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Investment, Uncertainty, and Cooperation

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Abstract

This paper explores some cooperative aspects of investments in uncertain, real options. Key production factors are assumed transferable. They may reflect property or user rights. Emission of pollutants and harvest of renewable resources are cases in point. Of particular interest are alternative projects or technologies that provide inferior but anti-correlated returns. Any such project stabilizes the aggregate proceeds. Therefore, given widespread risk exposure and aversion, that project's worth may embody an extra bonus.

The setting is formalized as a stochastic production game. Granted no economies of scale such games are quite tractable in analysis, computation, and realization. A core imputation comes in terms of contingent shadow prices that equilibrate competitive, endogenous markets. The said prices emerge as optimal dual solutions to coordinated production programs, featuring pooled resources - and also via adaptive procedures. Extra value - or an insurance premium - adds to any project whose yield is negatively associated with the aggregate.

Key words: investment, risk attitudes, insurance, covariance-pricing, cooperative games, core, stochastic optimization.

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Contents

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1 Introduction

The actual management of natural resources and ecosystems inspires great concerns with sustainability and welfare. Notably, the possible depletion of shared stocks, and the emission of harmful pollutants into commons, gives rise to legitimate worries about efficiency and equity.

Typically the related industries must make heavy investments ex ante that cannot easily be undone or recouped ex post. Also, net returns may be rather uncertain in magnitude and somewhat distant in time.[1] Together these facts beg for thorough investment analysis, emphasizing precaution and the value of keeping options alive.

For such analysis, presuming relevant data, several disciplines have much to offer. First, and maybe foremost, comes economics of finance and insurance. Second, since one cannot avoid computation altogether, there is, in principle, no escape from optimization theory. Third, and somewhat surprisingly, since exchanges may be implicit or lurking in the background, so-called production (or market) games can elucidate multi-agent interaction.

It is seldom though, to find all these ingredients in one and the same study. Most analysts contend with the restricted perspective (and the partial analysis) that suits a single agent, situated within well defined markets. Easily ignored then is the endogenous nature of allocations and prices. Also troublesome is the possible absence of markets for inputs, products or risks. Such absence greatly affects the willingness to invest in large-scale projects, promising fairly unpredictable and "belated" dividends.

This paper deals with projects of precisely that sort. Besides, it accommodates many agents, each acting in three roles: as consumer, investor and producer. At the outset each owns a separate project, offering him uncertain returns. Our main purpose is to "compute" the total and marginal value of investing in those projects. As will become clear, such values are interdependent, subject to contingencies, and determined endogenously. Especially interesting are projects whose net returns swing "out-of-phase" with the aggregate. Their yield is "up" precisely when the total proceeds are "down". That convenient feature confers extra mark-ups on their values.

Technologies of such "counter-cyclical" sort abound, but they tend to cost more. Examples include:

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[1]Important references include [1], [2], [6], [7], [8], and [17].

• selective gears for harvesting multi-species, multi-cohort fish stocks;
• diversification over crops and livestock, and ecological production in agriculture;
• electricity production based on wind, biomass or solar energy.

In each case the alternative technology is costlier to operate, but it better safeguards the environment.

For more concrete examples consider *two* arrangements for electricity generation. In the *first*, suppose all plants are driven by hydro-power. Some depend on highly correlated, short term precipitation; they are well furnished in chilly, wet years. Other, more expensive plants merely tap melting water under a glacier; they are best off in warm, dry years. Given aggregate supply commitments, the two groups can mutually insure each other. Such insurance ex post affects investments ex ante. In the *second* arrangement, suppose the base load be delivered by thermal/nuclear plants. Hydro-power then acts only as swing producer; it serves peak demand and receives a marked-up price. Again, prudent investments will be affected throughout the sector.

To study such features Section 2 introduces the prototypical agent. Section 3 places several of them into a two-stage, cooperative setting, affected by much uncertainty. Brought out there are core solutions ex ante and ex post, both determined by shadow prices. Section 4 offers some novel qualitative results, akin to covariance-pricing in finance. Section 5 illustrates a few insights about parallel projects, and Section 6 briefly discusses comparative risk advantages. Section 7 considers attainment of equilibrium in environmental games, and Section 8 concludes.

This paper addresses at least *three* types of readers. Included are first, *economists*, not quite aware of the rich opportunities production games offer, be it in positive or normative way; second, *finance/insurance analysts*, little concerned with Lagrangian duality; and finally, *optimizers*, not fully conversant with the cooperative aspects of such duality.

2 The Risk-Exposed Agent

Considered here is a prototypical agent, being at once consumer, investor, and producer. His decisions evolve step-wise as described next:

Right now install *capacity* or *capital* k.
$\rightsquigarrow$ Next, observe the *state* $\omega \in \Omega$ and a contingent resource *endowment* $e(\omega)$.
$\rightsquigarrow$ Thereafter, adjust the capital stock by Δk and acquire *resources* r.
$\rightsquigarrow$ Go on to produce *output* $f(k, \omega, \Delta k, r)$.
$\rightsquigarrow$ Finally, *consume* c and collect *payoff* $\pi(k, \omega, \Delta k, r, c)$.

Often $k \leq \bar{k}$ where the prescribed upper bound $\bar{k}$ reflects natural limits or some historical right bestowed on the agent at hand. Examples are manifold. The table below indicates some instances:

sector :	**choice** k :	**threshold** $\bar{k}$:	**uncertain** ω or $e(\omega)$
hydropower	water reservoir	environmental limit	precipitation
fisheries	catch capacity	historical right	stock abundance
pollution	abatement capacity	target level	nature's tolerance
agriculture	density of lifestock	regulation	disease/pest

In any case, the realization ω remains unknown at the time when k is chosen. By contrast, $\Delta k, r, c$ are contingent decisions; that is, they depend on the pair (k, ω).

At this point *two* modelling issues come up. The first concerns *dimensionality*, the second *uncertainty*. Regarding the first, we may easily accommodate several sorts of capacities, resources and products, increasing thereby the dimension of the decision spaces. Doing so entails, in principle, no additional cost, be it in analysis or presentation (albeit of course in computation).[2] In fact, the reader may choose freely whether to regard some items as vectors or real numbers. For intuition it simplifies though, to deal merely with one-dimensional spaces.[3]

The second modelling issue concerns perception of risk and uncertainty. This bears of course on what is known, unknown and unknowable [16]. We take a "simplistic" stand here, assuming that uncertainty amounts (and reduces) to a probabilistic description of possible ω-values. Then, at one extreme end, one may posit that the *probability distribution* of ω be known by each and every concerned party. This hypothesis is certainly convenient, but hardly realistic. So, in lack of such knowledge, at the other extreme (and decisively more realistic) end, one may have to contend with sequential realizations of ω, and the attending build-up of empirical statistics. We shall deal with both scenarios. In either case, E denotes the expectation operator with respect to ω. To bypass purely technical concerns with measurability and integrability, assume Ω finite. Also, attainment of extremal values is tacitly assumed.

While still facing uncertainty, the agent wants to maximize the expectation of his payoff $\pi(k, \omega, \Delta k, r, c)$. If operating in autarchy, isolated from others, he should

$$\text{maximize } \ E\pi(k, \omega, \Delta k, r, c) \ \text{ subject to } k \leq \bar{k}, \ r \leq e(\omega), \ c \leq f(k, \omega, \Delta k, r). \quad (1)$$

Here and elsewhere we do not mention evident sign restrictions like k, r, $c \geq 0$. Note that problem (1) has two stages. First, k must be sunk before knowing ω. Second, after ω and $e(\omega)$ are unveiled, the decision maker had better

$$\text{maximize } \pi(k, \omega, \Delta k, r, c) \text{ subject to } r \leq e(\omega) \text{ and } c \leq f(k, \omega, \Delta k, r).$$

Several sorts of uncertainty may prevail, be it in preferences, productivity, or resource abundance. Format (1) is generic and quite general, able to accommodate manifold instances. In particular, if capital adjustment comes at prohibitive cost, Δk is not mentioned. Similarly, when e is missing, r should be omitted.

[2] Indeed, when $k, \bar{k}$ are vectors in the same space, c reside in another linear space, and r maybe in a third, corresponding vector inequalities should be understood to hold componentwise. And, juxtaposition of two compatible vectors then implies that the standard inner product be executed.

[3] In extremis, one may read this paper as dealing merely with financial markets.

Admittedly, for realistic analysis, the planning horizon should extend beyond two stages. It had better do so partly, because uncertainty is unveiled gradually, over many steps - and partly, because there are repeated opportunities to take recourse actions. Such complexity notwithstanding, if one is willing to work in suitably high dimensions, problem format (1) is rich enough to comprise as many stages and commodities as deemed necessary; see [12]. For our purpose it suffices to think of all future intervals as compressed into a single second period. Doing so simplifies things, and more so when several agents come on stage simultaneously - as described in the next section. The short time horizon seems particularly appropriate there because agents, who face substantial uncertainty, hesitate to make more than limited contractual commitments into the future.

3 Stochastic Production Games

Accommodated henceforth is a fixed, finite set I of agents, each of the prototypical sort just described. Agent $i \in I$ proceeds step-wise, in the following order: First, he installs capacity k_i. Next, he observes ω and $e_i(\omega)$. Thereafter, he makes capital adjustment Δk_i, uses resource r_i, and produces output $f_i(k_i, \omega, \Delta k_i, r_i)$. Finally, it is time for him to consume c_i and collect payoff $\pi_i(k_i, \omega, \Delta k_i, r_i, c_i)$.

The interaction among several such agents is modelled next by various *cooperative games*. Each such game associates a real *value* V_S to every *coalition* $S \subseteq I$. The mapping $S \mapsto V_S$ is called the *characteristic function* of the game in question. A payment scheme $(u_i) \in \mathbb{R}^I$ is then said to be in the *core* iff it entails

$$\begin{array}{lll} \textit{Pareto efficiency}: & \sum_{i \in I} u_i = V_I & \text{and} \\ \textit{no blocking}: & \sum_{i \in S} u_i \geq V_S & \text{for all } S \subset I. \end{array}$$

The efficiency constraint requires that the overall value V_I be attained and fully shared. The no blocking constraint captures that a dissatisfied coalition S, offered merely $\sum_{i \in S} u_i < V_S$, would reject that proposal (or defect from the others).

Instead of autarchy, as described in Section 2, suppose now that the agents can trade/exchange inputs and outputs among themselves. If $\mathbf{k}_S := (k_i)_{i \in S}$ and ω are given at the second stage, then coalition S can achieve *ex post* value

$$v_S(\mathbf{k}_S, \omega) := \max \left\{ \begin{array}{l} \sum_{i \in S} \pi_i(k_i, \omega, \Delta k_i, r_i, c_i) \text{ subject to} \\ \sum_{i \in S} r_i \leq \sum_{i \in S} e_i(\omega), \ \ \sum_{i \in S} c_i \leq \sum_{i \in S} f_i(k_i, \omega, \Delta k_i, r_i), \end{array} \right. \tag{2}$$

maximum taken with respect to $(\Delta k_i, r_i, c_i)_{i \in S}$. *Ex ante* the same coalition could shoot for value

$$v_S(\bar{k}_S) := \max \left\{ E v_S(\mathbf{k}_S, \omega) : \sum_{i \in S} k_i \leq \sum_{i \in S} \bar{k}_i =: \bar{k}_S \right\}. \tag{3}$$

Clearly, these values, be it $V_S = v_S(\mathbf{k}_S, \omega)$ or $V_S = v_S(\bar{k}_S)$, are superadditive; that is,

$$V_S \geq V_{\mathbf{S}} + V_{S \setminus \mathbf{S}} \text{ whenver } \mathbf{S} \subsetneq S \subseteq I.$$

This inequality indicates gains to be had by coordinating investment and consumption. So we ask: *Can all cooperative benefits be achieved? If so, in what manner? Can some voluntary contract be instrumental? Is it easy to implement? And how does it affect investment decisions in the first place?*

Under some natural, standing assumptions all these questions have positive or constructive answers - as will be brought out next. Those assumptions include that *for each ω the contingent functions π_i, f_i be concave in the other variables, and that each π_i be increasing in (r_i, c_i).*

First-stage cooperation (3) anticipates that second-stage cooperation (2) will follow. Such anticipation makes it natural to disentangle one from the other:

Proposition 1. (Ex post contingent core solutions) *For the grand coalition $S = I$, given capacity profile $\mathbf{k} := (k_i)_{i\in I}$ and realization ω, let $\Lambda_r(\mathbf{k},\omega)$, $\Lambda_c(\mathbf{k},\omega)$,be a Lagrange multipliers associated to the first and second constraint in* (2). *This means that the standard Lagrangian $L(\mathbf{k},\omega,\Delta\mathbf{k},\mathbf{r},\mathbf{c},\lambda) :=$*

$$\sum_{i\in I}\left\{\pi_i(k_i,\omega,\Delta k_i,r_i,c_i)+\lambda_r\left[e_i(\omega)-r_i\right]+\lambda_c\left[f_i(k_i,\omega,\Delta k_i,r_i)-c_i\right]\right\},$$

after inserting $\lambda := [\lambda_r,\lambda_c] = [\Lambda_r(\mathbf{k},\omega),\Lambda_c(\mathbf{k},\omega)]$, should satisfy

$$\max_{\Delta\mathbf{k},\mathbf{r},\mathbf{c}} L(\mathbf{k},\omega,\Delta\mathbf{k},\mathbf{r},\mathbf{c},\lambda)\le v_I(\mathbf{k}_I,\omega). \tag{4}$$

If so, the state-dependent payment scheme $u_i(\mathbf{k},\omega) :=$

$$\max_{\Delta k_i,r_i,c_i}\left\{\pi_i(k_i,\omega,\Delta k_i,r_i,c_i)+\lambda_r(\mathbf{k},\omega)\left[e_i(\omega)-r_i\right]+\Lambda_c(\mathbf{k},\omega)\left[f_i(k_i,\omega,\Delta k_i,r_i)-c_i\right]\right\}$$

belongs to the core of the second-stage, contingent game having characteristic function $S\mapsto v_S(\mathbf{k}_S,\omega)$, as defined in (2). *When $\sum_{i\in I}[e_i(\omega), f_i(k_i,\omega,0,e_i(\omega))] >> 0$, there do exist Lagrange multipliers.* □

Let next

$$L(\mathbf{k},\omega,\Delta\mathbf{k},\mathbf{r},\mathbf{c},\lambda,\mu) := L(\mathbf{k},\omega,\Delta\mathbf{k},\mathbf{r},\mathbf{c},\lambda)+\sum_{i\in I}\mu(\bar{k}_i-k_i).$$

Proposition 2. (Ex ante core solution) *For the grand coalition $S = I$, suppose $\omega\mapsto\lambda_r(\omega),\lambda_c(\omega)$ are Lagrange multiplier profiles associated to the first and second constraint in (2), and that μ is associated to $\sum_{i\in I}k_i\le\sum_{i\in I}\bar{k}_i =: \bar{k}_I$. This means that*

$$\max_{(k_i)} E\max_{\Delta\mathbf{k},\mathbf{r},\mathbf{c}} L(\mathbf{k},\omega,\Delta\mathbf{k},\mathbf{r},\mathbf{c},\lambda,\mu)\le v_I(\bar{k}_I). \tag{5}$$

Then the payment scheme $U_i(\bar{k}_i,\sum_{j\neq i}\bar{k}_j) :=$

$$\max_{k_i} E\max_{\Delta k_i,r_i,c_i}\left\{\begin{array}{l}\pi_i(k_i,\omega,\Delta k_i,r_i,c_i)+\\ \lambda_r(\omega)\left[e_i(\omega)-r_i\right]+\\ \lambda_c(\omega)\left[f_i(k_i,\omega,\Delta k_i,r_i)-c_i\right]+\\ \mu(\bar{k}_i-k_i)\end{array}\right\} \tag{6}$$

constitutes an overall core solution for the game having characteristic function $S \mapsto v_S(\bar{k}_S)$, *as defined in* (3). *If* $\sum_{i \in I} \left[e_i(\omega), f_i(\bar{k}_i, \omega, 0, e_i(\omega)) \right] >> 0$ *almost surely, and* $\bar{k}_I > 0$, *there do exist Lagrange multipliers* $\omega \mapsto \lambda_r(\omega), \lambda_c(\omega)$ *and* μ. □

One should not be lured into thinking that commonplace multipliers, furnished by *necessary* optimality conditions for problems (2), (3), automatically generate core imputations via (6). Rather, what imports here is that assumptions (4), (5) have the nature of *sufficient* optimality conditions. To satisfy these it largely helps to have π_i, f_i concave for each ω.

The above propositions, proven in Appendix, show how agents, having *convex preferences* and *stochastic assets*, can pool inputs, outputs and endowments to smoothen and insure individual payoffs across eventualities and time [14]. Individual projects can thus share risks - and occasionally even eliminate them. In particular, this holds when parties are few and risks idiosyncratic, so that neither the law of large numbers nor the Arrow-Lind theorem apply [3]. The main instruments for risk sharing are endogenous prices. These emerge as Lagrange multipliers, and they equilibrate intrinsic markets [22]. At those markets all parties are construed as price-takers.

One may rightly claim that Propositions 1&2 merely "rediscover" - or, just apply - known welfare properties of competitive equilibrium, stemming from its residence within the core. That viewpoint is certainly useful, but not quite necessary. In fact, Shapley-Shubik's cooperative perspective on market games largely suffices [22]. Important and novel in that perspective is presence of two stages - and associated recourse options, exercised as events unfold. Particularly interesting are differential impacts of uncertainty on various projects. We address that issue next.

4 Covariance-Pricing of Projects

Let $\Delta\hat{k}_i, \hat{r}_i, \hat{c}_i$ denote optimal (so-called recourse) decisions, assumed unique and well defined, in (6). Write briefly $f_i(k_i, \omega)$ for $f_i(k_i, \omega, \Delta\hat{k}_i, \hat{r}_i)$. The two terms $\lambda_r e_i$ and $\lambda_c f_i(k_i)$ are in focus here. These record the reimbursements to i for his endowment and output respectively. To inquire about the nature of these pecuniary items, recall that two random variables X, Y are declared *negatively dependent* (or briefly $-$dependent) if for all values x, y

$$\Pr\left[X \le x, Y \le y\right] \le \Pr\left[X \le x\right] \Pr\left[Y \le y\right].$$

Equivalently, there is negative dependence if $\Pr\left[X \le x \middle| Y \le y\right] \le \Pr\left[X \le x\right]$. In short, knowing Y small, reduces the likelihood that X also be small. *Positive dependence* (briefly +dependence) obtains when $\Pr\left[X \le x, Y \le y\right] \ge \Pr\left[X \le x\right] \Pr\left[Y \le y\right]$ for all x, y.

Proposition 3. (Dependence and covariance [19]) *If* X, Y *are negatively (positively) dependent and* $D : \mathbb{R} \to \mathbb{R}$ *is strictly decreasing, then* $X, D(Y)$ *are oppositely dependent, and the sign of* $cov(X, Y)$ is -1 ($+1$ *respectively*). □

The multipliers λ_r and λ_c in (6) are random, but depend also on the aggregate endowment e_I and output $f_I := \sum_{i \in I} f_i$ respectively. Ceteris paribus that dependence shows up as "inverse demand curves" $e_I \mapsto \lambda_r(e_I)$, $f_I \mapsto \lambda_c(f_I)$. It is commonplace that such curves be decreasing. Not surprisingly, this property obtains here also:

Proposition 4. (Monotonicity of shadow prices) *The inverse demand curves* $e_I \mapsto \lambda_r(e_I)$, $f_I \mapsto \lambda_c(f_I)$ *are both decreasing. That is, almost surely*

$$[e_I - \bar{e}_I]\,[\lambda_r(e_I) - \lambda_r(\bar{e}_I)] \le 0 \text{ whenever } e_I \neq \bar{e}_I, \text{ and} \tag{7}$$

$$\left[f_I - \bar{f}_I\right]\left[\lambda_c(f_I) - \lambda_c(\bar{f}_I)\right] \le 0 \text{ whenever } f_I \neq \bar{f}_I. \tag{8}$$

Proof. The reduced Lagrangian function $\mathcal{L}(\mathbf{k}, \omega, \lambda) := \max_{\Delta\mathbf{k},\mathbf{r},\mathbf{c}} L(\mathbf{k}, \omega, \Delta\mathbf{k}, \mathbf{r}, \mathbf{c}, \lambda)$ is concave with respect to e_I and f_I. Consequently, the corresponding partial derivatives $\lambda_r = \frac{\partial}{\partial e_I}\mathcal{L}(\mathbf{k}, \omega)$ and $\lambda_c = \frac{\partial}{\partial f_I}\mathcal{L}(\mathbf{k}, \omega)$ are decreasing. □

Combining Proposition 3 with strict versions of inequalities (7), (8) we get:

Proposition 5. (Dependence between shadow prices individual supply) *Quite naturally suppose that the inverse demand curve* $e_I \mapsto \lambda_r(e_I)$ *be strictly decreasing. Then, if* e_I, e_i *are one-sided dependent,* λ_r, e_i *are oppositely dependent. So, if* e_I, e_i *are* $\mp$ *dependent, then* $cov(\lambda_r, e_i)$ *has opposite sign, whence*

$$E\left[\lambda_r e_i\right] \quad > (<) \;\; E\left[\lambda_r\right] E\left[e_i\right].$$

In finance jargon, $\mp$*dependence between* e_I, e_i*,confers a "*β*-value" on* e_i *of opposite sign. In short* $\mp$*dependence, yields a corresponding* $\pm$ *bonus.*
If $f_I \mapsto \lambda_c(f_I)$ *is strictly decreasing, quite similar statements obtain for* λ_c *and* f_I*.*
□

If for example, under autarchy, standard present-value calculations proves project i "in the money", but f_I, f_i are positively dependent, then it better be fairly "deep in the money." Ex ante k_i is valued at the margin by the formula

$$E\frac{\partial}{\partial k_i}\left[\pi_i(k_i, \omega, \Delta\hat{k}_i, \hat{r}_i, \hat{c}_i) + \lambda_c(\omega) f_i(k_i, \omega, \Delta\hat{k}_i, \hat{r}_i)\right].$$

Enters here, as a separate part, the commonplace covariance format of pricing:

$$E\left[\lambda_c \frac{\partial}{\partial k_i} f_i(k_i)\right] = E\left[\lambda_c\right] E\left[\frac{\partial}{\partial k_i} f_i(k_i)\right] + cov\left[\lambda_c, \frac{\partial}{\partial k_i} f_i(k_i)\right].$$

5 Parallel Production

For illustration consider now a simple, tractable instance where project i contributes a steady income flow φ_i over the time interval $[0, T_i]$. Since revenues are discounted continuously at rate $\rho > 0$, that project furnishes present value

$$\varphi_i \int_0^{T_i} e^{-\rho t} dt = \frac{\varphi_i}{\rho}\left\{1 - e^{-\rho T_i}\right\}.$$

Let $T_i := r_i/\varphi_i$ where $r_i \geq 0$ denotes the amount of resources devoted to project i. Consequently, presuming that capacity adjustment be impossible,

$$f_i(k_i, \omega, r_i) := \frac{\varphi_i}{\rho} \{1 - \exp[-\rho r_i/\varphi_i]\}$$

where $\varphi_i = \Phi_i(k_i, \omega) \geq 0$. For simple notation write $\varphi_I := \sum_{i \in I} \varphi_i$. Also for simplicity, take $\pi_i(k_i, \omega, c_i, r_i) = c_i$. Agent i receives endowment $e_i(\omega)$ at the second stage, this yielding aggregate $e_I := \sum_{i \in I} e_i$. At that stage, given $\mathbf{k}_S = (k_i)_{i \in S}$ and ω, coalition $S \subseteq I$ could achieve

$$v_S(\mathbf{k}_S, \omega) := \max_{(r_i)} \left\{ \sum_{i \in S} f_i(k_i, \omega, r_i) : \sum_{i \in S} r_i \leq \sum_{i \in S} e_i(\omega) \right\}$$

by pooling its members' objectives, technologies, and endowments. Let $\lambda_r(\omega) \geq 0$ be the Lagrange multiplier associated to $\sum_{i \in I} r_i \leq \sum_{i \in I} e_i(\omega)$. This state-dependent shadow price should satisfy

$$\sum_{i \in I} \max_{r_i} \{f_i(k_i, \omega, r_i) + \lambda_r(\omega) [e_i(\omega) - r_i]\} \leq v_I(\mathbf{k}, \omega).$$

Simple calculations show that $\lambda_r = \exp(-\rho e_I/\varphi_I)$ hence λ_r is strictly decreasing in e_I, and Proposition 5 applies:

• *If e_I, e_i are negatively (positively) dependent, then $cov(\lambda_r, e_i)$ has opposite sign.*

Simple calculations also show that

• *the aggregate endowment e_I is distributed according to production flows; that is, the optimal*

$$\hat{r}_i = \frac{\varphi_i}{\varphi_I} e_I.$$

Agents with relatively large φ_i will thus take substantial parts of e_I. Such linear sharing is known from mutual insurance when φ_i denotes the *risk tolerance* of agent i; see [4], [19], [23]. The advantages of pooling not perfectly correlated risks are evident. In fact, accidentally "starving" agents are helped by more fortunate fellows. The receivers will reciprocate once providence smiles to them. The resulting ex post payment

$$u_i(k, \omega) := \sup_{r_i} \{f_i(k_i, \omega, r_i) + \lambda_r(\omega) [e_i(\omega) - r_i]\}$$

gives the second-stage core solution. This payment has two terms: first, the production part

$$f_i(k_i, \omega, \hat{r}_i) = \varphi_i \int_0^{e_I/\varphi_I} e^{-\rho t} dt;$$

second, the net *financial transfer*

$$\lambda_r [e_i - \hat{r}_i] = e^{-\rho e_I/\varphi_I} \left[e_i - \frac{\varphi_i}{\varphi_I} e_I \right].$$

Since these transfers sum to zero, the second-stage, total value equals $\sum_{i\in I} f_i(k_i, \omega, \hat{r}_i)$. As one might expect,

• *the total "cake" increases by cooperation:*

$$\sum_{i\in I} f_i(k_i, \omega, \hat{r}_i) = \varphi_I \int_0^{e_I/\varphi_I} e^{-\rho t} dt > \sum_{i\in I} \varphi_i \int_0^{e_i/\varphi_i} e^{-\rho t} dt = \sum_{i\in I} f_i(k_i, \omega, e_i).$$

The last inequality holds for all $\mathbf{k}$ and ω. It stems from $\int_0^T e^{-\rho t} dt$ being strictly concave in T. Therefore, by Jensen's inequality

$$\frac{1}{\varphi_I} \sum_{i\in I} \varphi_i \int_0^{e_i/\varphi_i} e^{-\rho t} dt < \int_0^{e_I/\varphi_I} e^{-\rho t} dt,$$

this attesting to the advantage of pooling resources. Uncertainty is likely to enforce this feature, and especially so when all risks are idiosyncratic. To wit, for illustration,

• *if each individual endowment e_i is random, but their sum e_I is not, the gains from cooperation are twofold: They stem first, from substitutions and second, from smoothing:*

$$E \sum_{i\in I} f_i(k_i, \omega, \hat{r}_i) = \varphi_I \int_0^{e_I/\varphi_I} e^{-\rho t} dt = \varphi_I \int_0^{\sum_i Ee_i/\varphi_I} e^{-\rho t} dt$$
$$> \sum_{i\in I} \varphi_i \int_0^{Ee_i/\varphi_i} e^{-\rho t} dt > E \sum_{i\in I} \varphi_i \int_0^{e_i/\varphi_i} e^{-\rho t} dt = E \sum_{i\in I} f_i(k_i, \omega, e_i).$$

6 Comparative Risk Advantage

David Ricardo - studying international trade, scarce resources, and division of labor - demonstrated that trading nations all gain by specializing in goods of comparative advantages [5]. This section takes up a similar issue. Here however, merely *one* good comes into play. Accordingly, one might expect that production largely and best be undertaken by the most efficient agent. Under uncertainty this need not be so. Indeed, rather inefficient producers may warrant premiums as suppliers of stability and insurance. This feature becomes particularly pronounced when some inefficient party's risk is out of line with others.

To illustrate in a simple setting, assume there is no capacity limit ($\bar{k}_i = +\infty$), no capacity adjustments ($\Delta k_i = 0$), and no endowment ($e_i = 0$). Also, instead of payoff consider cost $C_i := -\pi_i$ with

$$C_i(k_i, \omega, c_i) = \kappa_i k_i + \kappa \left[d_i - c_i\right]^+ . \quad (9)$$

The operator $[r]^+ := \max\{0, r\}$ preserves the positive part of the real number r. The parameter d_i is construed as i's "inelastic demand". If endogenous supply c_i falls short of demand d_i, the residual amount $d_i - c_i$ must be procured from external sources at unit cost κ. Let

$$f_i(k_i, \omega) = k_i P_i^{b_i(\omega)}$$

where the parameter $P_i \in (0,1)$ is prescribed, and $b_i(\omega) \in \{0,1\}$ is a binomial variable. The latter takes the value 1 with probability $p_i \in [0,1)$, leaving then only the proportion P_i of k_i intact. With complementary probability $\bar{p}_i := 1 - p_i$, all of k_i remains productive. *Under autarchy* agent i will maximize expected cost

$$E \max_{c_i} C_i(k_i, \omega, c_i) = \kappa_i k_i + \kappa \left\{ p_i \left[d_i - P_i k_i \right]^+ + \bar{p}_i \left[d_i - k_i \right]^+ \right\}$$

with respect to k_i. In that optic his best choice

$$k_i = \begin{cases} 0 & \text{if } \kappa_i > \kappa \{ p_i P_i + \bar{p}_i \} \\ d_i & \text{if } \kappa p_i P_i \leq \kappa_i \leq \kappa \{ p_i P_i + \bar{p}_i \} \\ d_i / P_i & \text{if } \kappa_i < \kappa p_i P_i. \end{cases}$$

Next open up for cooperation; that is, for *free trade.* That opening will minimize

$$F(\mathbf{k}) := \sum_{i \in I} \kappa_i k_i + \kappa E \left\{ \min_{(c_i)} \sum_{i \in I} \left[d_i - c_i \right]^+ : \sum_{i \in I} c_i \leq \sum_{i \in I} k_i P_i^{b_i(\omega)} \right\}$$

with respect to (k_i). Equivalently, trade minimizes

$$\tilde{F}(\mathbf{k}) := \sum_{i \in I} \kappa_i k_i + \kappa E \left\{ \min_{(c_i)} \left[\sum_{i \in I} (d_i - c_i) \right]^+ : \sum_{i \in I} c_i \leq \sum_{i \in I} k_i P_i^{b_i(\omega)} \right\}. \tag{10}$$

Indeed, a minimizing (c_i) in $F(\mathbf{k})$ satisfies $c_i \leq d_i$ for all i, whence $\sum_{i \in I} [d_i - c_i]^+ = \left[\sum_{i \in I} (d_i - c_i) \right]^+$ and $\tilde{F}(\mathbf{k}) \leq F(\mathbf{k})$. Conversely, given a minimizing profile $(\tilde{c}_i)$ in $\tilde{F}(\mathbf{k})$ there exists for each i a $c_i \leq d_i$ such that $\sum_{i \in I} c_i = \sum_{i \in I} \tilde{c}_i$. Hence $\left[\sum_{i \in I} d_i - \tilde{c}_i \right]^+ = \sum_{i \in I} [d_i - c_i]^+$ so that $F(\mathbf{k}) \leq \tilde{F}(\mathbf{k})$.

For the sake of transparency, suppose finally that there be only two agents. Objective (10) then takes the reduced form

$$\kappa_1 k_1 + \kappa_2 k_2 + \kappa E \left[d_1 + d_2 - k_1 P_1^{b_1(\omega)} - k_2 P_2^{b_2(\omega)} \right]^+. \tag{11}$$

We assume $\kappa_1 < \kappa_2 < \kappa$ and briefly discuss three cooperative cases next:
1) *Absent uncertainty*, when $p_1 = p_2 = 0$, we get $k_1 = d_1 + d_2$ and $k_2 = 0$. Then, to no surprise, *the most efficient agent produces all.*

2) *Only the efficient agent is at risk*; that is, $p_1 > 0, p_2 = 0$, in which case (11) specializes to

$$\kappa_1 k_1 + \kappa_2 k_2 + \kappa \left\{ p_1 \left[d_1 + d_2 - k_1 P_1 - k_2 \right]^+ + \bar{p}_1 \left[d_1 + d_2 - k_1 - k_2 \right]^+ \right\}.$$

Note that when total supply equals $d_1 + d_2$, the expected marginal production costs are $\kappa_1 - \kappa(p_1 P_1 + \bar{p}_1)$ for agent 1 and $\kappa_2 - \kappa$ for agent 2. If the latter is smaller, then, somewhat surprisingly, *the cost efficient agent better be inactive, leaving production entirely to his high-cost associate* (i.e. $k_1 = 0$, $k_2 = d_1 + d_2$). The latter is able to compensate qua insurer for his own handicaps qua producer.

3) *Both are risk exposed, but in perfectly opposed manner*; that is, $p_1 > 0, p_2 = 1 - p_1$, and $b_1(\omega) + b_2(\omega) = 1$. Then (11) reads

$$\kappa_1 k_1 + \kappa_2 k_2 + \kappa \left\{ p_1 \left[d_1 + d_2 - k_1 P_1 - k_2 \right]^+ + p_2 \left[d_1 + d_2 - k_1 - k_2 P_2 \right]^+ \right\}$$

The structure of the optimal solution is similar to the preceding case, but outside procurement could become more attractive. In particular, if $\kappa_1 > \kappa(p_1 P_1 + \bar{p}_1)$ and $\kappa_2 > \kappa(p_2 P_2 + \bar{p}_2)$, it is not worth anyone's while to produce.

Nonsmooth objectives like (9), reflecting "hit-or-miss" situations, inspire new measures of risk, notably so-called *Conditional Value-at-Risk* [21].

7 Environmental Games and Quota Trade

As noted, the core imputation (6) reduces essentially to competitive equilibrium in endogenous markets for capital and contingent commodities. One can hardly presume that human-like players, holding imperfect information/competence, will reach such equilibrium right away. More realistically, they must adapt and learn. The classical branch of economics that deals with competitive markets, fails however, to account for necessary adaptation and learning.

The simplicity of our setting invites reconsideration of these matters. But first it is time to address a related question, already invoked, namely: to what extent is uncertainty described or formalized? At this point the necessary prerequisites are few and reasonable. To wit, suppose a discrete-time process $\omega^t, t = 0, 1, \ldots$ of independent random variates, all distributed as the underlying ω, can be simulated or observed step by step. Then, since endowments and outputs are fixed ex post - and since income effects are negligible or ignored - there should be good prospects for reaching a stable equilibrium over time. Indeed, recent studies explore the convergence of repeated, bilateral exchange towards an efficient steady state [9], [10], [11], [15].

At this point the only coupling constraint $k_I := \sum_{i \in I} k_i \leq \sum_{i \in I} \bar{k}_i =: \bar{k}_I$ becomes crucial. Suppose that $\bar{k}_I$ reflects an aggregate upper bound on the catch of valuable fish, say - or on the emission of greenhouse gases. Hence $k_I = \bar{k}_I$ holds throughout, and payoffs $\pi_i(\bar{k}_I, \cdot)$ could depend on $\bar{k}_I$.

An environmental game thus unfolds in which players trade quotas ex ante and contingent commodities ex post. While adjustment of quotas is sluggish, the other variables are easily and quickly changed. So, to simplify, suppose that second-stage markets clear "instantaneously," and that $\Delta k_i = 0$ there. As a by-product this clearing generates Lagrange multipliers $\lambda_c(\omega)$. It also gives reduced functions $\pi_i(k_i, \omega) := \pi_i(k_i, \omega, \hat{r}_i, \hat{c}_i)$ and $f_i(k_i, \omega) := f_i(k_i, \omega, \hat{r}_i, \hat{c}_i)$. Let $s_t > 0$ be a sequence of step sizes selected a priori subject to

$$\sum_{t=0}^{\infty} s_t = +\infty, \quad \sum_{t=0}^{\infty} s_t^2 < +\infty.$$

Begin at time $t = 0$, with $s = s_0$, $\omega = \omega^0$, $k_i = \bar{k}_i$, and select two agents $i, j \in I$ at random. From there on the process could evolve iteratively as follows:

• The two agents hold stocks k_i and k_j, respectively. Calculate their realized marginal returns on capital:

$$m_i := \frac{\partial}{\partial k_i}\left[\pi_i(k_i,\omega) + \lambda_c(\omega) f_i(k_i,\omega)\right], \quad m_j := \frac{\partial}{\partial k_j}\left[\pi_j(k_j,\omega) + \lambda_c(\omega) f_j(k_j,\omega)\right].$$

Transfer $s(m_i - m_j)$ to i from j, this giving the two parties new holdings:

$$k_i \leftarrow k_i + s(m_i - m_j) \quad \text{and} \quad k_j \leftarrow k_j + s(m_j - m_i).$$

• Increase time t by 1, update $s \leftarrow s_t$, and observe a new independent $\omega \leftarrow \omega^t$.
• Continue to pick pairs of agents until convergence.

Convergence obtains as in [9], [10], [15]. Note that trade is voluntary and driven by perceived prospects for mutual improvements. It happens out of equilibrium, uses money as instrument, and requires no private information to be revealed. While still away from equilibrium, the price - and the associated monetary compensation - that goes along with a bilateral exchange could result from bargaining. If so, it would be hard to predict, but depend on the difference, as reflected in $m_i - m_i$, between i's willingness to buy and sell [18].

8 Concluding Remarks

While the preceding model were expressly stylized, extensions can easily incorporate much realism and detail. But the simple version brings out already that stochastic production (or market) games offer manifold opportunities to put much of economic theory, applied mathematics and computer science jointly to good use. A fortiori this holds in quasi-markets or market-like settings affected by sequential decisions and much uncertainty. Several theories, and attending practices, then come on stage simultaneously. And they supplement each other. Included are finance, insurance, stochastic optimization and Mont Carlo simulation. These disciplines have complementary concerns and perspectives. Together they facilitate a rich analysis - be it positive or normative - of how players would/should fare. Particularly important is absence or incompleteness of markets. Internal exchanges, of perfectly Walrasian sort, may then provide some mitigation. Presence of public goods/bads - or widespread externalities - need not preclude coordination and relative efficiency. Troublesome though, are economies of scale in which case Lagrange multipliers may not exist [12].

Especially important are concerns with expandability and reversibility of capacity choice [1], [2], [6], [7], [8], [17]. Such concerns could also be studied within the frames of stochastic production games. In particular, these allow estimates of the value of perfect information [20].

9 Appendix

Proof of Proposition 1: Recall that any bivariate, real-valued function $\mathbb{L}(\xi,\lambda)$ satisfies

$$\min_{\lambda}\max_{\xi}\mathbb{L}(\xi,\lambda) \geq \max_{\xi}\min_{\lambda}\mathbb{L}(\xi,\lambda).$$

(This inequality, named *weak duality* in optimization theory, reflects the last-mover advantage in zero-sum, two-person, noncooperative games.) In the present context, at the second stage, given $\mathbf{k} = (k_i)$ and ω, let $\xi := (\Delta k_i, r_i, c_i)_{i\in S}$, and associate the contingent Lagrangian

$$L_S(\xi,\lambda) := \sum_{i\in S} \{\pi_i(k_i,\omega,\Delta k_i, r_i, c_i) + \lambda_r\,[e_i(\omega) - r_i] + \lambda_c\,[f_i(k_i,\omega,\Delta k_i, r_i) - c_i]\}$$

to coalition S. Note that

$$\min_{\lambda\geq\mathbf{0}} L_S(\xi,\lambda) = \sum_{i\in S} \pi_i(k_i,\omega,\Delta k_i, r_i, c_i)$$

if $\sum_{i\in S} r_i \leq \sum_{i\in S} e_i(\omega)$ & $\sum_{i\in S} c_i \leq \sum_{i\in S} f_i(k_i,\omega,\Delta k_i, r_i)$; otherwise the minimal value equals $-\infty$. Consequently,

$$\max_{\xi}\min_{\lambda\geq\mathbf{0}} L_S(\xi,\lambda) = v_S(\mathbf{k}_S,\omega).$$

When $\Lambda \geq 0$ is a Lagrange multiplier, coalition S will not block the proposed payment scheme because

$$\sum_{i\in S} u_i(\mathbf{k},\omega) = \max_{\xi} L_S(\xi,\Lambda) \geq \min_{\lambda\geq\mathbf{0}}\max_{\xi} L_S(\xi,\lambda) \geq \max_{\xi}\min_{\lambda\geq\mathbf{0}} L_S(\xi,\lambda) = v_S(\mathbf{k}_S,\omega).$$

In particular, $\sum_{i\in I} u_i(\mathbf{k},\omega) \geq v_I(\mathbf{k},\omega)$. Since the converse inequality holds by assumption, Pareto efficiency also obtains. When $e_I(\omega)$ and $\sum_{i\in I} f_i(k_i,\omega,0,e_i(\omega))$ are both positive, the Slater condition holds - hence existence of multipliers is ensured. $\square$

Proof of Proposition 2: Associate here another Lagrangian $\mathcal{L}_S :=$

$$\sum_{i\in S} \left\{\pi_i(k_i,\omega,\Delta k_i, r_i, c_i) + \lambda_r\,[e_i(\omega) - r_i] + \lambda_c\,[f_i(k_i,\omega,\Delta k_i, r_i) - c_i] + \mu(\bar{k}_i - k_i)\right\}$$

to coalition S. Note that

$$\min_{\mu\geq 0} E \min_{\lambda\geq\mathbf{0}} \mathcal{L}_S = E \sum_{i\in S} \pi_i(k_i,\omega,\Delta k_i, r_i, c_i)$$

if $\sum_{i\in S} r_i \leq \sum_{i\in S} e_i(\omega)$ & $\sum_{i\in S} c_i \leq \sum_{i\in S} f_i(k_i,\omega,\Delta k_i, r_i)$ almost surely, and $\sum_{i\in S} k_i \leq \sum_{i\in S} \bar{k}_i$; otherwise the minimal expected value equals $-\infty$. Consequently, still writing $\xi = (\Delta k_i, r_i, c_i)_{i\in S}$,

$$\max_{\mathbf{k}}\min_{\mu\geq 0} E \max_{\xi}\min_{\lambda\geq\mathbf{0}} \mathcal{L}_S = v_S(\bar{k}_S).$$

When Λ, μ are Lagrange multipliers, coalition S will not block the proposed payment scheme (U_i) because $\sum_{i\in S} U_i =$

$$\max_{\mathbf{k}} E \max_{\xi} \mathcal{L}_S(\Lambda,\mu) \geq \min_{\mu\geq 0}\max_{\mathbf{k}} E \min_{\lambda\geq\mathbf{0}}\max_{\xi} \mathcal{L}_S \geq \max_{\mathbf{k}}\min_{\mu\geq 0} E \max_{\xi}\min_{\lambda\geq\mathbf{0}} \mathcal{L}_S = v_S(\bar{k}_S).$$

In particular, $\sum_{i\in I} U_i \geq v_I(\bar{k}_I)$. Since the converse inequality holds by assumption, Pareto efficiency again obtains. The presumed positivity guarantees that the Slater condition holds. $\square$

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