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Competition of Gas Pipeline Projects: A Multi-Player Game of Timing

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Abstract

We use game theory to construct a model of investment in gas pipeline projects competing for a regional gas market. The model is designed as a multi-player game with integral payoffs, in which times of entering the market act as players' strategies. For each player, we identify the location of player's best responses to strategies chosen by other players. On this basis, we reduce the original game to a game with a finite number of strategies for each player. We introduce a regularity condition and for a regular game of timing prove the existence of a Nash equilibrium. An application of this result to a symmetric game of timing allows us to give the entire description of the set of all Nash equilibrium points. Finally, we construct a finite algorithm for finding player's best responses and the Nash equilibrium points in the game. The presented approach can be used to analyze competition of large-scale technological and energy transportation projects in situations where the investment periods preceed the periods of sales and the appearence of every new supplier on the market drastically effects the market price.

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1 Introduction

The present paper is motivated by recent IIASA studies on emerging energy infrastructures in Europe and Asia, including model-based analyses of regional gas pipeline routes (see Klaassen, et. al., 1999; 2002; 2003; 2004; Golovina, et. al., 2002). We consider a multi-player game as a model of competition of several gas pipeline projects targeted to a regional gas market. We base our study on Klaassen, et. al., 2004, where a *game of timing* for two competing gas pipeline projects is analyzed in detail and subtle properties of the Nash equilibrium points in this game are revealed. The considered game of timing is clearly linked to the well-known problem of choosing a stopping rule for a stochastic process (Chow, et. al., 1971), and also the problem of determining the termination time in a differential game (see, e.g., Brykalov, 1997; and Brykalov, 1999). The proposed model can be used to analyze competition of large-scale technological or energy transportation projects in general situations where the investment periods preceed the periods of sales and the appearence of every new supplier on market drastically effects the market price.

The paper is organized as follows. In Section 2 we give a mathematical formulation of the underlying problem of competition of gas pipelines and describe the basic model, the multi-player game of timing; also basic assumptions are introduced. In Section 3 we formulate and prove our main results concerned with the characterization of all players' best responses and Nash equilibrium points. In Section 4 we employ the results of Section 3 in order to reveal the 'finite-strategy' nature of the game; namely we show that in terms of the Nash equilibrium points, the original game, in which every player has an infinite number of strategies, is equivalent to a game with finite numbers of players' strategies. In Section 5 we introduce a regularity condition and for a regular game of timing prove the existence of a Nash equilibrium. Applying this result to a symmetric game of timing, we provide the entire description of the set of all Nash equilibrium points in this game. In section 6, basing on the results of Section 3, we justify two algorithms for identifying the players' best responses and finding all Nash equilibrium points in the game. Section 7 provides conclusions.

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2 Multi-Player Game of Timing

In this section we define our basic object, the game of timing. The game involves n players; $n \geq 2$. We view the players as investors or managers of n competing gas pipeline projects, which are expected to operate at a regional gas market. The investment process starts at time $t = 0$. Player i chooses a duration t_i of the investment period for project i, or the commercialization time for this project. As soon as the commercialization time t_i is chosen and fixed, it is decided that at time t_i the construction of pipeline i will be finished and the period of transporting gas to market through this pipeline will start. In this situiation gas transported through pipeline i is not available for sale before the comercialization time t_i and it is on sale at every time $t \geq t_i$ (we assume that the period of sales is infinite).

Let $C_i(t_i)$ be the total investmet needed for finilizing the construction of pipeline i at time t_i. We will deal with the cost functions $C_i(t_i)$ $(i = 1, \ldots, n)$ defined on $[0, \infty)$ and taking nonnegative values, and also with the cost reduction rates

$$a_i(t_i) = -C_i'(t_i). \tag{1}$$

Remark 1 Usually, the prolongation of the construction period reduces the construction cost; therefore, the cost functions $C_i(t_i)$ are usually decreasing (see Klaassen, et. al., 2004). Here we consider a more general situation and do not assume the monotonicity of $C_i(t_i)$.

We introduce the following assumption.

Assumption 1. Each cost function $C_i : [0, \infty) \mapsto (0, \infty)$ $(i = 1, \ldots, n)$ satisfies the following conditions:

(i) C_i is continuous;

(ii) C_i is continuously differentiable everywhere except of (possibly) a finite number of points, at which both one-sided derivatives of C_i exist and are finite,

(iii) there exists a finite right derivative of C_i at $t_i = 0$.

Note that Assumption 1 is satisfied if all cost functions C_i are continuously differentiable.

Let for every $i = 1, \ldots, n$, every set $H \subset \{1, \ldots, i-1, i+1, \ldots n\}$ and every $t > 0$, $b_{iH}(t)$ denote the benefit rate player i receives due to sales of gas at time t on the condition that at this time all pipelines $j \in H$ and only those operate on the market together with pipeline i. At the initial time when player i makes his/her decision on choosing his/her commercialization time t_i, he/she views the benefit rate $b_{iH}(t)$ as 'virtual', since he/she does not know if t will actually follow t_i, i.e., if pipeline i will actally operate on the market at time t. The benefit rate $b_{iH}(t)$ is determined by the cost of extraction of gas in the gas field i at time t, the cost of transportation of gas from this gas field to the market at time t, and also by the market price of gas at time t. The market price of gas at time t depends on the total gas supply; the latter, in turn, depends on H, the set of pipelines operating on the market at time t; the notation $b_{iH}(t)$ reflects the resulting impact of H on the benefit rate for player i.

Thus, we will deal with the benefit rate functions $b_{iH} : [0, \infty) \mapsto (-\infty, \infty)$ defined for every $i = 1, \ldots, n$ and every set $H \subset \{1, \ldots, i-1, i+1, \ldots n\}$.

Remark 2 Usually, the values $b_{iH}(t)$ are positive. Here we do impose this constraint and assume that the benefit rate functions b_{iH} can take both positive and negative values.

We assume the following.

Assumption 2. For every $i = 1, \dots, n$ and every set $H \subset \{1, \dots, i-1, i+1, \dots n\}$ the benefit rate function b_{iH} satisfies the following conditions:

(i) b_{iH} is continuous everywhere except of (possibly) a finite number of points, at which both one-sided limits of its values exist and are finite,

(ii) at $t = 0$, b_{iH} is continuous from the right.

Let us note that the more pipelines operate on the market, the larger is the total gas supply and the smaller is the price of gas; this effects the benefit rates. We reflect this in the following assumption.

Assumption 3. If $G \subset H \subset \{1, \dots, i-1, i+1, \dots n\}$, $G \neq H$ and $i \notin G$, then $b_{iG}(t) > b_{iH}(t)$ for all $t \geq 0$.

The set H determinig the benefit rate function b_{iH} can be empty: $H = \emptyset$. In this situation no players, except of player i, operate on the market, and $b_{i\emptyset}(t)$ represents the 'monopoly' benefit rate for this player. We assume that at the initial time for each player the rate of cost reduction exceeds the player's 'monopoly' benefit rate (in this context, see Assumption 2.2 and Remark 2.1 in Klaassen, et. al., 2004):

Assumption 4. For every $i = 1, \dots, n$ it holds that

$$a_i(0) > b_{i\emptyset}(0). \tag{2}$$

For every $i = 1, \dots, n$, let

$$A_i = \bigcup_{H \subset \{1, \dots, i-1, i+1, \dots n\}} \{t \geq 0 : a_i(t) = b_{iH}(t)\}. \tag{3}$$

Geometrically, A_i represents the set of the t coordinates of all points, at which the graph of the cost reduction rate a_i intersects the graphs of the benefit rates b_{iH} for all $H \subset \{1, \dots, i-1, i+1, \dots n\}$.

Our next assumption is the following.

Assumption 5. For every $i = 1, \dots, n$ the set A_i is finite.

Assumption 5 describes a generic situation: the cases where the graph of a_i intersects the graph of b_{iH} infiniely many times for a certain H are, clearly, exceptional.

We denote by D_i the set of all points t of discontinuity of the functions a_i and b_{iH} for all $H \subset \{1, \dots, i-1, i+1, \dots n\}$. As follows from Assumptions 1 and 2, the sets D_i $(i = 1, \dots, n)$ are finite.

Given players' commercialization times $t_1, \dots, t_n$, we denote by $G_i(t)$, or, more accurately, by $G_i(t|t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n)$ the set of all opponents of player i that occupy the market at time t:

$$G_i(t) = G_i(t|t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n) = \{j \neq i : t_j \leq t\}. \tag{4}$$

At every time $t \geq t_i$, the actual benefit rate $b_i(t)$ for player i is clearly determined by $G_i(t)$:

$$b_i(t) = b_i(t|t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n) = b_{iG_i(t)}(t). \tag{5}$$

The total benefit for player i is

$$B_i(t_1, \dots, t_n) = \int_{t_i}^{\infty} b_i(t|t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n)dt.$$

It is natural to require that the integrals are finite. We formulate this condition in the following equivalent way:

Assumption 6. For every $i = 1, \dots, n$

$$\left| \int_0^\infty b_i(t|t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n) dt \right| < \infty.$$

The total profit $P_i(t_1, \dots, t_n)$ of player i is defined as his/her total benefit minus the total investment in the construction of pipeline i:

$$\begin{aligned} P_i(t_1, \dots, t_n) &= -C_i(t_i) + B_i(t_1, \dots, t_n) \\ &= -C_i(t_i) + \int_{t_i}^\infty b_i(t|t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n) dt. \end{aligned} \tag{6}$$

We consider the following n-person *game of timing.* The set of strategies of player i in this game ($i = 1, \dots, n$) is the set of all positive reals t_i representing admissible commercialization times for project i. Any collection of players' strategies, $(t_1, \dots, t_n)$, determines the payoff $P_i(t_1, \dots, t_n)$ (6) to each player i; the payoff represents the total profit of player i received during the entire life period of project i, which includes the entire investment period and entire period of sailes for pipeline i.

3 Players' Best Responses

Let us recall two definitions of game theory and apply them to the considered multi-person game of timing. A strategy t_i of player i is called a best response of this player to strategies $t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n$ of players $1, \dots, i-1, i+1, \dots, n$ if

$$P_i(t_1, \dots, t_{i-1}, t_i, t_{i+1}, \dots, t_n) = \max_{s_i > 0} P_i(t_1, \dots, t_{i-1}, s_i, t_{i+1}, \dots, t_n). \tag{7}$$

Note that a point t_i at which the maximum is reached is generally not unique; therefore, each player i can have several best responses to a given collection of strategies of other players. A collection $(t_1, \dots, t_n)$ of players' strategies is called a Nash equilibrium if for every $i = 1, \dots, n$, t_i is a best response of player i to the strategies $t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n$ of players $1, \dots, i-1, i+1, \dots, n$ (see, e.g., Owen, 1968). A Nash equilibrium illustrates a situation, in which none of the players feels a need to change his/her strategy provided all other players keep their choices.

Below we will use the formula

$$\frac{\partial}{\partial t_i} P_i(t_1, \dots, t_n) = a_i(t_i) - b_i(t_i|t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n), \tag{8}$$

which follows from (6) and (1).

The next proposition specifying the location of the players' best responses is key in our analysis.

Proposition 1 *Let t_i be a best response of player i to strategies $t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n$ of players $1, \dots, i-1, i+1, \dots, n$. Then $t_i \in A_i \cup D_i$.*

Proof. For fixed $t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n$, the function

$$\varphi(t) = P_i(t_1, \ldots, t_{i-1}, t, t_{i+1}, \ldots, t_n)$$

where $t \in [0, \infty)$ is continuous and piecewise continuously differentiable due to Assumptions 1, 2, and formulas (6), (5) and (4). On every closed interval a continuously differentiable function reaches its maximum value either at the end points of the interval, or at its interior points, at which the derivative of the function is zero. Therefore, $\varphi(t)$ reaches its maximum value on $[0, \infty)$ at points, at which its derivative is either zero or discontinuous, and also at $t = 0$. It follows from (8) that

$$\varphi'(t) = a_i(t) - b_i(t|t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n). \tag{9}$$

Taking into account (5) and (4), we see that all maximim points of $\varphi(t)$ lie in one of the sets A_i, D_i and $E = \{0, t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n\}$.

Let us show that the maximum points of $\varphi(t)$ do not lie in $E \setminus (A_i \cup D_i)$. Indeed, it follows from (2) that the derivative (9) is positive at $t = 0$, which implies that zero is not a maximum point of $\varphi(t)$. Let us fix some $j \neq i$ and show that $\varphi(t)$. does not attain its maximum at $t = t_j$ if $t_j \notin A_i \cup D_i$. Take a $\delta > 0$ such that the interval $(t_j - \delta, t_j + \delta)$ does not intersect A_i and D_i, and the intervals $(t_j - \delta, t_j)$, $(t_j, t_j + \delta)$ do not intersect the set $\{t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n\}$. Let us use (5) and (4) and define the subsets G and H of $\{1, \ldots, i-1, i+1, \ldots, n\}$ by

$$b_i(t|t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n) = \begin{cases} b_{iG}(t) & \text{if} \quad t \in (t_j - \delta, t_j), \\ b_{iH}(t) & \text{if} \quad t \in [t_j, t_j + \delta). \end{cases}$$

Obviously, $G \subset H$, $j \notin G$, $j \in H$, $G \neq H$ and $i \notin H$. By Assumption 3

$$b_{iG}(t) > b_{iH}(t) \tag{10}$$

for all t. As $(t_j - \delta, t_j)$ does not intersect A_i, $\varphi'(t)$ (9) does not change its sign on $(t_j - \delta, t_j)$. Similarly, we find that $\varphi'(t)$ does not change its sign on $(t_j, t_j + \delta)$. If $\varphi'(t)$ (9) is negative on $(t_j - \delta, t_j)$, then φ is decreasing on this interval, and obviously, t_j is not a maximum point of $\varphi(t)$. Suppose $\varphi'(t)$ (9) is positive on $(t_j - \delta, t_j)$. Then

$$a_i(t) - b_{iG}(t) > 0 \tag{11}$$

for $t \in (t_j - \delta, t_j)$. We know that $(t_j - \delta, t_j + \delta)$ does not contain points from A_i. Consequently, (11) holds for all $t \in (t_j - \delta, t_j + \delta)$. For $t \in [t_j, t_j + \delta)$ we have $\varphi'(t) = a_i(t) - b_{iH}(t) > a_i(t) - b_{iG}(t) > 0$; here we used (9) and (10). Therefore, $\varphi(t)$ increases on $[t_j, t_j + \delta)$. Again we see that t_j is not a maximum point for $\varphi(t)$. We conclude that all maximum points of $\varphi(t)$ lie in $A_i \cup D_i$. The proposition is proved.

Remark 3 In the proof of Proposition 1, we did not use the assumption that the sets A_i and D_i are finite (see Assumptions 1, 2 and 5); we only used the fact that all points of these sets were isolated. The assumption that the sets A_i and D_i are finite will be essential for the algorithms described below.

Corollary 1 *If $(t_1, \ldots, t_n)$ is a Nash equilibrium, then $t_i \in A_i \cup D_i$ for all $i = 1, \ldots, n$.*

Proposition 2 *Let $i \in \{1, \ldots, n\}$ and there exist a $T_i > 0$ such that*

$$a_i(t) < b_{i\{1,\ldots,i-1,i+1,\ldots,n\}}(t) \tag{12}$$

for all $t > T_i$. Then for any strategies $t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n$ of players $1, \ldots, i-1, i+1, \ldots, n$ there exists a best response $t_i \in A_i \cup D_i$ of player i to these strategies.

Proof. Let us fix strategies $t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n$ of players $1, \ldots, i-1, i+1, \ldots, n$. We see from (8),(5) and (4) that for all sufficiently large t_i

$$\frac{\partial}{\partial t_i} P_i(t_1, \ldots, t_n) = a_i(t_i) - b_{i\{1,\ldots,i-1,i+1,\ldots,n\}}(t_i).$$

Thus, (12) implies that

$$\frac{\partial}{\partial t_i} P_i(t_1, \ldots, t_n) < 0$$

for all sufficiently large t_i. This shows that the maximum value of

$$\varphi(t_i) = P_i(t_1, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots, t_n)$$

on $[0, \infty)$ equals its maximum value on $[0, T]$ for some sufficiently large T. The continuous function $\varphi(t_i)$ reaches its maximum value at some point $t_i \in [0, T]$. From (2) it follows that $t_i > 0$. By Proposition 1 $t_i \in A_i \cup D_i$. The proof is completed.

4 Reduced Game of Timing

In the game of timing, each player has an infinite set of strategies (see Section 2). In this section we state that the game of timing is equivalent to an n-player game, in which each player has a finite number of strategies; the equivalence is understood as the fact that the sets of the Nash equilibria in both games coincide.

We assume that all conditions of Section 2, including Assumptions 1 - 6, are satisfied. We define the *reduced game of timing* as the n-player game, in which $S_i = A_i \cup D_i$ is the set of strategies of player i ($i = 1, \ldots, n$), and the payoff to player i, corresponding to an arbitrary collection $(t_1, \ldots, t_n) \in S_1 \times \ldots \times S_n$ of players' strategies, is given by $P_i(t_1, \ldots, t_n)$ (6). In the reduced game of timing, a strategy t_i of player i is called a best response of this player to strategies $t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n$ of players $1, \ldots, i-1, i+1, \ldots, n$ if

$$P_i(t_1, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots, t_n) = \max_{s_i \in S_i} P_i(t_1, \ldots, t_{i-1}, s_i, t_{i+1}, \ldots, t_n). \tag{13}$$

A collection $(t_1, \ldots, t_n)$ of players' strategies is called a Nash equilibrium in the reduced game of timing if for every $i = 1, \ldots, n$, t_i is a best response of player i to the strategies $(t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n)$ of players $1, \ldots, i-1, i+1, \ldots, n$ in this game. In this section, the game of timing defined in Section 2 will be referred to as the *original game of timing.*

Proposition 3 *A collection $(t_1, \ldots, t_n)$ of positive values is a Nash equilibrium in the original game of timing if and only if $(t_1, \ldots, t_n)$ is a Nash equilibrium in the reduced game of timing.*

Proof. Let N^1 be the set of all Nash equilibria in the original game of timing and N^2 be the set of all Nash equilibria in the reduced game of timing. We must show that $N^1 = N^2$.

Denote by $F_i^1(t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n)$ the set of all best responses of player i to strategies $t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n$ of players $1, \ldots, i-1, i+1, \ldots, n$ in the original game of timing, and by $F_i^2(t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n)$ the set of all best responses of player i to strategies $t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n$ of players $1, \ldots, i-1, i+1, \ldots, n$ in the reduced game of timing. By Proposition 1

$$F_i^1(t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n) \subset S_i. \tag{14}$$

Let us prove that

$$N^1 \subset N^2. \tag{15}$$

By the definition of a Nash equilibrium in the original game of timing we have

$$N^1 = \{(t_1, \ldots, t_n) : t_i \in F_i^1(t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n) \ (i = 1, \ldots, n)\}. \tag{16}$$

and by the definition of a Nash equilibrium in the reduced game of timing we have

$$N^2 = \{(t_1, \ldots, t_n) : t_i \in F_i^2(t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n) \ (i = 1, \ldots, n)\}. \tag{17}$$

Let $(t_1, \ldots, t_n) \in N^1$. By (16) $t_i \in F_i^1(t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n)$ for every $i = 1, \ldots, n$. Hence by (14) $t_i \in S_i$ for every $i = 1, \ldots, n$. Therefore, $t_i \in F_i^2(t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n)$ for every $i = 1, \ldots, n$. Consequently by (17) $(t_1, \ldots, t_n) \in N^2$. Since $(t_1, \ldots, t_n)$ is an arbitrary point in N^1, we conclude that the (15) holds.

Let us prove the opposite relation:

$$N^2 \subset N^1. \tag{18}$$

Take an arbitrary $(t_1, \ldots, t_n) \in N^2$. By (17) for every $i = 1, \ldots, n$ we have $t_i \in F_i^2(t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n)$; equivalently, $t_i \in S_i$ and (13) holds. By Proposition 1 the right-hand sides in (13) and in (7) coincide. Hence, $t_i \in F_i^1(t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n)$ for every $i = 1, \ldots, n$. By (16) $(t_1, \ldots, t_n) \in N^1$. Thus, (18) is stated. Now (15) and (18) yield $N^1 = N^2$. The proposition is proved.

5 Regular Game of Timing

As it is known in theory of games (see, e.g., Owen, 1968) a finite-strategy n-person game may have no Nash equilibrium. The reduced finite-strategy n-person game of timing having the same Nash equilibria as the original game of timing (see Proposition 3) has a specific structure; the latter can be used to carry out a detalied analysis of the issue of the existence of a Nash equilibrium in this game. Klaassen, et. al., 2004, established the existence of a Nash equilibrium in a general 2-player game of timing and provided an ultimate description of the set of the Nash equilibrium points in this game. In this section we suggest the first step in the analysis of conditions sufficient for the existence of a Nash equilibrium point in the n-person game of timing. Again, we suppose that Assumptions 1 – 6 introduced in Section 2 are fulfilled.

The game of timing will be said to be *regular* if for every $i = 1, \ldots, n$ the cost reduction function a_i is continuous and there is a permutation $(i_1, \ldots, i_n)$ of the n-tuple $(1, \ldots, n)$ such that

(i) for every $k, j = 1, \ldots, n$ such that $j \neq k$ the benefit rate function $b_{i_k H_k^j}$ where

$$H_k^j = \{i_1, \ldots, i_j\} \setminus \{i_k\} \tag{19}$$

is continuous;

(ii) for every $k = 1, \ldots, n$ there is the unique $s_k^* > 0$ such that $a_{i_k}(s_k^*) = b_{i_k H_k^k}(s_k^*)$;

(iii) $s_k^* \leq s_{k+1}^*$ $(k = 1, \ldots, n-1)$;

(iv) for every $k, j = 1, \ldots, n$ such that $j < k$ one has $a_{i_k}(t) \geq b_{i_k H_k^j}(t)$ for all $t \in [s_j^*, s_{j+1}^*]$ if $k > 1$ and for all $t \in [0, s_1^*]$ if $k = 1$;

(v) for every $k, j = 1, \ldots, n$ such that $j > k$ one has $a_{i_k}(t) \leq b_{i_k H_k^j}(t)$; for all $t \in [s_{j-1}^*, s_j^*]$ if $j < n$ and for all $t \geq s_j^*$ if $j = n$.

The permutation $(i_1, \ldots, i_n)$ will be called a *regular permutation of players* and the n-tuple $(s_1^*, \ldots, s_n^*)$ will be called the *strategy permutation* associated with $(i_1, \ldots, i_n)$.

Proposition 4 *Let the game of timing be regular, $(i_1,\dots,i_n)$ be a regular permutation of players and $(s_1^*,\dots,s_n^*)$ be the strategy permutation associated with $(i_1,\dots,i_n)$. Then $(t_1^*,\dots,t_n^*)$ where $t_{i_k}^* = s_k^*$ $(k=1,\dots,n)$ is a Nash equilibrium.*

Proof. Let $k \in \{1,\dots,n\}$ and

$$\varphi(t) = P_{i_k}(t_1^*,\dots,t_{i_k-1}^*,t,t_{i_k+1}^*,\dots,t_n^*).$$

It is sufficient to show that $t_{i_k}^* = s_k^*$ is a maximum point of $\varphi(t)$ on $[0,\infty)$. By (8), (5) and (4)

$$\varphi'(t) = a_{i_k}(t) - b_{i_k}(t) \tag{20}$$

where

$$b_{i_k}(t) = b_{i_k G_{i_k}(t)}(t),$$

$$G_{i_k}(t) = \{i_j : t_{i_j}^* \le t,\ i_j \ne i_k\} = \{i_j : s_j^* \le t,\ i_j \ne i_k\}$$

Let H_k^j be defined by (19). Since $s_j^* \le s_{j+1}^*$ for all $j=1,\dots,n-1$, we have

$$G_{i_k}(t) = \begin{cases} \emptyset & \text{if } \ t \in [0,s_1^*), \\ H_k^{j-1} & \text{if } \ t \in [s_{j-1}^*, s_j^*) \quad (j=2,\dots,n), \\ H_k^n & \text{if } \ t \ge s_n^*. \end{cases}$$

Hence,

$$b_{i_k}(t) = \begin{cases} b_{i_k\emptyset}(t) & \text{if } \ t \in [0,s_1^*), \\ b_{i_k H_k^{j-1}}(t) & \text{if } \ t \in [s_{j-1}^*, s_j^*) \quad (j=2,\dots,n), \\ b_{i_k H_k^n}(t) & \text{if } \ t \ge s_n^*. \end{cases}$$

First let consider $j=k$. It follows from the definition of H_k^j that $b_{i_k H_k^k}(t) = b_{i_k H_k^{k-1}}(t)$ for all $t \in [0,\infty)$. Hence,

$$a_{i_k}(s_k^*) = b_{i_k H_k^k}(s_k^*) = b_{i_k H_k^{k-1}}(s_k^*). \tag{21}$$

The fact that s_k^* is uniquely defined by (21) and Assumptions 3 and 4 yield that $a_{i_k}(t) > b_{i_k H_k^{k-1}}(t)$ for all $t \in [0,s_k^*)$ and every $k=1,\dots,n$.

If $k>1$, then by the definition of a regular game of timing (see (iv)) for every $j = 1,\dots,n$ such that $j<k$ we have $a_{i_k}(t) \ge b_{i_k H_k^{j-1}}(t)$ for all $t \in [s_{j-1}^*, s_j^*]$ if $j>1$ and for all $t \in [0,s_j^*]$ if $j=1$. Therefore, $a_{i_k}(t) \ge b_{i_k}(t)$ for all $t \in [0,s_k^*]$. Now (20) shows that $\varphi(t)$ is increasing on $[0,s_k^*]$. Using a similar argument, we state that $\varphi(t)$ is decreasing on $[s_k^*,\infty)$. Hence, s_k^* is a maximum point of $\varphi(t)$ on $[0,\infty)$. The proposition is proved.

Now we will show that the symmetric game of timing, in which all players are identical, is regular; on this basis we will describe the set of all Nash equilibrium points in this game. A formal definition is the following. The game of timing will be said to be *symmetric* if there are real-valued continuous functions $a, b^1,\dots,b^n$ on $[0,\infty)$ such that

(i) $a_i = a$ for every $i=1,\dots,n$;

(ii) $b^j(t) > b^{j+1}(t)$ $(t \ge 0)$ for every $j=1,\dots,n-1$;

(iii) for every $i,j=1,\dots,n$, and every $(j-1)$-element set $H \subset \{1,\dots,i-1,i+1,\dots,n\}$ it holds that $b_{iH} = b^j$;

(iv) for every $j=1,\dots,n$ there is the unique $\tau_j > 0$ such that $a(\tau_j) = b^j(\tau_j)$; moreover, $a(t) > b^j(t)$ for all $t \in [0,\tau_j)$ and $a(t) < b^j(t)$ for all $t > \tau_j$.

Remark 4 Assumptions (ii) and (iv) imply that $\tau_j < \tau_{j+1}$ $(j=1,\dots,n-1)$.

Proposition 5 *Let the game of timing be symmetric. Then*

1) *the game of timing is regular;*

2) *for every permutation $(i_1, \ldots, i_n)$ of the n-tuple $\{1, \ldots, n\}$ the collection $(t_1, \ldots, t_n)$ of players' strategies where $t_k = \tau_{i_k}$ $(k = 1, \ldots, n)$ is a Nash equilibrium;*

3) *every Nash equilibrium $(t_1, \ldots, t_n)$ has the structure described in statement 2, i.e., $t_k = \tau_{i_k}$ $(k = 1, \ldots, n)$ where $(i_1, \ldots, i_n)$ is a permutation of the n-tuple $\{1, \ldots, n\}$.*

Proof. 1. Let a and $b^1, \ldots, b^n$ be the functions introduced in the definition of the symmetric game of timing. Now we refer to the definition of the regular game of timing. Clearly, for every $i = 1, \ldots, n$ the cost reduction function $a_i = a$ is continuous. Let us show that $(1, \ldots, n)$ is a regular permutation of players and

$$(s_1^*, \ldots, s_n^*) = (\tau_1, \ldots, \tau_n) \tag{22}$$

is the regular strategy permutation associated with $(1, \ldots, n)$. The continuity of the functions $b^1, \ldots, b^n$ and assumption (iii) in the definition of the symmetric game of timing implies that assumption (i) in the definition of the regular game of timing is satisfied. From assumptions (iii), (iv) in the definition of the symmetric game of timing and Remark 4 obviously follows the validiy of assumptions (ii) and (iii) in the definition of the regular game of timing. Let us prove the validity of assumption (iv) in the definition of the regular game of timing. Take arbitrary $k, j = 1, \ldots, n$ such that $j < k$. Consider an arbitrary $t \in [s_j^*, s_{j+1}^*] = [\tau_j, \tau_{j+1}]$ if $k > 1$ or an arbitrary $t \in [0, s_1^*] = [0, \tau_1]$ if $k = 1$. We must state that

$$a_k(t) \geq b_{kH_k^j}(t) \tag{23}$$

where (see (19))

$$H_k^j = \{1, \ldots, j\} \setminus \{k\} = \{1, \ldots, j\}.$$

By assumptions (i) and (iii) in the definition of the symmetric game of timing

$$a_k(t) = a(t), \quad b_{kH_k^j}(t) = b^{j+1}(t); \tag{24}$$

since $t \leq \tau_{j+1}$, assumption (iv) in the definition of the symmetric game of timing yields $a(t) \geq b^{j+1}(t)$. Combining with (24), we arrive at (23). Thus, assumption (iv) in the definition of the regular game of timing is fulfilled. Similarly, we state the validity of assumption (v) in the definition of the regular game of timing. Thus, $(1, \ldots, n)$ is a regular permutation of players and (22) is the regular strategy permutation associated with $(1, \ldots, n)$. Statement 1 is proved.

2. Reordering the players and using statement 1, we find that an arbitrary permutation $(i_1, \ldots, i_n)$ of the n-tuple $\{1, \ldots, n\}$ is a regular permutation of players and the collection $(t_1, \ldots, t_n)$ of players' strategies where $t_k = \tau_{i_k}$ $(k = 1, \ldots, n)$ the regular strategy permutation associated with $(i_1, \ldots, i_n)$. By Proposition 4 $(t_1, \ldots, t_n)$ is a Nash equilibrium.

3. Let $(t_1, \ldots, t_n)$ be an arbitrary Nash equilibrium. By Corollary 1

$$t_i \in A_i \cup D_i \quad (i = 1, \ldots, n); \tag{25}$$

here D_i the set of all points of discontinuity of a_i and b_{iH} with $H \subset \{1, \ldots, i-1, i+1, \ldots n\}$, and A_i is given by (3). Since $a_i = a$ is continuous and b_{iH} coincides with one of the continuous functions $b^1, \ldots, b^n$, the set D_i is empty and $A_i = \{\tau_1, \ldots, \tau_n\}$. Thus, (25) is specified as

$$\{t_1, \ldots, t_n\} \subset \{\tau_1, \ldots, \tau_n\}. \tag{26}$$

Suppose $\{t_1, \ldots, t_n\} = \{\tau_1, \ldots, \tau_n\}$. Then $t_k = \tau_{i_k}$ $(k = 1, \ldots, n)$ where $(i_1, \ldots, i_n)$ is a permutation of the n-tuple $\{1, \ldots, n\}$, and statement 3 is proved.

Now suppose $\{t_1, \ldots, t_n\}$ is a struct subset of $\{\tau_1, \ldots, \tau_n\}$. We exclude this situation by contradiction and thus finalize the proof. Obviously, we have $t_i = t_k = \tau_m$ for some $i, k, m \in \{1, \ldots, n\}$, $i \neq k$. Let

$$\varphi(t) = P_i(t_1, \ldots, t_{i-1}, t, t_{i+1}, \ldots, t_n). \tag{27}$$

Since $t_i = \tau_m$ maximizes $\varphi(t)$ over $(0, \infty)$, we have

$$\varphi'(\tau_m) = 0. \tag{28}$$

By (8), (5) and (4)

$$\varphi'(t) = a_i(t) - b_i(t) = a(t) - b_i(t) \tag{29}$$

where

$$b_i(t) = b_{iG_i(t)}(t),$$
$$G_i(t) = \{j \neq i : t_j \leq t\}. \tag{30}$$

By assumption (iii) in the definition of the symmetric game of timing we have

$$b_{iG_i(t)}(t) = b^{p(t)+1}(t)$$

where $p(t)$ is the number of elements in $G_i(t)$. Now (29) is specified as

$$\varphi'(t) = a(t) - b^{p(t)+1}(t). \tag{31}$$

Suppose $m = 1$. Since $k \in G_i(\tau_m)$, it holds that $p(\tau_m) \geq 1$. Then by assumptions (iii) and (ii) in the definition of the symmetric game of timing

$$b^{p(\tau_m)+1}(\tau_1) < b^1(\tau_1)$$

and (31)

$$\varphi'(\tau_1) = a(\tau_1) - b^{p(\tau_m)+1}(\tau_1) > a(\tau_1) - b^1(\tau_1) = 0;$$

the latter equality follows from assumption (iv) in the definition of the symmetric game of timing. The obtained inequality $\varphi'(\tau_1) > 0$ contradicts (28).

Now suppose $m > 1$. If $p(\tau_m) \geq m$, we arrive at a contradiction using the same argument as in the case of $m = 1$ (in which we simply replace 1 by m). Consider the case where $p(\tau_m) \leq m - 1$. Let $p(\tau_m) < m - 1$. By assumptions (iii) and (ii) in the definition of the symmetric game of timing

$$b_i(\tau_m) = b_{iG_i(\tau_m)}(\tau_m) = b^{p(\tau_m)+1}(\tau_m) > b^m(\tau_m).$$

By (31)

$$\varphi'(\tau_m) = a(\tau_m) - b^{p(\tau_m)+1}(\tau_m) < a(\tau_m) - b^m(\tau_m) = 0$$

(see assumption (iv) in the definition of the symmetric game of timing). The obtained inequality $\varphi'(\tau_m) < 0$ contradicts (28).

Finally, let $p(\tau_m) = m - 1$. Referring to the definition of $G_i(t)$ (see (30)), we find that $G_i(t) = G_i(\tau_{m-1}) \subset G_i(\tau_m) \setminus \{k\}$ for all $t \in (\tau_{m-1}, \tau_m)$. Hence,

$$p(t) = p(\tau_{m-1}) \leq p(\tau_m) - 1 = m - 2 \quad (t \in (\tau_{m-1}, \tau_m)).$$

Then by assumption (ii) in the definition of the symmetric game of timing

$$b^{p(t)+1}(t) \geq b^{m-1}(t) \quad (t \in (\tau_{m-1}, \tau_m)). \tag{32}$$

By assumption (iv) in this definition for all $t \in (\tau_{m-1}, \tau_m]$ we have $a(t) < b^{m-1}(t)$, which in view of (32) implies

$$a(t) < b^{p(t)+1}(t) \quad (t \in (\tau_{m-1}, \tau_m)).$$

Now (31) yields that $\varphi'(t) < 0$ for all $t \in (\tau_{m-1}, \tau_m)$. Therefore,

$$\varphi(\tau_{m-1}) = \varphi(\tau_m) - \int_{\tau_{m-1}}^{\tau_m} \varphi'(t)dt > \varphi(\tau_m).$$

Coming back to the definition of $\varphi(t)$ (see (27)), we conclude that $t_i = \tau_m$ is not a best response of player i to the strategies $t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n$ of players $1, \ldots, i-1, i+1, \ldots, n$. Therefore, $(t_1, \ldots, t_n)$ is not a Nash equilibrium, which contradicts the assumption. Statement 3 is proved.

6 Solution Algorithms

Here we describe an algorithm that allows a player to find all his/her best responses to a given collection of strategies of other players, and an algoritm for verifying if a given collection of strategies forms a Nash equilibrium. With the help of these algorithms, one can find the set of all Nash equilibria in the game of timing. The algorithms are based on Proposition 1, Proposition 2 and Corollary 1. We suppose that all conditions given in Section 2, including Assumptions 1 - 6, are valid. Recalling Proposition 2, we also assume the following.

Assumption 7. For every $i = 1, \ldots, n$ there exists a $T_i > 0$ such that for all $t > T_i$ the inequality (12) holds.

Consider the following

Best Response Algorithm.
The input data of the algorithm include:
(i) an integer i located between 1 and n;
(ii) the cost function C_i of player i;
(iii) the benefit rate functions b_{iH} of player i for all $H \subset \{1, \ldots, i-1, i+1, \ldots, n\}$;
(iv) strategies $t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n$ of players $1, \ldots, i-1, i+1, \ldots, n$.
The output of the algorithm is a finite set S of all best responses of player i to strategies $t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n$ of players $1, \ldots, i-1, i+1, \ldots, n$.
The algorithm is organized as follows.
Step 1. Use definition (1) to find the cost reduction function a_i.
Step 2. Use definition (3) to find the finite set A_i.
Step 3. Find the finite set D_i of all points of discontinuity of a_i and b_{iH} for all $H \subset \{1, \ldots, i-1, i+1, \ldots, n\}$.
Step 4. Compose the union $A_i \cup D_i$.
Step 5. With the help of (6), (5) and (4) compute the values $v(s) = P_i(t_1, \ldots, t_{i-1}, s, t_{i+1}, \ldots, t_n)$ for all $s \in A_i \cup D_i$.
Step 6. Form the output set S as the collection of all maximizers of $v(s)$ as s runs through the set $A_i \cup D_i$.

Propositions 1, and 2 prove that the output S of the Best Response Algorithm is indeed the set of all best responses of player i to strategies $t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n$ of players $1, \ldots, i-1, i+1, \ldots, n$.

Recalling the definition of a Nash equilibrium, we easily find that the next algorithm verifies if a given collection of players's strategies is a Nash equilibrium.

Nash Equilibrium Verification Algorithm.

The input data of the algorithm include:

(i) the cost functions C_i and benefit rate functions b_{iH} for all $i = 1, \ldots, n$ and all $H \subset \{1, \ldots, i-1, i+1, \ldots, n\}$;

(ii) a collection $(t_1, \ldots, t_n)$ of players's strategies.

The output of the algorithm is YES if $(t_1, \ldots, t_n)$ is a Nash equilibrium, and NO otherwise.

The algorithm is organized as follows.

Step 1. Put $i := 1$.

Step 2. For player i and the strategies $t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n$ of players $1, \ldots, i-1, i+1, \ldots, n$ apply the Best Response Algorithm and find the set S of all best responses of player i to these strategies.

Step 3. If $t_i \notin S$, finish the work of the algorithm with the output NO.

Step 4. If $t_i \in S$ and $i < n$, set $i := i + 1$ and return to Step 2.

Step 5. If $t_i \in S$ and $i = n$, finish the work of algorithm with the output YES.

Remark 5 By Corollary 1 the set of all Nash equilibria $(t_1, \ldots, t_n)$ is finite; more accurately, it lies in the finite set $N = (A_1 \cup D_1) \times \ldots \times (A_n \cup D_n)$. Applying the Nash Equilibrium Verification Algorithm to all $(t_1, \ldots, t_n) \in N$, we find all Nash equilibria in the game of timing.

7 Conclusion

The paper develops a game-theoretical approach to planning investments within a group of competing large-scale projects. To be particular, we focus on gas pipeline projects competing for a regional gas market. Our approach assumes that investment policies are determined by projects' commercialization times, at which the construction periods are terminated and periods of sales start. The assumption is motivated by the observation that for each project the choice of its commercialization time determines the entire construction plan, including the optimal regime of the allocation of resources. We define a multi-player game of timing as a model of the investment process. In this game player's strategies are commercialization times and player's payoffs are the entire profits gained during the entire life periods of the corresponding gas pipeline projects. Our key result specifies the location of the player's best responses to any choice of the opponents. Namely, we show that generically the player's best responses are concentrated in a finite number of pre-determined points in time. This finding reveals a 'finite-strategy' nature of the game of timing and implies that the number of Nash equilibrium points in this game is finite. For a regular game of timing we prove the existence of a Nash equilibrium and for a symmetric game of timing provide the entire description of the set of all Nash equilibrium points. Finally, we construct a finite algorithm for finding player's best responses, and another finite algorithm that verifies if a given collection of players' strategies forms a Nash equilibrium. Using the latter algorithm one can find all Nash equilibria through the direct examination of a finite number of options.

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