an element of debate in the growth literature. Ramsey (1928) found discounting the future at a positive rate, "..a practice which is ethically indefensible." Other economists, such as Harrod, have also taken a similar standpoint. Whether a positive rate of time preference is unethical and what the discount rate should be are questions beyond the scope of this paper. The fact is that most economists today introduce some discounting in optimal policy models. Some authors investigate and compare the theoretical features of both cases $\rho > 0$ and $\rho = 0$ (see e.g. Dasgupta, 1969; Sato and Davis, 1971).

c. Age composition and demo-economic policy

The optimal decision rules of demo-economic policy, studied in the previous sections, are based on the assumption of homogenous capital and labor. The validity of the optimal policies depends on the value of this underlying assumption.

Recently, some effort has been devoted to the analysis of more disaggreated policy models, in particular, age-specific models. The following discussion is based on work carried out by Samuelson and Arthur and McNicoll.

Samuelson considers only two age groups***. The young age group consists of the working population, while the old age group contains only retired people. In this simple model, the working population supports the retired population through "consumption" loans.

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*This is the Ramsey Rule. The rate of time preference also may be written as

$$-\frac{1}{p(t)}\ \frac{dp(t)}{dt}\ ,$$

with $p(t) = e^{-\rho t}u'[c(t)]$, i.e., the rate of time preference is the rate of decline in the discounted marginal utility.

**The utility function is, of course, of central importance for the optimum trajectory to equilibrium.

***At the micro-level, the introduction of aging is identical to the consideration of an explicit life cycle (e.g. childhood, work, childbearing, retirement).

Repayment can be expected when this working population retires. Therefore, each generation is supported by the following generation. The support or consumption transfers received by the retired population increases if the proportion of the young population expands, which is the case if the population growth rate rises. In Samuelson's two-age model with intergenerational transfer, therefore, the greater the population growth rate, the better, This conclusion is the opposite of the optimum defined in Solow's neoclassical model and its extensions (see for example equation (80)).

A combination of the Solow-model in (59) with the Samuelson model of overlapping generations yields an intermediate result. Recall equation (72). There total output is equal to consumption and investment (savings):

$$F(K_t, L_t) = C_t + K_{t+1} - K_t \quad . \tag{108}$$

Consumption at time t is the sum of the consumption of the young and the old populations. In the absence of mortality, and for time intervals equal to age intervals, the number of old people at time t is equal to the number of young people at time t - 1. Let L_{t-1} denote the young population at time t -1, and c_t^1 and c_t^2 the per capita consumption of the young and old populations respectively, then (108) becomes

$$F(K_t, L_t) = L_t c_t^1 + L_{t-1} c_t^2 + K_{t+1} - K_t. \tag{109}$$

Dividing by L_t yields

$$F(k_t, 1) = c_t^1 + \frac{1}{1+n} c_t^2 + \frac{K_{t+1} - K_t}{K_t} \cdot k_t \ ,$$

where n is the growth rate of labor. In the steady-state, capital and labor grow at the same rate, hence

$$F(k, 1) = c^1 + \frac{1}{1+n} c^2 + nk. \tag{110}$$

Compare (110) with (76). Note that each combination of n, c^1 and c^2 defines a golden state. Each golden state is characteriszed by a constant capital-labor ratio.

Following Phelps, Samuelson asks which golden state yields maximum utility from consumption. The utility function must of course reflect life-time consumption, i.e. $u = u[c^1,c^2]$. There is a unique relation between c^1 and c^2, given by (110):

$$c^1 = F(k,1) - \frac{1}{1+gn} c^2 - nk. \tag{111}$$

The utility function, therefore, is

$$u[c^1,c^2] = u[F(k,1) - \frac{1}{1+n} c^2 - nk ,c^2]. \tag{112}$$

and maximization with respect to k, c^1 and c^2 yields the following optimality conditions:

$$\frac{\delta u}{\delta k} = 0 = F'(k^*,1) - n \tag{113}$$

$$\frac{\delta u[{}^*c^1,{}^*c^2]}{\delta c^1} = (1 + n) \frac{\delta u[{}^*c^1,{}^*c^2]}{\delta c^2} \tag{114}$$

Condition (113) is Phelps's Golden Rule of capital accumulation. Equation (114) states that for utility to be a maximum, the discounted marginal utility of consumption must be the same for all ages. This relation is the "biological interest rate" relation of Samuelson. The two conditions together constitute the Golden Rule.

There is a Golden Rule state associated with every population growth rate n. One, therefore, may be interested in selecting a rate n which is socially desirable. This problem has been addressed by Phelps and has led him to the derivation of the Golden Rule of Procreation. Phelps included both per capita consumption and the population growth rate in the welfare function. Samuelson, however, (1975, p.534) kept to the function (112). Maximization of (112) with respect to n yields:

$$\frac{\delta u}{\delta n} = 0 = \frac{\delta u}{\delta c^1} \cdot \frac{\delta c^1}{\delta n} ,$$

or

$$0 = - k^* + \frac{{}^*c^2}{(1+n^{**})^2}$$

and

$$1 + n^{**} = \left[\frac{{}^*c^2}{k^*}\right]^{1/2} . \tag{115}$$

The growth rate n**that maximizes lifetime welfare is denoted by Samuelson as the Goldenest Golden Rule state. Moreover, at a growth rate n**, private life-time savings will be just sufficient to support the Goldenest Golden Rule state. Since k and c^2 are themselves functions of n in (113)and (114), (115)is an implicit function of n*. To find the true maximum, second-order conditions must be supplemented. Note that rapid or slow growth no longer is "better", but that the value of n** is determined by the utility function and the production function. (Compare this result with Phelps's Golden Rule of Procreation).

Arthur and McNicoll (1977b) have generalized the two-age lifecycle model to one with a continuous-age lifecycle. This generalization allows for an inclusion of child-dependency costs. The intergenerational transfer is not only from working population to old, but also from working population to children. Therefore, the net intergenerational transfer effect of growth is no longer necessarily positive. The inclusion of tranfers to younger people, therefore, tends to result in lower optimal growth.

The authors consider continuous intervals for both time and age. Equation (108)becomes

$$F(K_t,L_t) = C_t + \frac{dK}{dt} . \tag{116}$$

The population is assumed to be stable, i.e. with constant age-specific rates of fertility and mortality, constant age distribution, and growing at a constant rate n (see example, Keyfitz, 1968; Rogers, 1975). At stability, the population at exact age x at time t is

$$\ell(x,t) = e^{-nx}\ p(c)\ \ell(0,t), \tag{117}$$

where $\ell(0,t)$ is the number of births at time t, and
$p(x)$ is the probability of survival from birth to age x.

Let $c(x,t)$ denote age-specific per capita consumption at time t. Equation (116) may be expressed as follows:

$$F(K_t,L_t) = \int_0^{\omega} \ell(x,t)\ c(x,t)dx + \frac{dK_t}{dt}\ , \tag{118}$$

where ω is the last age group.

Dividing by the amount of labor L_t, where

$$L_t = [\int_0^{\omega} e^{-mx}\ p(x)\ \lambda(x)\ dx]\ell(0,t) \tag{119}$$

and $\lambda(x)$ is the age-specific labor force participation rate, we have that

$$F(k_t,1) = \bar{c}(t) + \frac{1}{K_t}\ \frac{dk_t}{dt}\ k_t, \tag{120}$$

where $\bar{c}(t)$ is the average consumption per worker at time t, i.e.,

$$\bar{c}(t) = \frac{\int_0^{\omega} e^{-nx}\ p(x)\ c(x,t)\ dx}{\int_0^{\omega} e^{-nx}\ p(x)\ \lambda(x)\ dx}\ .$$

In steady-state, capital and labor grow at the same rate n, and $c(x,t) = c(x)$, hence

$$F(k,1) - \bar{c} = nk. \tag{121}$$

Each combination of $\bar{c}$ and n defines a golden state. Note that (121) is the familiar Solow condition, $sF(k,1) = nk$ (see also Arthur and McNicoll, 1977a, p. 116).

Which golden state yields a maximum life-time utility from consumption? If $u[c(x)]$ is the utility from consumption at age x, then a baby just born has a probability $p(x)$ of enjoying this consumption. Maximization of

$$U(c(x)] = \int_0^{\omega} p(x)\ u[c(x)]dx \tag{122}$$

subject to (120) yields the following optimality conditions

$$\frac{\delta u[c^*(x)]}{\delta c(0)} = e^{nx} \frac{\delta u[c^*(x)]}{\delta c(x)} , \tag{123}$$

i.e. the discounted marginal utility of consumption must be the same for all ages. In other words, for the lifetime welfare to be a maximum, the disutility of one unit of consumption less at age 0 (loan) must be offset by the utility of the consumption of this unit at age x, multiplied by the interest (repayment). This relation is the "biological interest rate" condition, similar to (114).

As in the case of two ages, we search now for the growth rate n which yields the most golden Golden Rule state. Maximizing (122) with respect to n gives

$$\frac{\delta U[c^*(x)]}{\delta n} = \int_0^{\omega} p(x) \frac{[\delta u(c^*(x)]}{\delta c(x)} \frac{\delta c(x)}{\delta n} dx$$

$$= \frac{\delta U[c^*(x)]}{\delta c(0)} [\int_0^{\omega} e^{-nx} p(x) \frac{\delta c(x)}{\delta n} dx] .$$

Transforming this expression gives

$$\frac{\delta U[c^*(x)]}{\delta n} = \frac{\delta U[c^*(x)]}{\delta c(0)} [\bar{c}^*(A_C - A_L) - k];$$

here $\bar{c}^*$ is the optimal average consumption per worker and A_C and A_L are, respectively, the mean age of consuming and of the labor force. Therefore, the lifetime welfare effect of changing the population growth rate is equal to an intergenerational transfer effect (the difference in A_C and A_L*) and a capital widening effect. The latter effect is always negative, while the intergenerational transfer effect can be either positive or negative.

4. Conclusion

The purpose of this paper has been to review the existing links between formal representations of population and economic policies. This bringing together of the work of demographers and economists aims to contribute to better policy-making.

The common feature of the models reviewed is the underlying mathematical paradigm. Any dynamic policy problem may formally be stated as an optimal control problem, and the theory of optimal control provides the apparatus necessary to solve for the optimal values of the policy variables.

Two groups of demo-economic policy models have been examined in increasing order of complexity: planning-oriented models and theoretically-oriented models. Planning oriented models may be viewed as logical extensions of mathematical demographic growth models to the policy domain. The demographic growth model itself is embedded in the policy models as the homogenous part of the state equation. The discussion focused on two major issues in dynamic policy modelling: existence and design. Systems theory provides the necessary mathematical tools.

The second group of policy models is theoretically-oriented. These models originated in the theory of economic growth and have a much higher level of abstraction than the planning-oriented models. Studies of their underlying theoretical concepts, of the structure of their policy problem, and of the existence and stability of their optimal policies have received much attention.

This paper presents the current state-of-the-art in linking demo-economic growth and policy in formal models. Fundamental differences between both approaches do not permit a complete sythesis at this time. Complete synthesis may never be achieved and may even not be desirable since the two approaches serve a different purpose.

*The average age of consuming is three to four years below the average working age (Arthur and McNicoll, 1977b).

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