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Multiequilibrium Game of Timing and Competition of Gas Pipeline Projects[1]

G. Klaassen,[2] A. V. Kryazhimskii,[3] and A. M. Tarasyev[4]

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Abstract. The paper addresses the issue of the optimal investments in innovations with strong long-term aftereffects. As an example, investments in the construction of gas pipelines are considered. The most sensitive part of an investment project is the choice of the commercialization time (stopping time), i.e., the time of finalizing the construction of the pipeline. If several projects compete on the market, the choices of the commercialization times determine the future structure of the market and thus become especially important. Rational decisions in this respect can be associated with Nash equilibria in a game between the projects. In this game, the total benefits gained during the pipelines life periods act as payoffs and the commercialization times as strategies. The goal of this paper is to characterize multiequilibria in the game of timing. The case of two players is studied in detail. A key point in the analysis is the observation that, for all players, the best response commercialization times concentrate at two instants that are fixed in advance. This reduces decisionmaking to choosing between two fixed investment policies (fast and slow) with the prescribed commercialization times. A description of a simple algorithm that finds all the Nash equilibria composed of fast and slow scenarios concludes the paper.

Key Words. Optimal stopping problem, game of timing, multiequilibria, best reply curves, econometric estimation.

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1. Introduction

When several large-scale gas pipeline projects compete for a new gas market, the choices of the commercialization times (stopping times), i.e., the times of finalizing the construction of the pipelines, determine the future structure of the market and thus become especially important. In Ref. 1, which motivated the present study, a detailed pipeline model based on classical patterns of mathematical economics (see Refs. 2–3) was designed and a best reply dynamic adaptation algorithm originating from the theory of evolutionary games (see Refs. 4–9) was used to estimate numerically the commercialization times for the pipeline projects competing nowadays for the Turkey gas market.

Rational choices of the commercialization times can be viewed as Nash equilibria in a game between the projects. In the present paper, we study the structure of this game. In order to make the model easily tractable in terms of game theory (see Refs. 10–11), we introduce several simplifying assumptions; in particular, we reduce the number of competing projects to two. A background in the analysis of gas infrastructures (see Ref. 12) and problems of optimal timing (see Refs. 13–14) is employed.

The model takes into account the stages of construction and exploitation of the gas pipelines. In each level, the model is optimized and estimated using appropriate techniques of the theory of optimal control and theory of differential games (see Refs. 15–17). At the stage of exploitation, as gas supply policies compete on the market, decisionmaking is relatively clear: the competitors search for an equilibrium supply at any instant. Therefore, we focus on the stage of construction, at which investment policies compete and decisionmaking is concerned with strong long-term aftereffects. The competitors interact through choosing their commercialization times. A proper individual choice is the best response to the choices of the other competitor. Therefore, a pair of commercialization times is suitable to every competitor if and only if the commercialization time of every competitor responds best to the commercialization time of the other competitor. Such situations constitute Nash equilibria in the game under consideration. In this game, the total benefits gained during the pipelines life periods act as payoffs and the commercialization times act as strategies. Our goal is to characterize the equilibria in this game, which will be referred to further as game of timing.

In Section 2, we describe the general two-player game of timing, in which the cost and benefit functions determining the players payoffs are not specified. In Section 3, we find the Nash equilibria in the game. A key point in the analysis is the observation that, for all players, the best response commercialization times concentrate at two instants that are fixed in advance.

This reduces decisionmaking to choosing between two fixed investment policies (fast and slow) with the prescribed commercialization times.

In Section 4, we describe an algorithm that finds all the Nash equilibria in the game of timing. In Section 5, we study the game of timing for the model of operation of gas pipelines which was described in Ref. 1.

In Section 6, we give results of the model-based analysis for two case studies: competing gas pipeline projects in the Caspian region and planned pipeline routes to the gas market in China. Finally, Section 7 contains the proofs of the propositions formulated in Section 4.

2. Game of Timing

In this section, we construct a game-theoretic model of competition of two gas pipeline projects. We call it the game of timing. The pipelines are expected to operate at the same market. We associate players 1 and 2 with the investors/managers of projects 1 and 2, respectively. Assuming that the starting time for making investments is $t=0$, we consider virtual positive commercialization times t_1 and t_2 of projects 1 and 2 (i.e., the final times of the construction of the pipelines). Given a (virtual) commercialization time t_i, player i, $i=1, 2$, can estimate the cost $C_i(t_i)$ for finalizing project i at time t_i. The positive-valued cost functions $C_i(t_i)$, $i=1, 2$, are defined on the positive half axis. The following assumption will simplify our analysis.

Assumption 2.1. For each player i, the cost function $C_i(t_i)$ is smooth (continuously differentiable), monotonically decreasing, and convex.

A formal interpretation of Assumption 2.1 is that the derivative $C_i'(t_i)=dC_i(t_i)/dt_i$ is negative and increasing. A substantial interpretation is that the cost of the project falls down as the project commercialization period is prolonged; moreover, the longer is the commercialization period, the less sensitive, with respect to its prolongations, is the rate of cost reduction. In what follows, the rate of cost reduction for player i is understood as the positive-valued monotonically decreasing function

$$a_i(t_i) = -C_i'(t_i). \tag{1}$$

Let us argue for player 1 as the manager of pipeline 1. At any time $t>0$, the price of gas and costs for extraction and transportation of gas determine the benefit rate $b_1(t)$ of player 1 (note that this benefit rate is virtual, because t may precede the actual commercialization time of project 1). The costs for extraction and transportation of gas do not depend on the state of project 2, whereas the price of gas depends on the presence (absence) of player 2 on the marketplace. In the situation where both players operate on the market, the price of gas should obviously be smaller compared to the situation where player 1 occupies the market solely. Hence, the benefit rate $b_1(t)$ may take two values, $b_{11}(t)$ and $b_{12}(t)$, with

$$b_{11}(t) > b_{12}(t). \tag{2}$$

We call $b_{11}(t)$ the upper benefit rate and $b_{12}(t)$ the lower benefit rate of player 1 at time t. At time t (which virtually follows the commercialization time of player 1), player 1 virtually gets $b_{11}(t)$, if player 2 does not operate on the market, and gets $b_{12}(t)$, if player 2 operates on the market. Similarly, we introduce the upper and lower benefit rates of player 2 at time t, $b_{21}(t)$ and $b_{22}(t)$, with

$$b_{21}(t) > b_{22}(t). \tag{3}$$

A time t, player 2 gets $b_{21}(t)$, if player 1 does not operate on the market, and gets $b_{22}(t)$, otherwise. We assume that the positive-valued upper and lower benefit rates $b_{i1}(t)$ and $b_{i2}(t)$, $i = 1, 2$, are continuous functions defined on the positive half axis. We introduce also the following assumption.

Assumption 2.2. For every player i, $i = 1, 2$, the graph of the rate of cost reduction $a_i(t)$ intersects the graph of the upper benefit rate $b_{i1}(t)$ from above at a unique point $t_i^- > 0$ and stays below it afterward; similarly, the graph of $a_i(t)$ intersects the graph of $b_{i2}(t)$ from above at a unique point $t_i^+ > 0$ and stays below it afterward; more accurately,

$$a_i(t) > b_{i1}(t) \quad \text{for } 0 < t < t_i^-, \qquad a_i(t_i^-) = b_{i1}(t_i^-), \qquad a_i(t) < b_{i1}(t) \quad \text{for } t > t_i^-, \tag{4}$$

$$a_i(t) > b_{i2}(t) \quad \text{for } 0 < t < t_i^+, \qquad a_i(t_i^+) = b_{i2}(t_i^+), \qquad a_i(t) < b_{i2}(t) \quad \text{for } t > t_i^+. \tag{5}$$

Remark 2.1. Assumption 2.2 implies in particular that, if $t > 0$ is sufficiently small, the rate of cost reduction $a_i(t)$ is greater than the upper benefit rate $b_{i1}(t)$; if $t > 0$ is sufficiently large, the rate of cost reduction $a_i(t)$ is smaller than the lower benefit rate $b_{i2}(t)$.

Remark 2.2. Since $a_i(t)$ is decreasing and $b_{i1}(t) > b_{i2}(t)$ [see (2) and (3)], we have

$$t_i^- < t_i^+. \tag{6}$$

The graph of the rate of cost reduction $a_i(t)$ and the graphs of the upper and lower benefit rates $b_{i1}(t)$ and $b_{i2}(t)$ are shown schematically in Fig. 1.

The fact that t_2 is the commercialization time of player 2 implies that player 2 does not operate on the market at any time $t < t_2$ and operates on

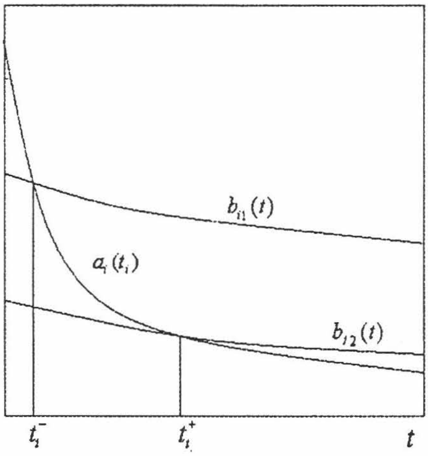


Fig. 1. Rate of cost reduction $a_i(t)$ and upper and lower benefit rates $b_{i1}(t)$ and $b_{i2}(t)$.

the market at every time $t \geq t_2$. Accordingly, the benefit rate $b_1(t)$ of player 1 equals $b_{11}(t)$ for $t < t_2$ and equals $b_{12}(t)$ for $t \geq t_2$. We stress the dependence of $b_1(t)$ on t_2 and write $b_1(t|t_2)$ instead of $b_1(t)$. Thus, given a commercialization time t_2 of project 2, the benefit rate of player 1 is found as

$$b_1(t|t_2) = \begin{cases} b_{11}(t), & \text{if } t < t_2, \\ b_{12}(t), & \text{if } t \geq t_2. \end{cases} \quad (7)$$

Similarly, a commercialization time t_1 of project 1 determines the benefit rate of player 2 as

$$b_2(t|t_1) = \begin{cases} b_{21}(t), & \text{if } t < t_1, \\ b_{22}(t), & \text{if } t \geq t_1. \end{cases}$$

The graphs of the benefit rates $b_1(t|t_2)$ and $b_2(t|t_1)$ are shown schematically in Fig. 2.

Given a commercialization time t_1 of player 1, and a commercialization time t_2 of player 2, the total benefits of players 1 and 2 are represented by the integrals

$$B_1(t_1, t_2) = \int_{t_1}^{\infty} b_1(t|t_2)dt, \quad (8)$$

$$B_2(t_1, t_2) = \int_{t_2}^{\infty} b_2(t|t_1)dt, \quad (9)$$

respectively.

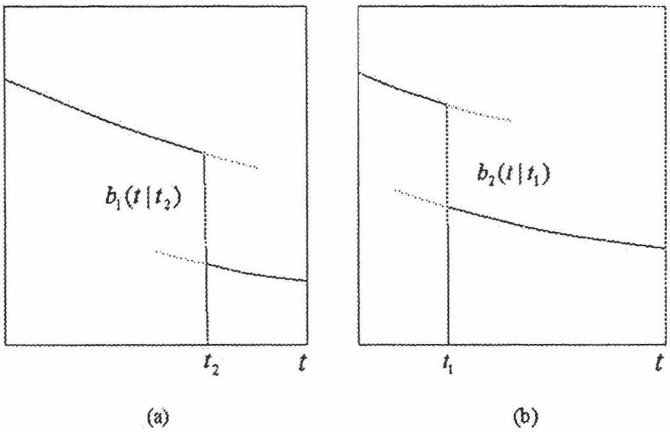


Fig. 2. (a) Benefit rate of player 1. (b) Benefit rate of player 2.

We make the following natural assumption.

Assumption 2.3. For every positive time t_1 and every positive time t_2, the integrals $B_1(t_1, t_2)$ and $B_2(t_1, t_2)$ are finite.

Remark 2.3. Assumption 2.3 is equivalent to the following: for every positive time t_1 and every positive time t_2, the integrals $\int_{t_2}^{\infty} b_{12}(t)dt$, $\int_{t_1}^{\infty} b_{22}(t)dt$ are finite.

Given a commercialization time t_1 of player 1 and a commercialization time t_2 of player 2, the total profit of player i is defined as

$$P_i(t_1, t_2) = -C_i(t_i) + B_i(t_1, t_2). \tag{10}$$

We are ready to define the game of timing for players 1 and 2 in line with the standards of game theory (see Ref. 11). In the game of timing, the strategies of player i, $i = 1, 2$, are the positive (virtual) commercialization times t_i, for project i; the payoff to player i, thanks to the strategies t_1 and t_2 of players 1 and 2, respectively, is the total profit $P_i(t_1, t_2)$.

3. Nash Equilibria

According to the standard terminology of game theory, a strategy t_1^* of player 1 is said to be a best response of player 1 to a strategy t_2 of player 2 if t_1^* maximizes the payoff $P_1(t_1, t_2)$ to player 1 over the set of all strategies t_1

of player 1,

$$P_1(t_1^*, t_2) = \max_{t_1>0} P_1(t_1, t_2).$$

Similarly, a strategy t_2^* of player 2 is said to be a best response of player 2 to a strategy t_1 of player 1 if t_2^* maximizes the payoff $P_2(t_1, t_2)$ to player 2 over the set of all strategies t_2 of player 2,

$$P_2(t_1, t_2^*) = \max_{t_2>0} P_2(t_1, t_2).$$

Any pair (t_1^*, t_2^*), where t_1^* is a strategy of player 1 and t_2^* a strategy of player 2, is said to be a Nash equilibrium in the game of timing if t_1^* is a best response of player 1 to t_2^* and t_2^* is a best response of player 2 to t_1^*. Our goal is to characterize the Nash equilibria in the game of timing.

We start with a simple observation concerned with the dependence of the player payoff on the strategy of the other player. For example, let us consider the payoff $P_1(t_1, t_2)$ to player 1. The differentiation of $P_1(t_1, t_2)$ with respect to t_1 yields

$$\begin{aligned}\partial P_1(t_1, t_2)/\partial t_1 &= a_1(t_1) - b_1(t_1|t_2)\\ &= \begin{cases} a_1(t_1) - b_{11}(t_1), & \text{if } t_1 < t_2,\\ a_1(t_1) - b_{12}(t_1), & \text{if } t_1 > t_2.\end{cases}\end{aligned} \tag{11}$$

Here, we have used (10), (1), (8), and (7). Note that the above partial derivative exists and is continuous at any $t_1 > 0$ except for $t_1 = t_2$. Geometrically, (11) means that $P_1(t_1, t_2)$ grows in t_1 on the time intervals where the graph of $a_1(t_1)$ lies above the graph of $b_1(t_1|t_2)$ an declines in t_1 on the time intervals where the graph of $a_1(t_1)$ lies below the graph of $b_1(t_1|t_2)$.

Let us take two arbitrary strategies of player 2, t_{21} and $t_{22} > t_{21}$. As (11) shows,

$$\partial P_1(t_1, t_{22})/\partial t_1 = \partial P_1(t_1, t_{21})/\partial t_1,$$

for $t_1 < t_{21}$ and $t_1 > t_{22}$, and

$$\partial P_1(t_1, t_{22})/\partial t_1 = \partial P_1(t_1, t_{21})/\partial t_1 - [b_{11}(t_1) - b_{12}(t_1)],$$

for $t_{21} < t_1 < t_{22}$. Recall that

$$b_{11}(t_1) - b_{12}(t_1) > 0;$$

see (2). We have stated that, beyond the time interval located between t_{21} and t_{22}, $P_1(t_1, t_{22})$ and $P_1(t_1, t_{21})$ have the same rate in t_1 and, that within this time interval, $P_1(t_1, t_{22})$ declines in t_1 faster than $P_1(t_1, t_{21})$. Thanks to (8) and (7),

$$P_1(t_1, t_{22}) = P_1(t_1, t_{21}), \qquad \text{for } t_1 \geq t_{22}.$$

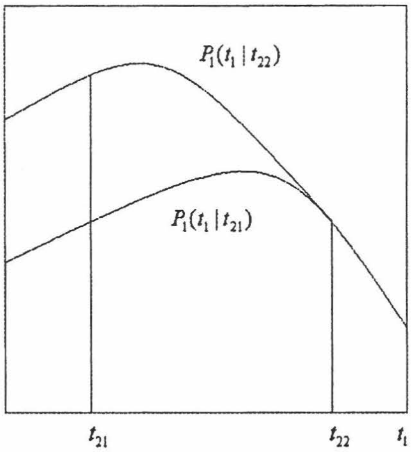


Fig. 3. Payoff $P_1(t_1, t_2)$ for $t_2 = t_{21}$ and $t_2 = t_{22} > t_{21}$.

Therefore,

$$P_1(t_1, t_{22}) > P_1(t_1, t_{21}), \qquad \text{for } t_1 < t_{22}.$$

Let us sum up the previous arguments in the following statement.

Proposition 3.1. For every $t_1 > 0$, the payoff $P_1(t_1, t_2)$ to player 1 increases in t_2; moreover, given a $t_{21} > 0$ and a $t_{22} > t_{21}$, one has $P_1(t_1, t_{22}) = P_1(t_1, t_{21})$ for $t_1 \geq t_{22}$ and $P_1(t_1, t_{22}) > P_1(t_1, t_{21})$ for $t_1 < t_{22}$.

The graphs of $P_1(t_1, t_2)$ for $t_2 = t_{21}$ and $t_2 = t_{22} > t_{21}$ are shown in Fig. 3. A symmetric argument leads to a similar observation for player 2.

Proposition 3.2. For every $t_2 > 0$, the payoff $P_2(t_1, t_2)$ to player 2 increases in t_1; moreover, given a $t_{11} > 0$ and a $t_{12} > t_{11}$, one has $P_2(t_{12}, t_2) = P_2(t_{11}, t_2)$ for $t_2 \geq t_{12}$ and $P_2(t_{12}, t_2) > P_2(t_{11}, t_2)$ for $t_2 < t_{12}$.

Remark 3.1. The fact stated in Propositions 3.1 and 3.2 is intuitively clear: for the investor/manager of a gas pipeline project, any prolongation of the commercialization period of the competing project is profitable.

Now, let us find the best responses (the best reply curve) of player 1 to a given strategy t_2 of player 2.

It is easy enough to identify the intervals of growth and decline of the payoff $P_1(t_1, t_2)$ as a function of t_1. We use (11) and refer to the points t_1^- and t_1^+, at which the graph of $a_1(t)$ intersects the graphs of $b_{11}(t)$ and $b_{12}(t)$; see (4), (5), and Fig. 2.

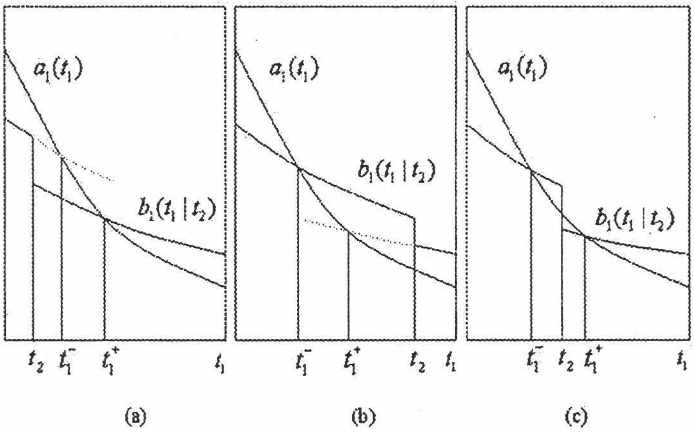


Fig. 4. Functions $a_1(t_1)$ and $b_1(t_1|t_2)$ for (a) $t_2 \le t_1^-$, (b) $t_2 \ge t_1^+$, (c) $t_1^- \le t_2 \le t_1^+$.

Assume first that $t_2 \le t_1^-$; recall that $t_1^- < t_1^+$ [see (6)]. Then, as (4), (5), and Fig. 2 show, the graph of $a_1(t_1)$ lies above the graph of $b_1(t_1|t_2)$ for $t_1 < t_1^+$ and lies below it for $t_1 > t_1^+$; at $t_1 = t_1^+$, the graphs intersect. Figure 4(a) illustrates the relations between the graphs.

Due to (11), $\partial P_1(t_1, t_2)/\partial t_1$ is positive for $t_1 < t_1^+$, $t_1 \ne t_2$, and negative for $t_1 > t_1^+$. Therefore, $t_1 = t_1^+$ is the unique maximizer of $P_1(t_1, t_2)$ in the set of all positive t_1; in other words, t_1^+ is the single best response of player 1 to strategy t_2 of player 2.

Let us assume that $t_2 \ge t_1^+$. Then, (4), (5), and Fig. 2 show that the graph of $a_1(t_1)$ lies above the graph of $b_1(t_1|t_2)$ for $t_1 < t_1^-$, and lies below it for $t_1 > t_1^-$; at $t_1 = t_1^+$, the graphs intersect. Figure 4(b) illustrates the relations between the graphs. Due to (11), $\partial P_1(t_1, t_2)/\partial t_1$ is positive for $t_1 < t_1^-$ and negative for $t_1 > t_1^-$, $t_1 \ne t_2$. Hence, $t_1 = t_1^-$ is the unique maximizer of $P_1(t_1, t_2)$ in the set of all positive t_1; i.e., t_1^- is the single best response of player 1 to t_2.

Now, let t_2 lie in the interval $[t_1^-, t_1^+]$. Then, (4), (5), and Fig. 2 show that the graph of $a_1(t_1)$ lies above the graph of $b_1(t_1|t_2)$ for $t_1 < t_1^-$, lies below it for $t_1^- < t_1 < t_2$, lies again above the graph of $b_1(t_1|t_2)$ for $t_2 < t_1 < t_1^+$, and lies again below it for $t_1 > t_1^+$. Figure 4(c) illustrates the relations between the graphs. Thanks to (11), we conclude that $P_1(t_1, t_2)$, as a function of t_1, strictly decreases on the interval $(0, t_1^-)$, strictly decreases on the interval (t_1^-, t_2), strictly increases on the interval (t_2, t_1^+), and strictly decreases on the interval (t_1^+, ∞). Therefore, the maximizers of $P_1(t_1, t_2)$ in the set of all positive t_1, i.e., the best responses of player 1 to t_2, are restricted to the two-element set $\{t_1^-, t_1^+\}$.

Let us identify the actual maximizers in this set. We refer to Proposition 3.1. Suppose that $t_2 < t_1^+$. Set $t_1 = t_1^+, t_{21} = t_2$, and $t_{22} = t_1^+$. We see that

$t_1 = t_{22} > t_{21}$. By Proposition 3.1,

$$P_1(t_1, t_{22}) = P_1(t_1, t_{21})$$

or

$$P_1(t_1^+, t_1^+) = P_1(t_1^+, t_2). \tag{12}$$

Since $P_1(t_1^+, t_2)$ is continuous in t_2, (12) holds for $t_2 = t_1^+$ as well. Now, we take arbitrary t_{21} and $t_{22} > t_{21}$ in the interval $[t_1^-, t_1^+]$. By Proposition 3.1,

$$P_1(t_1^-, t_{22}) > P_1(t_1^-, t_{21}).$$

Therefore, $P_1(t_1^-, t_2)$ strictly increases in t_2 on $[t_1^+, t_2^+]$. Consider the function

$$p(t_2) = P_1(t_1^-, t_2) - P_1(t_1^+, t_2), \tag{13}$$

defined on $[t_1^-, t_1^+]$. By (12), we have

$$p(t_2) = P_1(t_1^-, t_2) - P_1(t_1^+, t_1^+),$$

for all t_2 in the interval $[t_1^+, t_2^+]$. As long as $P_1(t_1^-, t_2)$ strictly increases in t_2 on $[t_1^-, t_1^+]$, $p(t_2)$ strictly increases on $[t_1^+, t_2^+]$. Earlier, we have stated that t_1^+ is the single best response of player 1 to any $t_2 \leq t_1^-$; in particular, this holds for $t_2 = t_1^-$, i.e.,

$$P_1(t_1^+, t_1^-) > P_1(t_1^-, t_1^-).$$

Hence,

$$p(t_1^-) = P_1(t_1^-, t_1^-) - P_1(t_1^+, t_1^-) < 0.$$

Earlier, we have stated that t_1^- is the single best response of player 1 to any $t_2 \geq t_1^+$; in particular, this holds for $t_2 = t_1^+$, i.e.,

$$P_1(t_1^-, t_1^+) > P_1(t_1^+, t_1^+).$$

Hence,

$$p(t_1^+) = P_1(t_1^-, t_1^+) - P_1(t_1^+, t_1^+) > 0.$$

We have found that $p(t_2)$ takes a negative value at the left endpoint of the interval $[t_1^-, t_1^+]$ and a positive value at the right endpoint of this interval. Since $p(t_2)$ is continuous, there exists a $\hat{t}_2$ in the interior of $[t_1^-, t_2^+]$, for which $p(\hat{t}_2) = 0$. The fact that $p(t_2)$ strictly increases on $[t_1^-, t_1^+]$ implies that the point $\hat{t}_2$ is unique; i.e., $p(t_2) < 0$ for $t_1^- \leq t_2 < \hat{t}_2$ and $p(t_2) > 0$ for $t_1^+ \geq t_2 > \hat{t}_2$. By the definition of $p(t_{12})$ and (13), we have

$$P_1(t_1^-, \hat{t}_2) = P_1(t_1^+, \hat{t}_2),$$

$$P_1(t_1^-, t_2) < P_1(t_1^+, t_2), \qquad \text{for } t_1^- \leq t_2 < \hat{t}_2,$$

$$P_1(t_1^-, t_2) < P_1(t_1^+, t_2), \qquad \text{for } t_1^+ \geq t_2 > \hat{t}_2.$$

Earlier, we have stated that all the best responses of player 1 to t_2 lie in the two-element set $\{t_1^-, t_1^+\}$. Therefore, we conclude that, if $t_2 = \hat{t}_2$, player 1 has two best responses, t_1^- and t_1^+, to t_2; if $t_1^- \leq t_2 < \hat{t}_2$, the unique best response of player 1 to t_2 is t_1^+; and if $t_1^+ \geq t_2 > \hat{t}_2$, the unique best response of player 1 to t_2 is t_1^-. Recall that the best response of player 1 to t_2 is t_1^+ if $t_2 < t_1^-$, and t_1^- if $t_2 > t_1^+$.

We summarize the above considerations as follows.

Proposition 3.3. In the interval (t_1^-, t_1^+), there exists a unique point $\hat{t}_2$ such that

$$P_1(t_1^-, \hat{t}_2) = P_1(t_1^+, \hat{t}_2). \tag{14}$$

The set of all best responses of player 1 to $\hat{t}_2$ is $\{t_1^-, t_1^+\}$. If $0 < t_2 < \hat{t}_2$, then the unique best response of player 1 to t_2 is t_1^+. If $t_2 > \hat{t}_2$, then the unique best response of player 1 to t_2 is t_1^-.

We call t_1^- the fast choice of player 1 and t_1^+ the slow choice of player 1. Proposition 3.3 claims that the slow choice of player 1 is the best response of player 1 to all fast strategies t_2 of player 2, namely, those satisfying $t_2 < \hat{t}_2$, and the fast choice of player 1 is the best response of player 1 to all slow strategies t_2 of player 2, namely, those satisfying $t_2 > \hat{t}_2$; finally, both fast and slow choices of player 1 respond best to $t_2 = \hat{t}_2$. We call $\hat{t}_2$ the switch point for player 1.

Let us consider the function that associates to each strategy t_2 of player 2 the set of all best responses of player 1 to t_2; we call it the best response function of player 1. The graph of the best response function of player 1 is shown in Fig. 5(a). It consists of the horizontal segment located strictly above the segment $(0, \hat{t}_2]$ on the t_2-axis at the level t_1^+, and the unbounded horizontal segment located strictly above the segment $[\hat{t}_2, \infty)$ on the t_2-axis at the level t_1^-. The points $(t_1^+, \hat{t}_2)$ and $(t_1^-, \hat{t}_2)$ lie on the graph.

A symmetric argument leads to a similar characterization of the best responses of player 1.

Proposition 3.4. In the interval (t_2^-, t_2^+), there exists a unique point $\hat{t}_1$ such that

$$P_2(\hat{t}_1, t_2^-) = P_1(\hat{t}_1, t_2^+). \tag{15}$$

The set of all best responses of player 2 to $\hat{t}_1$ is $\{t_2^-, t_2^+\}$. If $0 < t_1 < \hat{t}_1$, then the unique best response of player 2 to t_1 is t_2^+. If $t_1 > \hat{t}_1$, then the unique best response of player 2 to t_1 is t_2^-.

We call t_2^- the fast choice of player 2, t_2^+ the slow choice of player 2, and $\hat{t}_2$ the switch point for player 2. We introduce also the best response function

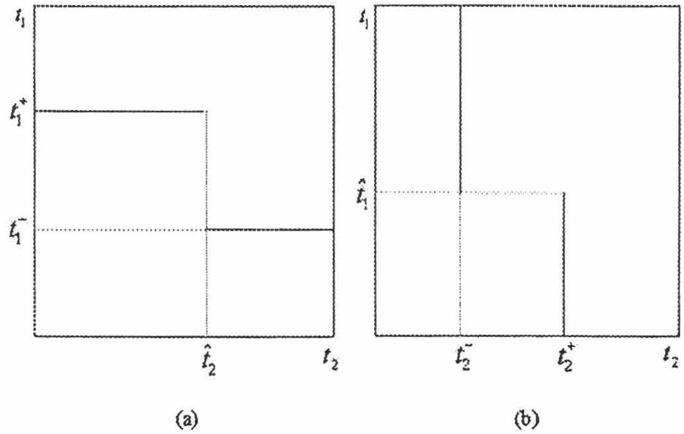


Fig. 5. (a) Best response function of player 1. (b) Best response function of player 2.

of player 2, which associates to each strategy t_1 of player 1 the set of all best responses of player 2 to t_1. The graph of the best response function of player 2 is shown in Fig. 5(b). Here, the independent variable t_1 is shown on the vertical axis and the best responses of player 2 are located on the horizontal axis. The graph of the best response function of player 2 consists of the vertical segment located to the right of the segment $(0, \hat{t}_1]$ on the t_1-axis at the distance t_2^+, and the unbounded vertical segment located to the right of the segment $[\hat{t}_1, \infty)$ on the t_1-axis at the distance t_2^-. The points $(\hat{t}_1, t_2^+)$ and $(\hat{t}_1, t_2^-)$ lie on the graph.

Now, we recall the definition of a Nash equilibrium and find easily that a strategy pair (t_1^*, t_2^*) is a Nash equilibrium if and only if the point (t_1^*, t_2^*) belongs to the intersection of the graphs of the best response functions of players 1 and 2. Figure 5 shows that the graphs necessarily intersect. Figure 6 gives an example of the intersection.

For each intersection point [i.e., each Nash equilibrium (t_1^*, t_2^*)], point t_1^* is the fast or slow choice of player 1, and point t_2^* is the fast or slow choice of player 2. In case t_1^* is the fast choice of player 1 and t_2^* the slow choice of player 2, we call (t_1^*, t_2^*), the fast-slow Nash equilibrium; similarly, we define the slow-fast, fast-fast, and slow-slow Nash equilibria.

Nash equilibria of different types arise under different relations between the players fast and slow choices and the switch points of their rivals. The list of all admissible cases is as follows:

$$\hat{t}_2 \geq t_2^+, \qquad \hat{t}_1 < t_1^+, \tag{16}$$

$$\hat{t}_2 \geq t_2^+, \qquad t_1^- < \hat{t}_1 < t_1^+, \tag{17}$$

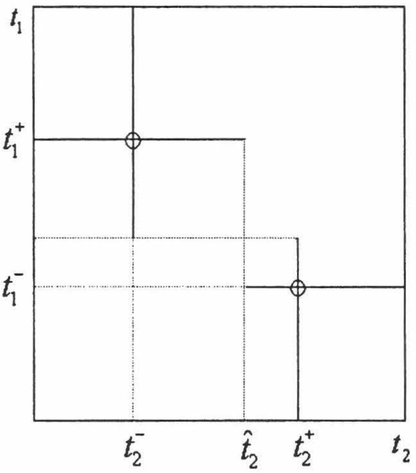


Fig. 6. Intersection of the graphs of best response functions of players 1 and 2.

$$\hat{t}_2 \le t_2^-, \qquad t_1^- < \hat{t}_1 < t_1^+, \tag{18}$$

$$t_2^- \le \hat{t}_2 < t_2^+, \qquad t_1^- < \hat{t}_1 \le t_1^+, \tag{19}$$

$$t_2^- < \hat{t}_2 \le t_2^+, \qquad t_1^- \le \hat{t}_1 < t_1^+, \tag{20}$$

$$t_2^- < \hat{t}_2 < t_2^+, \qquad \hat{t}_1 \le t_1^-, \tag{21}$$

$$t_2^- < \hat{t}_2 < t_2^+, \qquad \hat{t}_1 \ge t_1^+, \tag{22}$$

$$\hat{t}_2 < t_2^-, \qquad \hat{t}_1 \ge t_1^+. \tag{23}$$

An elementary analysis in the spirit of Fig. 6 leads to the full classification of the Nash equilibria in the game of timing.

Proposition 3.5. In cases (16), (17), and (21), the unique Nash equilibrium is slow-fast (t_1^-, t_2^+); see Fig. 7(a), (b), (c). In cases (18), (22), and (23), the unique Nash equilibrium is fast-slow (t_1^+, t_2^-); see Fig. 7(d), (e), (f). In cases (19) and (20), the game of timing has precisely two Nash equilibria, fast-slow (t_1^-, t_2^+) and slow-fast (t_1^+, t_2^-); see Fig. 7(g).

Remark 3.2. Proposition 3.1 shows that the game of timing admits only fast-slow and slow-fast equilibria.

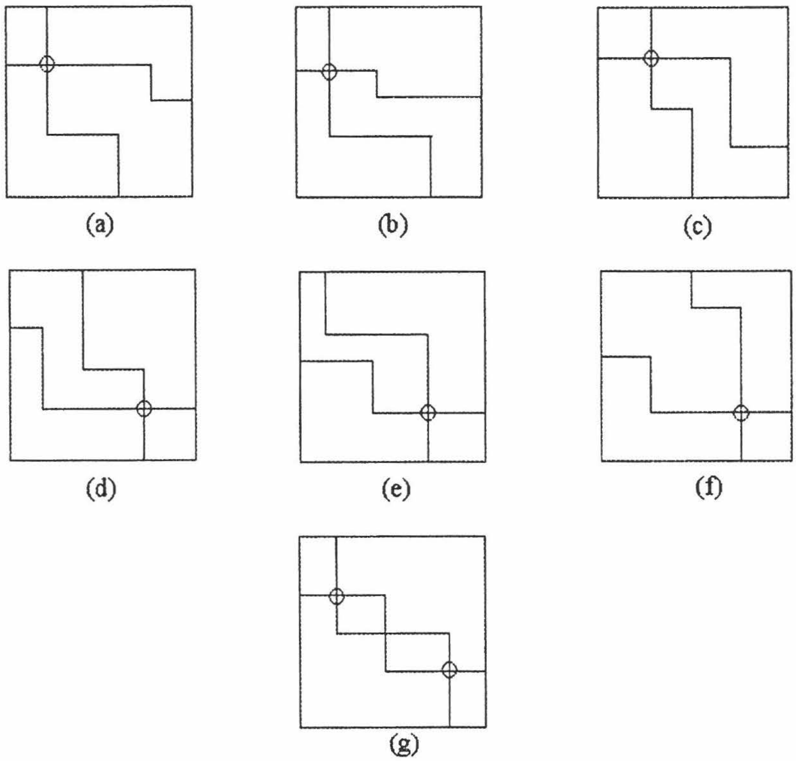


Fig. 7. (a) One equilibrium, slow-fast. (b) One equilibrium, slow-fast. (c) One equilibrium, slow-fast. (d) One equilibrium, fast-slow. (e) One-equilibrium, fast-slow. (f) One equilibrium, fast-slow. (g) Two equilibria, fast-slow and slow-fast.

Let us consider in more detail the most interesting situation where the game of timing has two Nash equilibria, fast-slow and slow-fast; i.e., (19) or (20) holds; see Fig. 7(g). By Proposition 3.1 and due to the inequalities $t_1^- < \hat{t}_2 \leq t_2^+$, we have

$$P_1(t_1^-, t_2^+) \geq P_1(t_1^-, \hat{t}_2);$$

moreover, the inequality is strict if and only if $\hat{t}_2 < t_2^+$. Using equality (14), Proposition 3.1 and the inequalities $t_1^+ > \hat{t}_2 \geq t_2^-$, we transform the right-hand side as follows:

$$P_1(t_1^-, \hat{t}_2) = P_1(t_1^+, \hat{t}_2) = P_1(t_1^+, t_2^-).$$

Thus, for the fast-slow and slow-fast equilibria (t_1^-, t_2^+) and (t_1^+, t_2^-), we have

$$P_1(t_1^-, t_2^+) \geq P_1(t_1^+, t_2^-).$$

Moreover, the inequality is strict if $\hat{t}_2 < t_2^+$. If this is so, player 1 prefers the fast-slow equilibrium; otherwise, the fast-slow and slow-fast equilibria are equivalent for this player. Similarly, we state that, if $\hat{t}_1 < t_1^+$, player 2 prefers the slow-fast equilibrium; otherwise, the equilibria are equivalent for this player. Thus, generally, each player prefers his fast equilibrium.

Let us give an exact formulation.

Proposition 3.6. Let the game of timing have two Nash equilibria; i.e., let (19) or (20) hold. Then:

(i) $P_1(t_1^-, t_2^+) \geq P_1(t_1^+, t_2^-)$; moreover, the inequality is strict if and only if $\hat{t}_2 < t_2^+$;

(ii) $P_2(t_1^-, t_2^+) \geq P_2(t_1^+, t_2^-)$; moreover, the inequality is strict if and only if $\hat{t}_1 < t_1^+$.

Remark 3.3. Let the game have two equilibria [i.e., (19) or (20) hold]. Assume that the fast-slow and slow-fast equilibria are equivalent to player 1, i.e.,

$$P_1(t_1^-, t_2^+) = P_1(t_1^+, t_2^-).$$

Then, by Proposition 3.6(i), $\hat{t}_2 \geq t_2^+$. As (19), (20) show, we actually have $\hat{t}_2 = t_2^+$, which is an exceptional situation for the case of two equilibria. Hence,

$$\hat{t}_1 < t_2^+ \leq \hat{t}_2 < t_1^+.$$

By Proposition 3.6(ii),

$$P_2(t_1^-, t_2^+) > P_2(t_1^+, t_2^-).$$

In other words, the slow-fast equilibrium is strictly preferable for player 2. In the symmetric case, where the fast-slow and slow-fast equilibria are equivalent to player 2, i.e.,

$$P_1(t_1^-, t_2^+) = P_1(t_1^+, t_2^-),$$

we find similarly that the fast-slow equilibrium is strictly preferable for player 1. Thus, in those exceptional cases where one of the players has no preference in choosing an equilibrium, the other player strictly prefers his fast equilibrium.

Remark 3.4. Let us assume that the parameters of projects 1 and 2 are identical; i.e.,

$$C_1(t) = C_2(t) \quad \text{and} \quad B_1(t, s) = B_2(s, t), \qquad \text{for all positive } t \text{ and } s.$$

Then, the game of timing takes a symmetric form. The players have the same fast and slow choices and switch times,

$$t_1^- = t_2^-, \qquad t_1^+ = t_2^+, \qquad \hat{t}_2 = \hat{t}_1.$$

Hence, (19) and (20) hold. By Proposition 3.5, the game of timing has the fast-slow and slow-fast equilibria. The inequality $\hat{t}_2 < t_2^+$ is equivalent to $\hat{t}_2 < t_1^+$, which holds trivially [see (6)]. By Proposition 3.6, we conclude that

$$P_1(t_1^-, t_2^+) > P_1(t_1^+, t_2^-).$$

Similarly, we find that

$$P_2(t_1^-, t_2^+) > P_2(t_1^+, t_2^-).$$

Thus, in the symmetric game of timing, player 1 prefers the fast-slow equilibrium and player 2 prefers the slow-fast equilibrium. Obviously, the situation does not change if the parameters of projects 1 and 2 are sufficiently close to each other. The question of a practical choice of an equilibrium in the case where the players have different preferences arises. Here, we do not argue on this; we note only that game theory does not provide any clear recommendations in this respect.

4. Solution Algorithm

For convenience, we represent the obtained classification of the Nash equilibria in table form (see Table 1).

We conclude the general part of our study with the description of an algorithm that finds the Nash equilibria in the game of timing. The algorithm refers to the definitions of the players fast and slow choices t_i^- and t_i^+, $i = 1, 2$, the players switch times $\hat{t}_i$, $i = 1, 2$, and Table 1.

Step 1. Use definitions (4) and (5) for finding the players fast and slow choices t_i^-, and t_2^+, $i = 1, 2$.

Step 2. Use definitions (14) and (15) for finding the players switch times $\hat{t}_i$, $i = 1, 2$.

Step 3. Use Table 1 for identifying the Nash equilibria.

Table 1. Classification of Nash equilibria in the game of timing (table form of Proposition 3.5).

Case	Number of equilibria	Types of equilibria	Notation
$\hat{t}_1 < t_1^-$ $\hat{t}_2 \geq t_2^+$	1	slow-fast	(t_1^+, t_2^-)
$t_1^- < \hat{t}_1 < t_1^+$ $\hat{t}_2 \geq t_2^+$	1	slow-fast	(t_1^+, t_2^-)
$t_1^- < \hat{t}_1 < t_1^+$ $\hat{t}_2 \leq t_2^-$	1	fast-slow	(t_1^-, t_2^+)
$t_1^- < \hat{t}_1 \leq t_1^+$ $t_2^- \leq \hat{t}_2 < t_2^+$	2	fast-slow slow-fast	(t_1^-, t_2^+) (t_1^+, t_2^-)
$t_1^- \leq \hat{t}_1 < t_1^+$ $t_2^- < \hat{t}_2 \leq t_2^+$	2	fast-slow slow-fast	(t_1^-, t_2^+) (t_1^+, t_2^-)
$\hat{t}_1 \leq t_1^-$ $t_2^- < \hat{t}_2 < t_2^+$	1	slow-fast	(t_1^+, t_2^-)
$\hat{t}_1 \geq t_1^+$ $t_2^- < \hat{t}_2 < t_2^+$	1	fast-slow	(t_1^-, t_2^+)
$\hat{t}_1 \geq t_1^+$ $\hat{t}_2 < t_2^-$	1	fast-slow	(t_1^-, t_2^+)

5. Gas Pipeline Game

In this section, we apply the suggested solution method to a model described in Ref. 1. Wishing to demonstrate a clear analytic result, we consider a simplified version of the model. Namely, we eliminate the price of liquid natural gas, which acts as an upper bound for the price of gas in the original model; we do not introduce the upper bounds for the rates of supply or the pipelines capacities; we assume that the costs for extraction and transportation of gas are functions of time only; finally, we analyze the competition of two pipeline projects (as our theory prescribes).

The model is as follows. The cost $C_i(t_i)$ for finalizing the construction of pipeline i, $i = 1, 2$, at time t_i is defined to be the minimum of the integral investment

$$I_i(r_i) = \int_0^{t_i} e^{-\lambda t} r_i(t) dt.$$

Here, λ is a positive discount. The minimum is taken over all admissible open-loop investment strategies $r_i(t)$ of player i. An admissible open-loop investment strategy of player i (for a commercialization time t_i) is modeled as an integrable control function,

$$r_i(t) > 0, \tag{24}$$

that brings the accumulated investment $x_i(t)$ from 0 to the prescribed commercialization level $\bar{x}_i > 0$ at time t_i. Thus, for the initial and final values of the accumulated investment, we have

$$x_i(0) = 0, \qquad x_i(t_i) = \bar{x}_i. \tag{25}$$

The dynamics of $x_i(t)$ is modeled as

$$\dot{x}_i(t) = -\sigma x_i(t) + r_i^{\gamma}(t). \tag{26}$$

Here, σ is a positive obsolescence coefficient and γ is a delay parameter, located strictly between 0 and 1. In the terminology of control theory (see Ref. 15), the cost $C_i(t_i)$ is defined to be the optimal value in the problem of minimizing the performance index $I_i(r_i)$ for the control system (26), (24), subject to the boundary constraints (25).

The upper and lower benefit rates $b_{i1}(t)$ and $b_{i2}(t)$ for player i at time $t > 0$ are found as equilibrium payoffs in the static supply game modeling the instantaneous gas market. In the supply game arising at time t, the strategies y_i of player i are nonnegative rates of supply and the payoff to player i is defined as

$$p_i(y_1, y_2 | t) = e^{-\lambda t}[\pi(t, y) - c_i(t)]y_i. \tag{27}$$

Here, y is the total rate of supply, $\pi(t, y)$ is the price of gas, and $c_i(t) > 0$ is the cost for extraction and transportation of gas for player i. The price of gas is modeled as

$$\pi(y|t) = (g(t)/y)^{\beta},$$

where $g(t) > 0$ is the consumer GDP at time t and β is the inverse to the price elasticity of gas demand; we have

$$0 < \beta < 1.$$

The total supply y equals y_i if player i occupies the market solely and equals $y_1 + y_2$ if both players operate on market.

The next proposition gives the expressions for the costs $C_i(t_i)$, rates of cost reduction $a_i(t_i)$, and upper and lower benefit rates $b_{i1}(t_i)$ and $b_{i2}(t_i)$, $i = 1, 2$. We need the following assumption.

Assumption 5.1. It holds that

$$1 - (2-\beta)c_i(t)/[c_1(t) + c_2(t)] > 0, \qquad i = 1, 2. \tag{28}$$

Remark 5.1. Condition (28) implies that the costs $c_1(t)$ and $c_2(t)$ are relatively close to each other. Indeed, in the extremal case where $c_1(t) = c_2(t) = c(t)$, (28) is equivalent to the trivial inequality $\beta > 0$. Another interpretation of condition (28) is that β is close to 1. Indeed, in the limiting case where $\beta = 1$, (28) is equivalent to the trivial inequality

$$1 - c_i(t)/[c_1(t) + c_2(t)] > 0.$$

Proposition 5.1. For player i, $i = 1, 2$, the following formulas hold:

(a) The cost $C_i(t_i)$ is given by

$$C_i(t_i) = \rho^{\alpha - 1} e^{-\lambda t_i} \bar{x}_i^{\alpha} / (1 - e^{-\rho t_i})^{\alpha - 1}, \tag{29}$$

where

$$\alpha = 1/\gamma, \qquad \rho = (\alpha\sigma + \lambda)/(\alpha - 1). \tag{30}$$

(b) The rate of cost reduction $a_i(t_i)$ is given by

$$a_i(t) = \rho^{\alpha - 1} \bar{x}_i^{\alpha} e^{-\lambda t} (\lambda + \nu e^{-\rho t}) / (1 - e^{-\rho t})^{\alpha}, \tag{31}$$

where

$$\nu = \alpha\sigma. \tag{32}$$

(c) The upper benefit rate $b_{i1}(t_i)$ is given by

$$b_{i1}(t) = e^{-\lambda t}(1 - \beta)^{1/\beta - 1} g(t) / c_i^{1/\beta - 1}(t). \tag{33}$$

(d) If Assumption 5.1 holds, the lower benefit rate $b_{i2}(t_i)$ is given by

$$b_{i2}(t) = e^{-\lambda t}(2 - \beta)^{1/\beta - 1} \left\{ 1 - \frac{(2 - \beta) c_i(t)}{[c_1(t) + c_2(t)]} \right\}^2 g(t) / [c_1(t) + c_2(t)]^{1/\beta - 1}. \tag{34}$$

(e) Under Assumption 5.1, the following inequality is valid:

$$b_{i1}(t) > b_{i2}(t); \tag{35}$$

see (2) and (3).

In what follows, we assume that $c_i(t)$, $i = 1, 2$, and $g(t)$ are defined on the positive half axis and are continuous. We also fix the functions described in Proposition 5.1 and introduce the next assumption.

Assumption 5.2. For $i = 1, 2$, the functions

$$h_{i1}(t) = g(t) / c_i(t)^{1/\beta - 1}, \tag{36a}$$

$$h_{i2}(t) = \left\{ 1 - \frac{(2 - \beta) c_i(t)}{[c_1(t) + c_2(t)]} \right\}^2 g(t) / [c_1(t) + c_2(t)]^{1/\beta - 1}, \tag{36b}$$

$t > 0$, increase and tend to infinity as t tends to infinity, and the integral $\int_0^\infty e^{-\lambda t} h_{i1}(t)dt$ is finite.

Remark 5.2. Assumption 5.2 holds if $g(t)$ [the consumer GDP] and the costs $c_i(t)$ grow exponentially,

$$g(t) = g^0 e^{\zeta t}, \qquad c_i(t) = c_i^0 e^{\omega t}, \quad i = 1, 2, \tag{37}$$

ζ and ω are nonnegative, and

$$0 < \kappa < \lambda, \tag{38}$$

where

$$\kappa = \zeta - (1/\beta - 1)\omega. \tag{39}$$

Note that g_0 is the consumer GDP at time 0 and c_i^0 is the cost for transportation and extraction for player i at time 0.

The theory described earlier for the general case is applicable for the model considered. Namely, the following is true.

Proposition 5.2. Let Assumptions 5.1 and 5.2 hold. Then, Assumptions 2.1 and 2.2 hold. Moreover, the fast choice t_i^- of player i, $i = 1, 2$, is the unique solution of the algebraic equation

$$\rho^{\alpha-1} \bar{x}_i^\alpha / (1 - \beta)^{1/\beta - 1} = [(1 - e^{-\rho t})^\alpha / (\lambda + \nu e^{-\rho t})] h_{i1}(t); \tag{40}$$

the slow choice t_i^+ of player i is the unique solution of the algebraic equation

$$\rho^{\alpha-1} \bar{x}_i^\alpha / (2 - \beta)^{1/\beta - 1} = [(1 - e^{-\rho t})^\alpha / (\lambda + \nu e^{-\rho t})] h_{i2}(t). \tag{41}$$

Thus, under Assumptions 5.1 and 5.2, the general algorithm for the resolution of the game of timing (see Section 4) is specified as follows.

Step 1. Solve equations (40) and (41) for finding the players fast and slow choices t_i^- and t_2^+, respectively, $i = 1, 2$.

Step 2. Use equalities (14) and (15) for finding the players switch times $\hat{t}_i$, $i = 1, 2$.

Step 3. Use Table 1 for identifying the Nash equilibria in the game of timing.

As a specific example, let us consider the case described in Remark 5.2. Thus, in what follows, we assume that $g(t)$ and $c_i(t)$, $i = 1, 2$, are given by (37) and inequality (38) is satisfied. Formulas (33) and (34) for $b_{i1}(t)$ and $b_{i2}(t)$ are specified as

$$b_{i1}(t) = b_{i1}^0 e^{-\psi t}, \qquad b_{i2}(t) = b_{i2}^0 e^{-\psi t},$$

where

$$\psi = \lambda - \kappa,$$
$$b_{i1}^0 = (1-\beta)^{1/\beta-1} g^0/(c_i^0)^{1/\beta-1},$$
$$b_{i2}^0 = (2-\beta)^{1/\beta-1}[1-(2-\beta)c_i^0/(c_1^0+c_i^0)]^2 g^0/(c_1^0+c_2^0)^{1/\beta-1}.$$

Using the definition of the total benefit $B_i(t_1, t_2)$ of player i [see (8) and (9)] and the expression (29) for the cost $C_i(t_i)$, we find an explicit formula for the total profit $P_i(t_1, t_2)$ [see (10)] of player i, which is determined by the player strategies t_1 and t_2. We have

$$P_1(t_1, t_2) = -\rho^{\alpha-1} e^{-\lambda t_1} \bar{x}_1^\alpha/(1-e^{-\rho t_1})^{\alpha-1} + \begin{cases} b_{11}^0 e^{-\psi t_1}/\psi + (b_{12}^0 - b_{11}^0) e^{-\psi t_2}/\psi, & \text{if } t_1 \leq t_2, \\ b_{12}^0 e^{-\psi t_1}/\psi, & \text{if } t_1 \geq t_2, \end{cases} \tag{42}$$

$$P_2(t_1, t_2) = -\rho^{\alpha-1} e^{-\lambda t_2} \bar{x}_2^\alpha/(1-e^{-\rho t_2})^{\alpha-1} + \begin{cases} b_{21}^0 e^{-\psi t_1}/\psi + (b_{22}^0 - b_{21}^0) e^{-\psi t_1}/\psi, & \text{if } t_2 \leq t_1, \\ b_{22}^0 e^{-\psi t_2}/\psi, & \text{if } t_2 \geq t_1. \end{cases} \tag{43}$$

Figure 8 shows the Maple-simulated landscape of $P_1(t_1, t_2)$ for

$$\alpha = 1.5, \quad \lambda = 0.3, \quad \sigma = 0.3, \quad g^0 = 3.5, \quad \bar{x}_1 = 0.7,$$
$$\beta = 0.5, \quad c_1^0 = c_2^0 = 0.2.$$

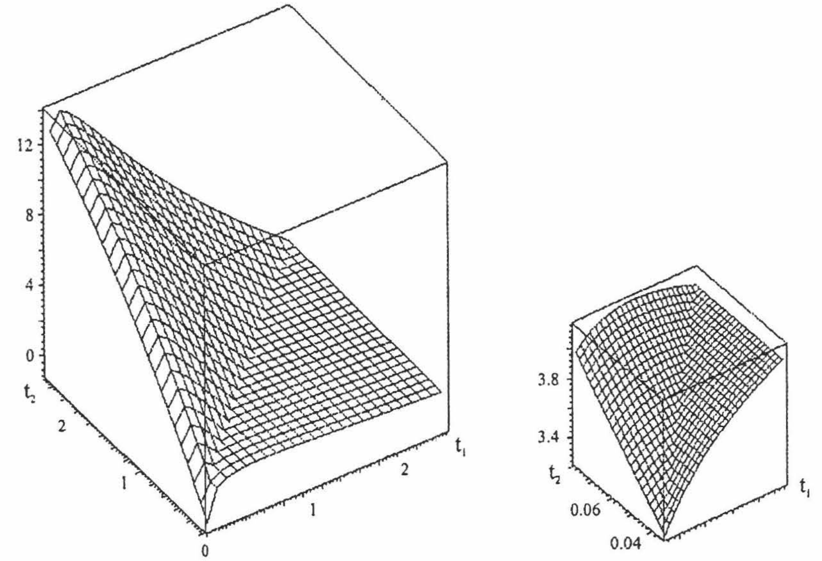
Recall that, by Proposition 3.3, the critical points $\hat{t}_2$ and $\hat{t}_1$ needed for the identification of the type of the equilibria in the game of timing (see Table 1) are found from the equalities

$$P_1(t_1^-, \hat{t}_2) = P_1(t_1^+, \hat{t}_2) \quad \text{and} \quad P_1(\hat{t}_1, t_2^-) = P_1(\hat{t}_1, t_2^+),$$

respectively. In the situation considered now, the critical points are given explicitly. The next proposition is true.

Proposition 5.3. For $i = 1, 2$, we have

$$\hat{t}_i = -(1/\psi)\log[\psi G_i/(b_{i2}^0 - b_{i1}^0)], \tag{44}$$



(a) (b)

Fig. 8. Payoff landscape for player 1: (a) for large t_2, the fast choice t_1^- replies best; (a) for small t_2, the slow choice t_1^+ replies best.

where

$$G_i = -\rho^{\alpha-1}e^{-\lambda t_i^+}\bar{x}_1^\alpha/(1-e^{-\rho t_1^+})^{\alpha-1} + b_{i2}^0 e^{-\psi t_1^+}/\psi$$
$$+\rho^{\alpha-1}e^{\lambda t_i^-}\bar{x}_1^\alpha/(1-e^{-\rho t_i^-})^{\alpha-1} - b_{i1}^0 e^{-\psi t_i^-}/\psi. \quad (45)$$

The next proposition specifies Proposition 5.2.

Proposition 5.4. Let $g(t)$ and $c_i(t)$, $i=1, 2$, be given by (37) and let inequality (38) be satisfied. Then, for every player i, $i=1, 2$, the following assertions hold.

(a) The fast choice t_i^- of player i is the unique solution of the algebraic equation

$$l_i w_i = e^{\kappa t}(1-e^{-\rho t})^\alpha/(\lambda + \nu e^{-\rho t}), \quad (46)$$

where

$$l_i = \rho^{\alpha-1}/(1-\beta)^{1/\beta-1}g^0, \quad (47)$$

$$w_i = \bar{x}_i^\alpha (c_i^0)^{1/\beta-1}. \quad (48)$$

(b) The slow choice t_i^+ of player i is the unique solution of the algebraic equation

$$l_i z_i = e^{\kappa t}(1 - e^{-\rho t})^{\alpha}/(\lambda + \nu e^{-\rho t}), \tag{49}$$

where l_i is defined by (47) and

$$z_i = \bar{x}_i^{\alpha}(c_1^0 + c_2^0)^{1/\beta - 1}/[1 - (2 - \beta)c_i^0/(c_1^0 + c_2^0)]^2. \tag{50}$$

Thus, under the assumptions of Remark 5.2, the suggested solution algorithm for the game of timing (Section 4) takes the following form.

Step 1. Solve equations (46) and (49) for finding the players fast and slow choices t_i^- and t_i^+, respectively, $i = 1, 2$.
Step 2. Use formula (44) for finding the players switch times $\hat{t}_i$, $i = 1, 2$.
Step 3. Use Table 1 for identifying the Nash equilibria in the game of timing.

Figure 9 shows the Maple-simulated graphs of the fast choice t_1^- and slow choice t_1^+ of player 1 as functions of $\bar{x}_1$ and $c_1^0 = c_2^0$ for different values of β and

$$\alpha = 1.5, \quad \lambda = 0.3, \quad \sigma = 0.3, \quad g^0 = 3.5.$$

6. Case Study

In this section, we describe the game of timing in application to the Caspian and China gas markets. The values of the model parameters are

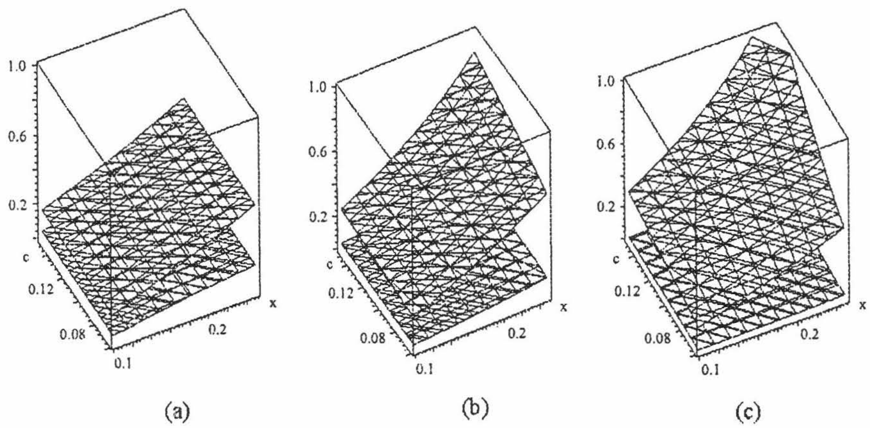


Fig. 9. Graphs of the fast choice t_1^- and slow choice t_1^+ of player 1 as functions of $\bar{x}_1$ and c_1^0: (a) $\beta = 0.97$; (b) $\beta = 0.75$; (c) $\beta = 0.5$.

based on preliminary expert estimates. Our first case study deals with the competition of two major gas pipeline projects in the Caspian region, the Blue Stream Project of the Russian GAZPROM Company (project 1), which is aimed at delivering Russian gas to Turkey under the Black Sea; and the Trans-Caspian Project (project 2) directed from Turkmenistan underneath the Caspian Sea through Azerbaijan and Georgia to Turkey. In this case study, the parameters of the model are chosen as follows: discount rate $\lambda = 0.1$; obsolescence coefficient $\sigma = 0.3$; delay coefficient, $\gamma = 0.65$; inverse to the price elasticity of gas demand $\beta = 0.55$; initial level of the consumer GDP $g^0 = 214.6$; growth rate of the consumer GDP $\zeta = 0.1$; growth rate of the extraction costs $\omega = 0.15$; initial extraction costs $c_1^0 = 67.3$, $c_2^0 = 78.4$; commercialization levels of the accumulated investments $\bar{x}_1 = 4.0$, $\bar{x}_2 = 2.5$. It is assumed that the projects start in 2001.

For these parameters, there exist two Nash equilibria in the game of timing, the fast-slow equilibrium

$$(t_1^-, t_2^+) = (2002.8, 2005.2)$$

and the slow-fast equilibrium

$$(t_1^+, t_2^-) = (2004.6, 2002.2).$$

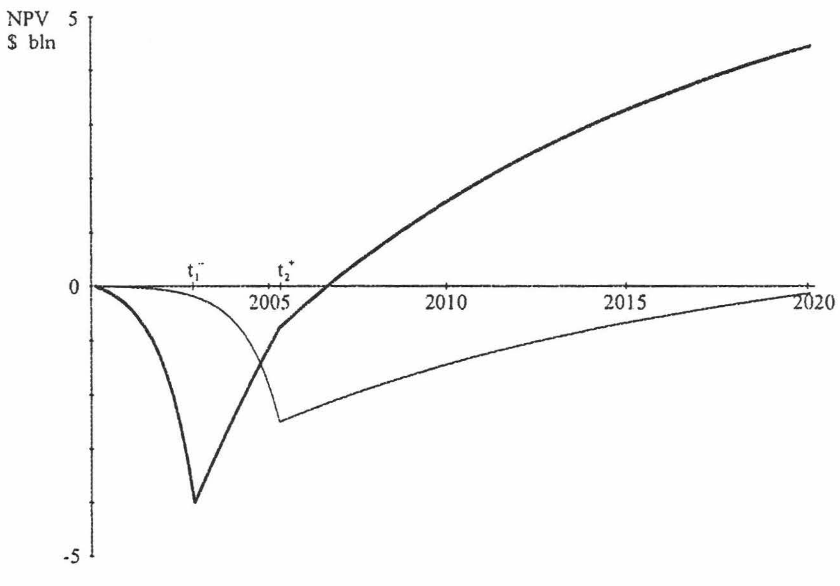


Fig. 10. NPV dynamics for the fast-slow scenario of the gas pipeline competition in the Caspian region.

Figures 10 and 11 depict the dynamics of the net present values NPV, $P_i = P_i(t, t_1, t_2)$ (in billion dollars)

$$P_i(t, t_1, t_2) = \begin{cases} -C_i(t), & \text{if } 0 \leq t < t_i, \\ -C_i(t_i) + \int_{t_i}^{t} b_i(\tau | t_j) d\tau, & \text{if } t \geq t_i, \end{cases} \tag{51}$$

with $i, j = 1, 2, i \neq j$, for project 1 (Blue Stream) and project 2 (Trans-Caspian) under the fast-slow Nash equilibrium investment scenario $(t_1, t_2) = (t_1^-, t_2^+)$ and the slow-fast Nash equilibrium investment scenario $(t_1, t_2) = (t_1^+, t_2^-)$, respectively. The heavy line and the fine line show the NPV dynamics of Blue Stream and Trans-Caspian, respectively.

Our second case study is related to the planned projects of gas pipelines from Russia to China. Two potential competitors on the North China gas market are the Kovikta-Zabaikalsk-Kharbin pipeline (project 1) stretched from the Irkutsk region to North China, and the Sakhalin-Khabarovsk-Kharbin pipeline (project 2). The following values of the model parameters are chosen:

$$\lambda = 0.1, \quad \sigma = 0.3, \quad \gamma = 0.58, \quad \beta = 0.46, \quad g^0 = 1157,$$
$$\zeta = 0.0668, \quad \omega = 0.05, \quad c_1^0 = 57, \quad c_2^0 = 68, \quad \bar{x}_1 = 6, \quad \bar{x}_2 = 3.$$

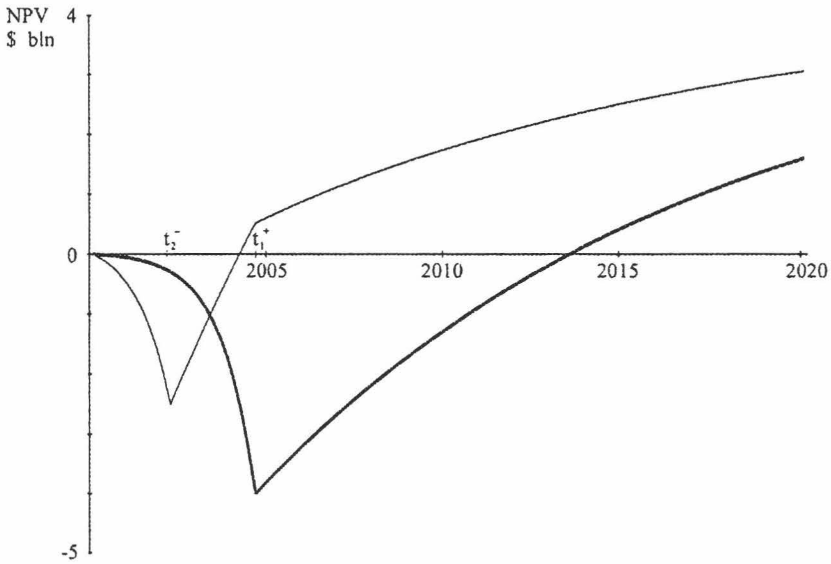


Fig. 11. NPV dynamics for the slow-fast scenario of the gas pipeline competition in the Caspian region.

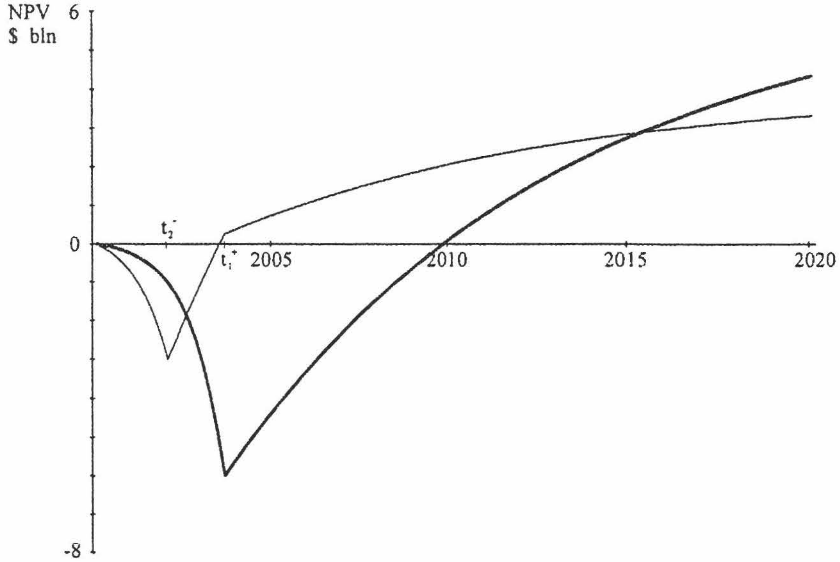


Fig. 12. NPV dynamics for the Nash equilibrium scenario of the planned pipeline projects to the gas market in China.

The initial year for the projects is set in 2001.

In this case study, there exists a unique slow-fast Nash equilibrium

$$(t_1^+, t_2^-) = (2003.6, 2002).$$

Figure 12 shows the dynamics of the NPV (51), $P_i = P_i(t, t_1, t_2)$, $i = 1, 2$, for projects 1 and 2 under the slow-fast Nash equilibrium investment scenario $(t_1, t_2) = (t_1^+, t_2^-)$. The heavy line and the fine line depict the NPV dynamics of the Kovikta-Zabaikalsk-Kharbin project and the Sakhalin-Khabarovsk-Kharbin project, respectively.

The results demonstrated on Figs. 10–12 have been calculated using the G-TIME software package elaborated and tested by O. I. Nikonov and Y. V. Minoullin.

7. Appendix: Proof of the Main Results

Here, we prove Propositions 5.1 to 5.4.

Proof of Proposition 5.1.

Step 1. Formula (29) was obtained in Ref. 14.

Step 2. The differentiation of (29) gives

$$C_i'(t_i) = \rho^{\alpha-1}\frac{-\lambda e^{-\lambda t_i}\bar{x}_i^{\alpha}(1-e^{-\rho t_i})^{\alpha-1} - e^{-\lambda t_i}\bar{x}_i^{\alpha}(\alpha-1)(1-e^{-\rho t_i})^{\alpha-2}\rho e^{-\rho t_i}}{(1-e^{-\rho t_i})^{2\alpha-2}}$$
$$= \frac{\rho^{\alpha-1}e^{-\lambda t_i}\bar{x}_i^{\alpha}(1-e^{-\rho t_i})^{\alpha-2}}{(1-e^{-\rho t_i})^{2\alpha-2}}[-\lambda(1-e^{-\rho t_i})-(\alpha-1)\rho e^{-\rho t_i}]$$
$$= -\rho^{\alpha-1}\bar{x}_i^{\alpha}\frac{e^{-\lambda t_i}}{(1-e^{-\rho t_i})^{\alpha}}[\lambda(1-e^{-\rho t_i})+(\alpha-1)\rho e^{-\rho t_i}]$$
$$= -\rho^{\alpha-1}\frac{e^{-\lambda t_i}\bar{x}_i^{\alpha}}{(1-e^{-\rho t_i})^{\alpha}}[\lambda+(\rho(\alpha-1)-\lambda)e^{-\rho t_i}]$$
$$= -\rho^{\alpha-1}\frac{e^{-\lambda t_i}\bar{x}_i^{\alpha}(\lambda+\nu e^{-\rho t_i})}{(1-e^{-\rho t_i})^{\alpha}}. \tag{52}$$

For the last transformation, we have used the equality

$$\rho(\alpha-1)-\lambda=\alpha\sigma$$

following from (30) and the notation (32). For $a_i(t_i) = -C_i'(t_i)$ [see (1)], we have (31).

Step 3. Assume that player i occupies the market solely. Then, the price is given by

$$\pi(y|t) = (g(t)/y_i)^{\beta},$$

and the payoff $p_i(y_1, y_2|t)$ to player i equals

$$p_i(y_i|t) = e^{-\lambda t}[g(t)^{\beta}y_i^{1-\beta} - c_i(t)y_i]. \tag{53}$$

The supply game is reduced to an optimization problem, and $b_{i1}(t)$ is found as the maximum of $p_i(y_i|t)$ over all positive y_i. Since $p_i(y_i|t)$ is strictly concave in y_i, its maximum is reached at the unique point $y_i(t)>0$ such that

$$dp_i(y_i(t)|t)/dy_i = e^{-\lambda t}g(t)^{\beta}[(1-\beta)y_i^{-\beta}(t) - c_i(t)] = 0.$$

Hence,

$$y_i(t) = [g(t)/c_i(t)^{1/\beta}](1-\beta)^{1/\beta}.$$

Recall that

$$b_{i1}(t) = p_i(y_i(t)|t)$$

and substitute $y_i = y_i(t)$ into (53). We get

$$b_{i1}(t) = e^{-\lambda t}[g^{\beta}(t)/y_i^{\beta} - c_i(t)]y_i(t)$$
$$= e^{-\lambda t}[c_i(t)/(1-\beta) - c_i(t)][g(t)/c_i(t)^{1/\beta}](1-\beta)^{1/\beta},$$

and finally,

$$b_{i1}(t) = e^{-\lambda t}\beta(1-\beta)^{1/\beta-1}g(t)/c_i^{1/\beta-1}(t);$$

i.e., (33) holds.

Step 4. Now, let Assumption 5.1 hold and let both players operate on the market. Then,

$$\pi(y|(t) = [g(t)/(y_1 + y_2)]^\beta,$$

and for the payoff to player i, we have

$$p_i(y_1, y_2|t) = e^{-\lambda t}[g(t)^\beta y_i/(y_1 + y_2)^\beta - c_i(t)y_i]. \tag{54}$$

Let us show that the instantaneous supply game has a unique Nash equilibrium under Assumption 5.1.

Since $p_i(y_1, y_2|t)$, $i = 1, 2$, is strictly concave in y_i, a point (y_1, y_2) is a Nash equilibrium if and only if

$$\partial p_i(y_1, y_2|t)/\partial y_i = 0, \tag{55}$$

or explicitly,

$$g^\beta(t)/y^\beta - \beta g^\beta(t)y_i/y^{\beta+1} - c_i(t) = 0. \tag{56}$$

Here, as above,

$$y = y_1 + y_2.$$

For the sum of the left-hand sides for $i = 1, 2$, we have

$$2g^\beta(t)/y^\beta - \beta g^\beta(t)/y^\beta - [c_1(t) + c_2(t)] = 0.$$

Hence,

$$(2-\beta)g^\beta(t) = [c_1(t) + c_2(t)]y^\beta$$

and

$$y^\beta = (2-\beta)g^\beta(t)/[c_1(t) + c_2(t)]. \tag{57}$$

Rewriting (56) as

$$\beta g^\beta(t)y_i = g^\beta(t)y - c_i(t)y^{\beta+1}$$

and using (57), we get

$$\begin{aligned} y_i &= [y/\beta g^\beta(t)][g^\beta(t) - c_i(t)y^\beta] \\ &= [y/\beta g^\beta(t)](g^\beta(t) - \{(2-\beta)c_i(t)/[c_1(t)+c_2(t)]\}g^\beta(t)) \\ &= \{(2-\beta)g^\beta(t)/[c_1(t)+c_2(t)]\}^{1/\beta}(1/\beta)\{1-(2-\beta)c_i(t)/[c_1(t)+c_2(t)]\} \\ &= [(2-\beta)^{1/\beta}/\beta]\{1-(2-\beta)c_i(t)/[c_1(t)+c_2(t)]\}g(t)/[c_1(t)+c_2(t)]^{1/\beta}. \qquad (58) \end{aligned}$$

The latter is necessary for (y_1, y_2) to be a Nash equilibrium in the supply game. Hence, if the Nash equilibrium exists, it is unique. Point (y_1, y_2) given by (58) has positive components due to Assumption 5.1; See (28). Moreover, (y_1, y_2) satisfies (56), where $y = y_1 + y_2$, which is equivalent to (55). Hence, (y_1, y_2) is the Nash equilibrium. We have stated that a unique Nash equilibrium exists. Denote it $(y_1(t), y_2(t))$. By (58), we get

$$y_i(t) = [(2-\beta)^{1/\beta}/\beta]\{1-(2-\beta)c_i(t)/[c_1(t)+c_2(t)]\}g(t)/[c_1(t)+c_2(t)]^{1/\beta}.$$

By definition,

$$b_{i2}(t) = p_i(y_1(t), y_2(t)|t).$$

Substituting $y_i = y_i(t)$, $i = 1, 2$, into (54) and noticing that

$$y = y_1(t) + y_2(t)$$

is given by (57), we get

$$\begin{aligned} b_{i2}(t) &= e^{-\lambda t}\left[\frac{g^\beta(t)}{y^\beta} - c_i(t)\right]y_i(t) \\ &= e^{-\lambda t}\{[c_1(t)+c_2(t)]/(2-\beta) - c_i(t)\}y_i(t) \\ &= e^{-\lambda t}\left\{\frac{[c_1(t)+c_2(t)]}{(2-\beta)}\right\}\left(1 - \frac{(2-\beta)c_i(t)}{c_1(t)+c_2(t)}\right)[(2-\beta)^{1/\beta}/\beta]\times \\ &\qquad \{1-(2-\beta)c_i(t)/[c_1(t)+c_2(t)]\}\frac{g(t)}{[c_1(t)+c_2(t)]^{1/\beta}}, \end{aligned}$$

and finally,

$$b_{i2}(t) = e^{-\lambda t}(2-\beta)^{1/\beta-1}\{1-(2-\beta)c_i(t)/[c_1(t)+c_2(t)]\}^2 g(t)/[c_1(t)+c_2(t)]^{1/\beta-1}.$$

Formula (34) is proved.

Step 5. By definition, we have

$$\begin{aligned} b_{i2}(t) &= p_i(y_1(t), y_2(t)|t) \\ &= e^{-\lambda t}[\pi(t, y_1(t) + y_2(t)) - c_i(t)]y_i(t) \\ &= \{g(t)/[y_1(t) + y_2(t)]\}^{\beta} y_i(t) - c_i(t)y_i(t) \\ &< [g(t)/y_i(t)]^{\beta} y_i(t) - c_i(t)y_i(t) \\ &\leq \sup_{y_i > 0} \{[g(t)/y_i]^{\beta} y_i - c_i(t)y_i\} \\ &= b_{i1}(t). \end{aligned}$$

Inequality (35) is stated. Proposition 5.1 is proved. □

Proof of Proposition 5.2. Let us check Assumption 2.1. The function $C_i(t_i)$ [see (29)] is continuously differentiable. The expression (52) for $C_i'(t)$ shows that $C_i'(t) < 0$. Hence, $C_i(t)$ is monotonically decreasing. Consider the ratio in the right-hand side. The numerator $e^{-\lambda t_i}\bar{x}_i$ decreases in t_i and the denominator $(1 - e^{-\rho t_i})^{\alpha}$ increases in t_i. Hence, the ratio decreases in t_i. Since the square bracket decreases in t_i, its product with the ratio decreases in t_i. As a result, we conclude that $C_i'(t_i)$ increases in t_i. We have shown that Assumption 2.1 is satisfied.

Let us turn to Assumption 2.2. For the rate of cost reduction, we have the expression (31) whose denominator tends to 0 when t approaches 0. Hence, $a_i(t)$ tends to infinity as t approaches 0. Therefore, for all $t > 0$ sufficiently small, we have

$$a_i(t) > b_{i1}(t) > b_{i2}(t).$$

The expression for $a_i(t)$ and $b_{i2}(t)$ [see (34)] show that

$$b_{i2}(t)/a_i(t) = h_0(t)h_{i2}(t),$$

where $h_{i2}(t)$ is given in (36) and $h_0(t)$ is such that, for some $\tau > 0$ and $\epsilon > 0$, the lower bound $\inf_{t \geq \tau} h_0(t) > \epsilon$ holds. By Assumption 5.2, $h(t)$ tends to infinity as t tends to infinity. Therefore, for all t sufficiently large, we have

$$b_{i1}(t) > b_{i2}(t) > a_i(t).$$

Since the functions $a_i(t)$, $b_{i1}(t)$, $b_{i2}(t)$ are continuous, there exist a $t_i^- > 0$ that solves the equation

$$a_i(t) = b_{i1}(t) \tag{59}$$

and a $t_i^+ > 0$ that solves the equation

$$a_i(t) = b_{i2}(t). \tag{60}$$

In order to state that Assumption 2.2 holds, it is now sufficient to show that t_i^- and t_i^+ are unique. We specify equation (59) by substituting the expressions for $a_i(t)$ and $b_{i1}(t)$; see (31) and (33). We get

$$\rho^{\alpha-1}\bar{x}_i^\alpha e^{-\lambda t}(\lambda + v\bar{x}_i^\alpha e^{-\rho t})/(1-e^{-\rho t})^\alpha = e^{-\lambda t}(1-\beta)^{1/\beta-1}g(t)/c_i^{1/\beta-1}(t).$$

Cancelling $e^{-\lambda t}$ and using the definition of $h_{i1}(t)$ [see (36)], we arrive at equation (40). The right-hand side of (40) strictly increases in t due to Assumption 5.2. Hence, equation (59) has the unique root t_i^-.

For equation (60), we argue similarly. Specify (60) by substituting the expression for $a_i(t)$ and $b_{i1}(t)$; see (31) and (34). We get

$$\begin{aligned}&\rho^{\alpha-1}\bar{x}_i^\alpha e^{-\lambda t}(\lambda + ve^{-\rho t})/(1-e^{-\rho t})^\alpha\\ &= e^{-\lambda t}(2-\beta)^{1/\beta-1}\{1-(2-\beta)c_i(t)/[c_1(t)+c_2(t)]\}^2 g(t)/[c_1(t)+c_2(t)]^{1/\beta-1}.\end{aligned}$$

Using the definition of $h_{i2}(t)$ [see (36)], we arrive at equation (41). The right-hand side of (41) strictly increases in t due to Assumption 5.2. Hence, equation (60) has the unique root t_i^+.

Proposition 5.2 is proved. □

Proof of Proposition 5.3. Let $i=1$ (for $i=2$, the argument is similar). Using formula (42) for $P_1(t_1, t_2)$ and taking into account that $\hat{t}_2$ lies between t_i^- and t_i^+ (see Proposition 3.3), we specify the equality

$$P_1(t_1^-, \hat{t}_2) = P_1(t_1^+, \hat{t}_2)$$

into

$$\begin{aligned}&-\rho^{\alpha-1}e^{-\lambda t_1^-}\bar{x}_1^\alpha/(1-e^{-\rho t_1^-})^{\alpha-1} + b_{11}^0 e^{-\psi t_1^-}/\psi + (b_{12}^0 - b_{11}^0)e^{-\psi\hat{t}_2}/\psi\\ &= -\rho^{\alpha-1}e^{-\lambda t_1^+}\bar{x}_1^\alpha/(1-e^{-\rho t_1^+})^{\alpha-1} + b_{12}^0 e^{-\psi t_1^+}/\psi.\end{aligned}$$

Resolving with respect to $\hat{t}_2$, we get

$$\begin{aligned}(b_{12}^0 - b_{11}^0)e^{-\psi\hat{t}_2}/\psi &= -\rho^{\alpha-1}e^{-\lambda t_1^+}\bar{x}_1^\alpha/(1-e^{-\rho t_1^+})^{\alpha-1} + b_{12}^0 e^{-\psi t_1^+}/\psi\\ &\quad + \rho^{\alpha-1}e^{-\lambda t_1^-}\bar{x}_1^\alpha/(1-e^{-\rho t_1^-})^{\alpha-1} - b_{11}^0 e^{-\psi t_1^-}/\psi,\end{aligned}$$

or

$$\hat{t}_2 = -(1/\psi)\log[\psi G_1/(b_{12}^0 - b_{11}^0)],$$

where

$$\begin{aligned}G_1 &= -\rho^{\alpha-1}e^{-\lambda t_1^+}\bar{x}_1^\alpha/(1-e^{-\rho t_1^+})^{\alpha-1}\\ &\quad + b_{12}^0 e^{-\psi t_1^+}/\psi + \rho^{\alpha-1}e^{-\lambda t_1^-}\bar{x}_1^\alpha/(1-e^{-\rho t_1^-})^{\alpha-1} - b_{11}^0 e^{-\psi t_1^-}/\psi.\end{aligned}$$

The representation (44), (45) is stated. □

Proof of Proposition 5.4.

Step 1. Due to the form of $g(t)$ and $c_i(t)$ [see (37)], equation (40), which determines the fast choice t_i^- of player i, is specified as

$$\rho^{\alpha-1}\bar{x}_i^{\alpha}/(1-\beta)^{1/\beta-1} = [(1-e^{-\rho t})^{\alpha}/(\lambda+ve^{-\rho t})]\,g^0 e^{\kappa t}/(c_i^0)^{1/\beta-1},$$

or

$$\rho^{\alpha-1}\bar{x}_i^{\alpha}(c_i^0)^{1/\beta-1}/(1-\beta)^{1/\beta-1}g^0 = e^{\kappa t}(1-e^{-\rho t})^{\alpha}/(\lambda+ve^{-\rho t}).$$

Using the notations (47) and (48), we arrive at equation (46).

Step 2. Due to (37), equation (41) determining t_i^+ is specified as

$$\begin{aligned}&\rho^{\alpha-1}\bar{x}_i^{\alpha}/(1-\beta)^{1/\beta-1}\\&= [(1-e^{-\rho t})^{\alpha}/(\lambda+ve^{-\rho t})][1-(2-\beta)c_i^0/(c_1^0+c_2^0)]^2 g^0 e^{\kappa t}/(c_1^0+c_2^0)^{1/\beta-1}.\end{aligned}$$

Using the notations (47) and (50), we arrive at equation (49). □

8. Conclusions

The paper is devoted to the analysis of a two-player game, in which the players strategies are the times of terminating the individual dynamical processes. The formal setting is related to the management of large-scale innovation projects, whose key feature is that the profits gained through the implementation of the projects are highly sensitive to the projects commercialization times. The basic reason for that is that the price formation mechanism changes rapidly the price as a new project is commercialized and the supply sharply increases. This situation is analyzed in the context of the competition of two projects on the construction of gas pipelines. In the game between the projects, the total profits gained during the pipelines life periods act as payoffs and the commercialization times as strategies. The reduction of project management to choices of the commercialization times is justified by the assumption that the individual regulation mechanisms, comprising investments into the construction of the gas pipelines and regulation of supply, work optimally provided the commercialization times are given. The analysis of the game leads to the restriction of the player rational choices to no more than two prescribed combinations of commercialization times, which constitute the Nash equilibria in the game. Typically, two Nash equilibria arise and the projects compete for a fast commercialization scenario; its complement, a slow commercialization scenario, is less profitable, representing the best response to the fast scenario of the competitor. A simple algorithm for finding the Nash equilibria is described.

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