PP-77-8

THE AVERAGING METHOD APPLIED TO THE INVESTIGATION OF SUBSTANTIAL TIME VARYING SYSTEMS OF A HIGHER ORDER

N. Vorontsov

September 1977

Abstract

This paper presents an analytical method to solving a system of differential equations describing a substantial time-varying feedback automatic control system. The method provides good results for systems whose characteristic equations have dominant complex conjugated roots. As an example, a five-order system was investigated. The results of the computer simulation proved the validity of applying this method to the above mentioned class of systems.

The Averaging Method Applied to the Investigation of Substantial Time Varying Systems of a Higher Order

It is well known that the averaging method originated in the area of planetary mechanics and that its development was connected mainly with problems in this area. To solve these problems different methods of averaging were applied (those of Gauss, Delone, and others). The main idea of the averaging method is that the right parts of complicated differential equations which describe oscillation or rotation were substituted by flattened averaged functions which contain neither time in an obvious form nor quickly changing parameters.
The resulting equations from averaging can be either exactly integrated or simplified, and after that an important conclusion concerning the movement investigated both of qualitative and quantitative character may be drawn.

The well known works of the Dutch scientist Van-der-Pole are considered as a basis for the systematical application of the averaging method for investigating non-linear oscillatory processes in the field of mechanics, radio- and electro- engineering. He found a basic method for solving non-linear differential equations describing oscillatory processes in a one-dimensional system. Due to its simplicity and descriptive character one then began to apply his method for investigating oscillatory processes

Significant results were obtained in the development of averaging by N.M. Krylov and N.N. Bogoliubov [1], who proved that the method could be applied when the right parts of averaging differential equations are quazi periodical time functions. They also found a general approach for investigating non-linear equations. The main idea of this approach is to substitute variables in such a way that allows for the separation of "slow changeable" from "quick changeable" variables. Such a substitution allows for the solution of these equations to be represented as an asymptotical series, the first part of which coincides with the solution of Van-der-Pole's method.

The Krylov-Bogoliubov method is worked out by applying those systems which are described by non-linear equations:

$$\ddot{x} + \varepsilon f(x,\dot{x}) + w_0^2 x = 0 \quad , \qquad (1)$$

here ε is a small parameter, and

w_0 is some constant.

The first approximate solution to equation (1) is

$$x(t) = a(t)\mathrm{Sin}[W_0 t + \Phi(t)] \quad . \qquad (2)$$

Here the amplitude $a(t)$ and phase $\Phi(t)$ can be determined by equations

$$\dot{a}(t) = - \frac{\varepsilon}{2\pi w_0} \int_0^{2\pi} f(a\mathrm{Sin}\Psi, aw\cos\Psi)\cos\Psi d\Psi \quad , \qquad (3)$$

$$\dot{\Phi}(t) = \frac{\varepsilon}{2\pi a w_0} \int_0^{2\pi} f(a\mathrm{Sin}\Psi, aw\ \cos\Psi)\mathrm{Sin}\Psi d\Psi \quad , \qquad (4)$$

where

$$\Psi = w_0 t + \Phi \quad .$$

The main advantage of equations (3) and (4) which result from averaging is that they do not contain time in an obvious form in their right parts.

The best results of this method used for solving equation (1) were obtained when the small parameter ε is nearly or equal to zero.

Apart from the equations of the (1) type there is a wide class of linear second-order differential equations with time varying parameters similar in form to equation (1) for which solution the averaging method can also be applied [2, 3].

The second-order differential equation

$$\ddot{x} - 2\sigma\dot{x} + (\sigma^2 + w^2)x = 0 \quad , \tag{5}$$

which describes a linear time varying system similar to (1) can be shown as

$$\ddot{x} + [\sigma^2 + w^2 - w_0^2)x - 2\sigma\dot{x}] + w_0^2 x = 0 \quad , \tag{6}$$

where

$$\varepsilon f(x,\dot{x}) = (\sigma^2 + w^2 - w_0^2)x - 2\sigma\dot{x} \quad , \tag{7}$$

and

$$\sigma = \sigma(t) \quad , \; w = w(t) \quad ,$$

are time functions.

In this case equations (3) and (4) may be written as follows:

$$\dot{a} = -\frac{1}{2\pi w_0}\int_0^{2\pi} [(\sigma^2 + w^2 - w_0^2)a\mathrm{Sin}\Psi - 2\sigma a w_0 \cos\Psi] \cdot \cos\Psi d\Psi = \sigma a \quad , \tag{8}$$

$$\dot{\Phi} = \frac{1}{2\pi a w_0} \int_0^{2\pi} [(\sigma^2 + w^2 - w_0^2) a \sin\Psi - 2\sigma a w_0 \cos\Psi] \cdot \sin\Psi d\Psi = \frac{1}{2w_0} (\sigma^2 + w^2 - w_0^2) \quad . \tag{9}$$

As a result of integrating the equations (8) and (9), the approximate solution of equation (2) can be found:

$$x(t) = a_0 \exp(\int_0^t \sigma_1 dt) \sin(\int_0^t w_1 dt + \Phi_0) \quad , \tag{10}$$

here a_0, Φ_0 are constants of integration, and

$$\sigma_1 = \sigma \quad , \quad w_1 = \frac{1}{2w_0} (\sigma + w + w_0) \quad .$$

After double differentiation of (10) we have

$$\dot{x} = a_0 \{\exp(\int_0^t \sigma_1 dt) \cos(\int_0^t w_1 dt + \Phi_0) w_1^2 + \sigma_1 \cdot \exp(\int_0^t \sigma_1 dt) \sin(\int_0^t w_1 dt + \Phi_0)\} \quad , \tag{11}$$

$$\ddot{x} = a_0\{- \exp(\int_0^t \sigma_1 dt) \sin(\int_0^t w_1 dt + \Phi_0) w_1^2 + \sigma_1 \cdot \exp(\int_0^t \sigma_1 dt) \cos(\int_0^t w_1 dt + \Phi_0) w_1 + \dot{w}_1 \cdot \exp(\int_0^t \sigma_1 dt) \cos(\int_0^t w_1 dt + \Phi_0) + \dot{\sigma}_1 \exp(\int_0^t \sigma_1 dt) \cdot \sin(\int_0^t w_1 dt + \Phi_0) + \sigma_1^2 \exp(\int_0^t \sigma_1 dt) \sin(\int_0^t w_1 dt + \Phi_0) + \sigma_1 \exp(\int_0^t \sigma_1 dt) \cos(\int_0^t w_1 dt + \Phi_0) w_1\} \quad . \tag{12}$$

Equation (5) when taking into account (11) and (12) under the conditions

$$\begin{aligned} &\text{i)} \quad \int_0^t w_1 dt = \frac{\pi}{2} - \Phi_0 \quad , \\ &\text{ii)} \quad t = 0 \quad , \end{aligned} \tag{13}$$

provides the following system:

$$\begin{aligned} &\dot{\sigma}_1 + \sigma_1^2 - 2\sigma_1\sigma - w_1^2 + \sigma^2 + w^2 = 0 \quad , \\ &2\sigma_1 w_1 + \dot{w}_1 - 2\sigma w_1 = 0 \quad . \end{aligned} \tag{14}$$

Having solved this system of equations relative to σ_1 and w_1^2, and neglecting the values of higher order we obtain

$$\begin{aligned} &\text{i)} \quad \sigma_1 = \sigma - \frac{\dot{w}_1}{2w_1} \quad , \\ &\text{ii)} \quad w_1^2 = \dot{\sigma} + w^2 \quad . \end{aligned} \tag{15}$$

The differentiating of (15, ii) gives

$$2w_1\dot{w}_1 = \ddot{\sigma} + 2w\dot{w} \approx 2w\dot{w} \quad .$$

Finally, (15) can be shown as

$$\begin{aligned} &\sigma_1 = \sigma - \frac{w\dot{w}}{2(\dot{\sigma} + w^2)} \quad , \\ &w_1^2 = \dot{\sigma} + w^2 \quad . \end{aligned} \tag{16}$$

In the case where the functions $\sigma(t)$ and $w(t)$ change slowly $\dot{\sigma} \approx \dot{w} \approx 0$, and so according to (16) $\sigma_1 = \sigma$, $w_1 = w$.

Thus, the approximate solution to equation (5) may be written in the following form:

$$x(t) = a_0 \exp\left(\int_0^t \sigma(t)dt\right)\mathrm{Sin}\left(\int_0^t w(t)dt + \Phi\right) \quad , \qquad (17)$$

where $\sigma(t)$ and $w(t)$ are monotonous time functions.

In the work [3] it was shown that this method could be applied to investigating time varying systems which are described with high-order linear differential equations with time varying coefficients.

$$\sum_{k=0}^{n} a_k S^k X = 0 \quad , \qquad (18)$$

where

$$S = \frac{d}{dt} \quad , \quad a_k = a_k(t) \quad , \quad (K=0,1\ldots,n) \quad .$$

From the solution of equation (18) in the form (17) one may presume that the dominant roots in a characteristic equation

$$F(S) = \sum_{k=0}^{n} a_k S^k = 0 \quad , \qquad (19)$$

correspond to the second-order equation with monotonous coefficients

$$S^2 - 2\sigma S + \sigma^2 + w^2 = 0 \quad , \qquad (20)$$

where

$$\sigma = \sigma(t) \quad , \; w = w(t) \quad .$$

The main advantage of this method is the fact that restrictions are only put on the speed of real and imaginary changing parts of dominant complex conjugated roots, but not on the speed of changing parameters in the system. This is especially important for investigating systems which have a rather high speed of changing their parameters.

Suppose that the differential equation of a considered time varying system in reference to output x has the following form:

$$a_5(t)x^V + a_4(t)x^{IV} + a_3(t)\dddot{x} + a_2(t)\ddot{x} + a_1(t)\dot{x} + a_0(t)\,x = 0 \quad . \tag{21}$$

And the system has such parameters that the trajectories of moving roots of a proper characteristic equation for the whole interval considered t_0, $t(t_0 = 0,\ t = 4$ sec) can be illustrated as shown in Figure 1. (The curves are calculated by applying the method of "fixed parameters".)

For the solution of a differential equation (21) in the form of (17) we write the second-order equation (20) corresponding to the dominant complex conjugated roots (β_1, β_2).

Having determined the law of changing parameters $\sigma(t)$ and $w(t)$, the equation (20) on the whole time interval considered enables one to find the system's response relative to the output x.

The dependence of variables $\sigma(t)$ and $w(t)$ from time is shown in Table 1.

In order to obtain a solution to equation (21) in a closed analytical form we approximate the step functions of Table 1 by polynomials of the third-order. So for this variable one may write

$$\sigma(t) = C_0 t^3 + C_1 t^2 + C_2 t + C_3 \quad . \tag{22}$$

The coefficients of the polynomial should be chosen so that the expression (23) has the minimal value

$$\delta = \sum_{i=1}^{m} [\sigma_i - (C_0 t_i^3 + C_1 t_i^2 + C_2 t_i + C_3)]^2 \quad . \tag{23}$$

By putting the coordinates of points (σ_i, t_i) in the equation (22) one gets m equations with four unknown variables.

$$\begin{aligned} C_0 t_1^3 + C_1 t_1^2 + C_2 t_1 + C_3 &= \sigma_1 \quad , \\ C_0 t_2^3 + C_1 t_2^2 + C_2 t_2 + C_3 &= \sigma_2 \quad , \ldots\ldots\ldots \\ C_0 t_i^3 + C_1 t_i^2 + C_2 t_i + C_3 &= \sigma_3 \quad , \ldots\ldots\ldots \\ C_0 t_m^3 + C_1 t_m^2 + C_2 t_m + C_3 &= \sigma_m \quad . \end{aligned} \tag{24}$$

According to Legendr's principle the system of conditional equations (24) in which the number of unknown variables is less than the number of equations can be reduced to a system of linear equations

$$
\begin{aligned}
&C_3 + C_2 \sum_{i=1}^{m} t_i + C_1 \sum_{i=1}^{m} t_1^2 + C_0 \sum_{i=1}^{m} t_i^3 = \sum_{i=1}^{m} \sigma_i \quad , \\
&C_3 \sum_{i=1}^{m} t_i + C_2 \sum_{i=1}^{m} t_i^2 + C_1 \sum_{i=1}^{m} t_i^3 + C_0 \sum_{i=1}^{m} t_i^4 \\
&= \sum_{i=1}^{m} \sigma_i t_i \quad , \\
&C_3 \sum_{i=1}^{m} t_i^2 + C_2 \sum_{i=1}^{m} t_i^3 + C_1 \sum_{i=1}^{m} t_i^4 + C_0 \sum_{i=1}^{m} t_i^5 \\
&= \sum_{i=1}^{m} \sigma_i t_i^2 \quad , \\
&C_3 \sum_{i=1}^{m} t_i^3 + C_2 \sum_{i=1}^{m} t_i^4 + C_1 \sum_{i=1}^{m} t_i^5 + C_0 \sum_{i=1}^{m} t_i^6 \\
&= \sum_{i=1}^{m} \sigma i t_i^3 \quad ,
\end{aligned}
\tag{25}
$$

where the number of unknown variables is equal to the number of equations. At the same time the minimum of δ is guaranteed. The system of equations (25) can be solved by one of the known methods. In the same way one can approximate the function $w(t)$. As a result we find

$$
\begin{aligned}
\sigma(t) &= 0{,}1455t^3 - 0{,}7306t^2 + 1{,}0919t - 3{,}8284 \quad , \\
w(t) &= -0{,}0042t^3 + 0{,}0792t^2 - 0{,}0493 + 3{,}0116 \quad .
\end{aligned}
$$

The curves $\sigma(t)$ and $w(t)$ are represented in Figure 2.

The approximate solution to equation (21) in the form (17) was found for the following initial conditions:

$$
\begin{aligned}
x(0) &= 0{,}105\frac{1}{\text{sec}} \quad , \quad \sigma(0) = -3{,}8284 \\
\dot{x}(0) &= 0 \quad , \quad w(0) = 3{,}0116
\end{aligned}
\tag{26}
$$

Here a_0 and Φ_0 can be found as a result of differentiating (17) and substituting the initial conditions (26) both in the expression received after the differentiation and the equation (17).

Finally, the approximate expression for the response function related to the output in the five-order system can be written:

$$\begin{aligned} x(t) = 0{,}1701 \exp(&0{,}0364t^4 - 0{,}2435t^3 + 0{,}5459t^2 \\ &- 3{,}82845t)\operatorname{Sin}(- 0{,}0011t^4 + 0{,}0264t^3 \\ &- 0{,}0246t^2 + 3{,}9116t + 0{,}6661) \end{aligned} \tag{27}$$

The estimated accuracy of solving equation (21) is done by comparing the results (21) with the results of the computer simulation.

To solve the equation (21) by a numerical method on a digital computer one should bring this equation to a normal linear form [4]. In order to do this we shall introduce new unknown functions

$$z_1 = x \;,\quad z_2 = \dot{x} \;,\quad z_3 = \ddot{x} \;,\quad z_4 = \dddot{x} \;,$$
$$z_5 = x^{IV} \;.$$

These unknown functions z_1, z_2, z_3, z_4, z_5 suit the linear system

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ \dot{z}_3 &= z_4 \end{aligned} \tag{28}$$

cont'd.

$$\dot{z}_4 = z_5 \qquad \text{cont'd (28)}$$

$$\dot{z}_5 = \frac{[-a_5(t)z_1 - a_4(t)z_2 - a_3(t)z_3 - a_2(t)z_4 - a_1(t)z_5]}{a_0(t)}$$

This system can be written in a matrix form:

$$\dot{z} = B(t)z \quad . \tag{29}$$

Here the matrix is

$$B(t) = \left\{ \begin{matrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \frac{-a_5(t)}{a_0(t)} & \frac{-a_4(t)}{a_0(t)} & \frac{-a_3(t)}{a_0(t)} & \frac{-a_2(t)}{a_0(t)} & \frac{-a_1(t)}{a_0(t)} \end{matrix} \right.$$

The structural scheme of the received normal linear system can be shown in Figure 3.

We integrate the system of linear differential equations with time varying coefficients (29) under the following initial conditions:

$$z_1(0) = 0{,}105\frac{1}{\text{sec}} \quad ; \quad z_2(0) = 0 \quad ; \quad z_3(0) = 0 \quad ;$$

$$z_4(0) = 0 \quad ; \quad z_5(0) = 0 \quad .$$

In Figure 4 the curves are drawn showing the response functions of the system. (1 was obtained by applying the averaging method, and 2 as a result of digital simulation).

It is obvious that the error which is introduced by averaging method takes place in the first stage of the system's response. Later on both curves will practically coincide.

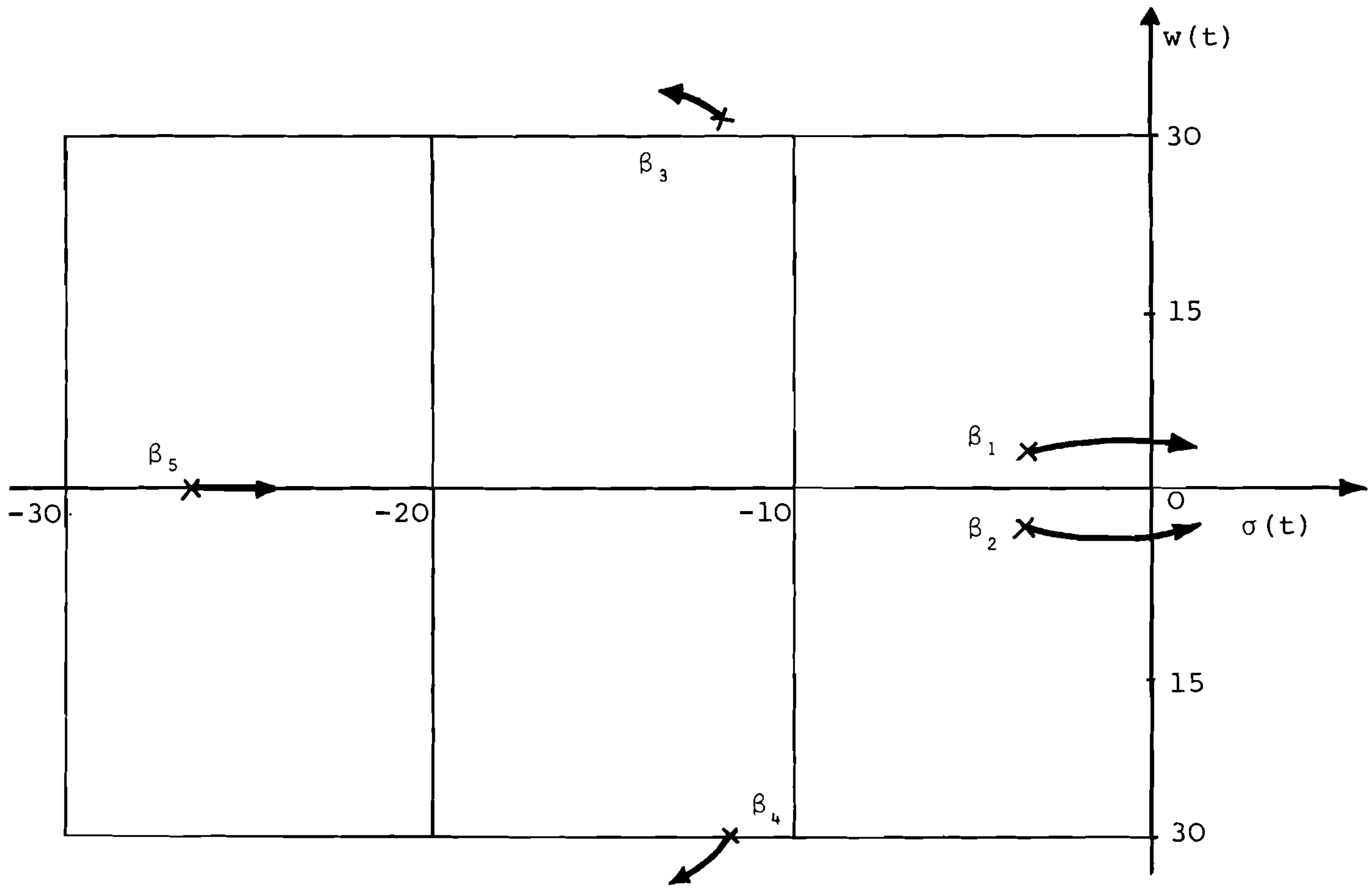

Figure 1. Trajectories of Moving Roots of the Characteristic Equation

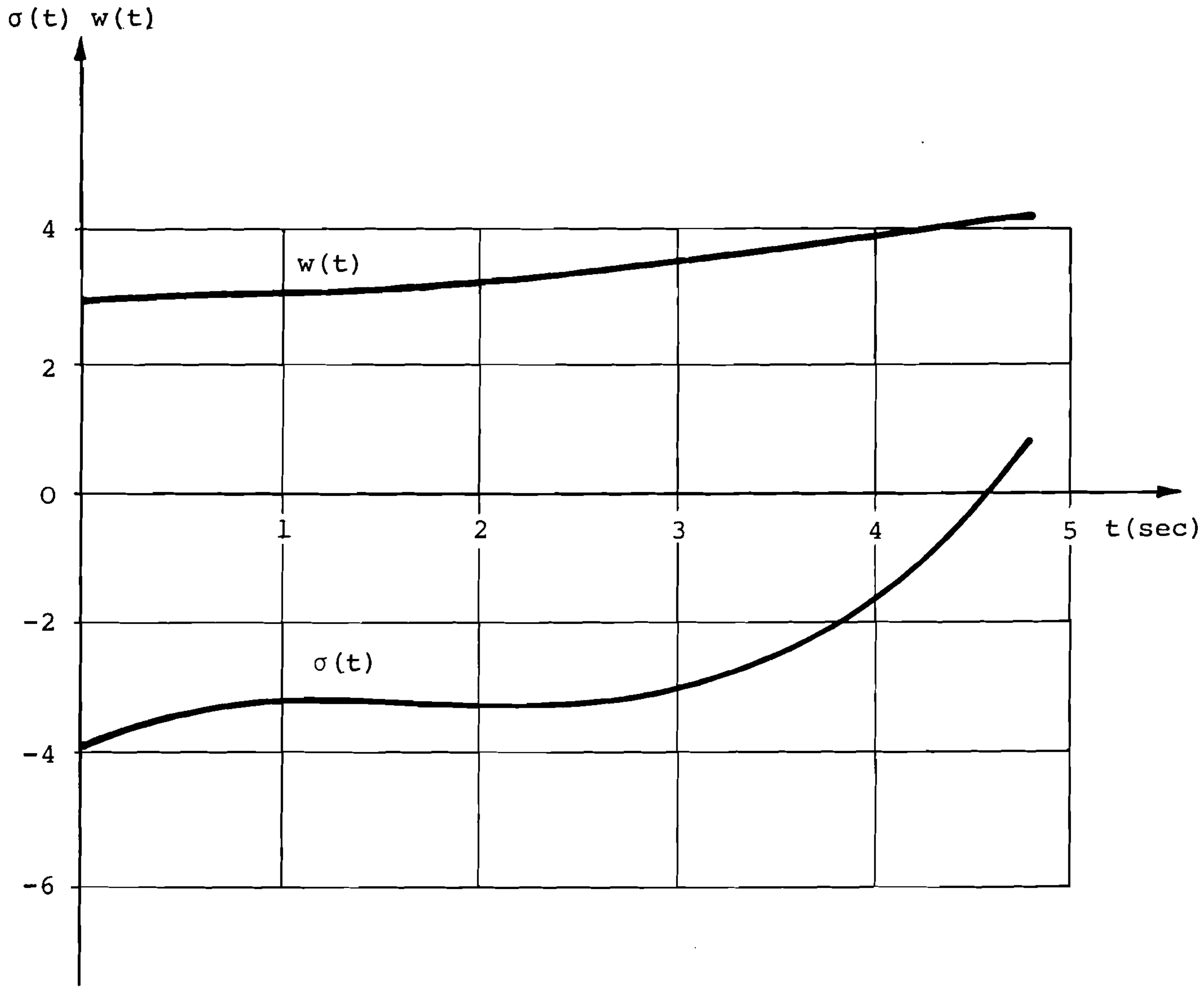

Figure 2. The Law of Changing Variables σ(t) and w(t).

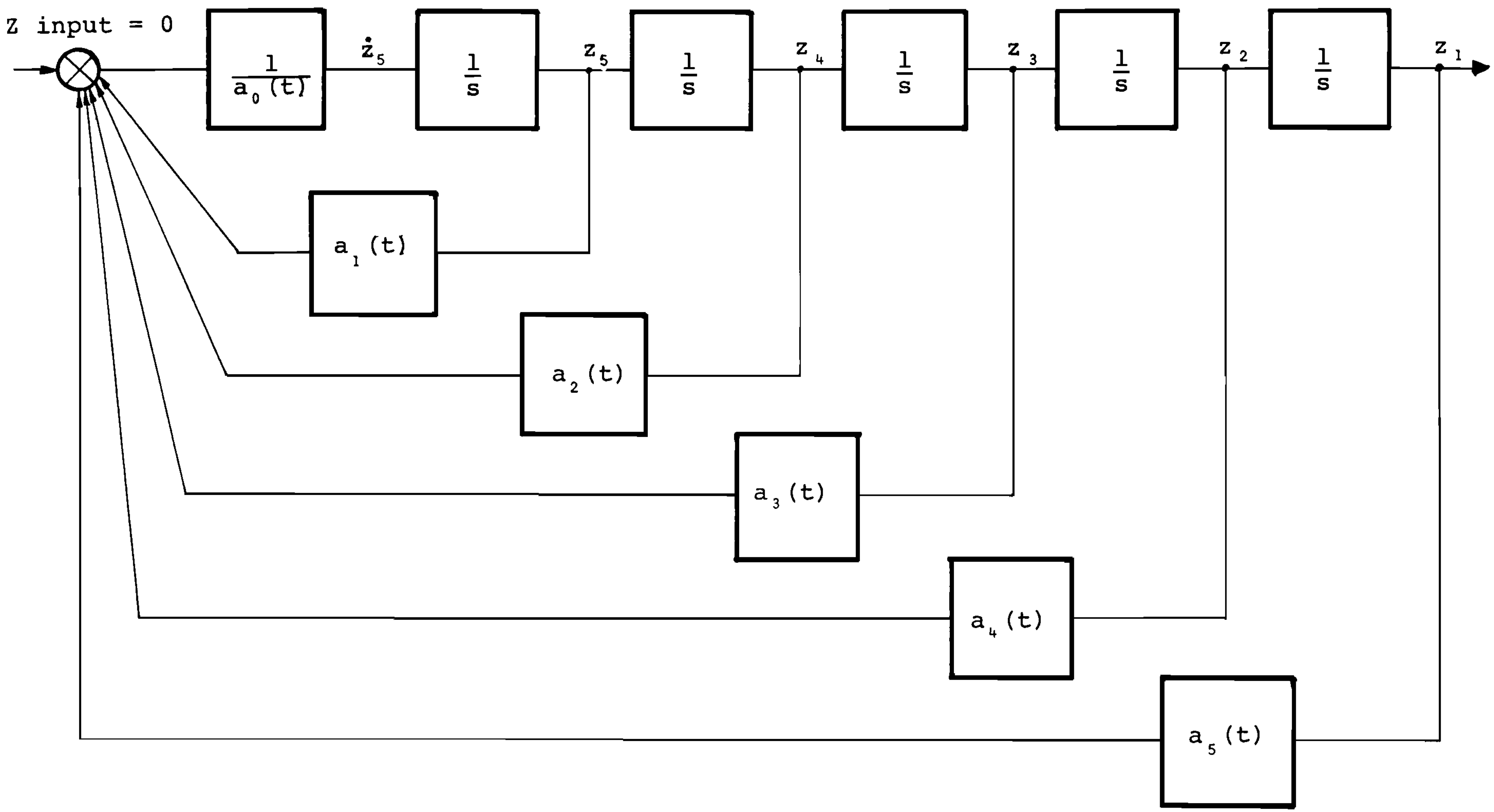

Figure 3. The Structural Scheme of the System Reduced to a Normal Form.

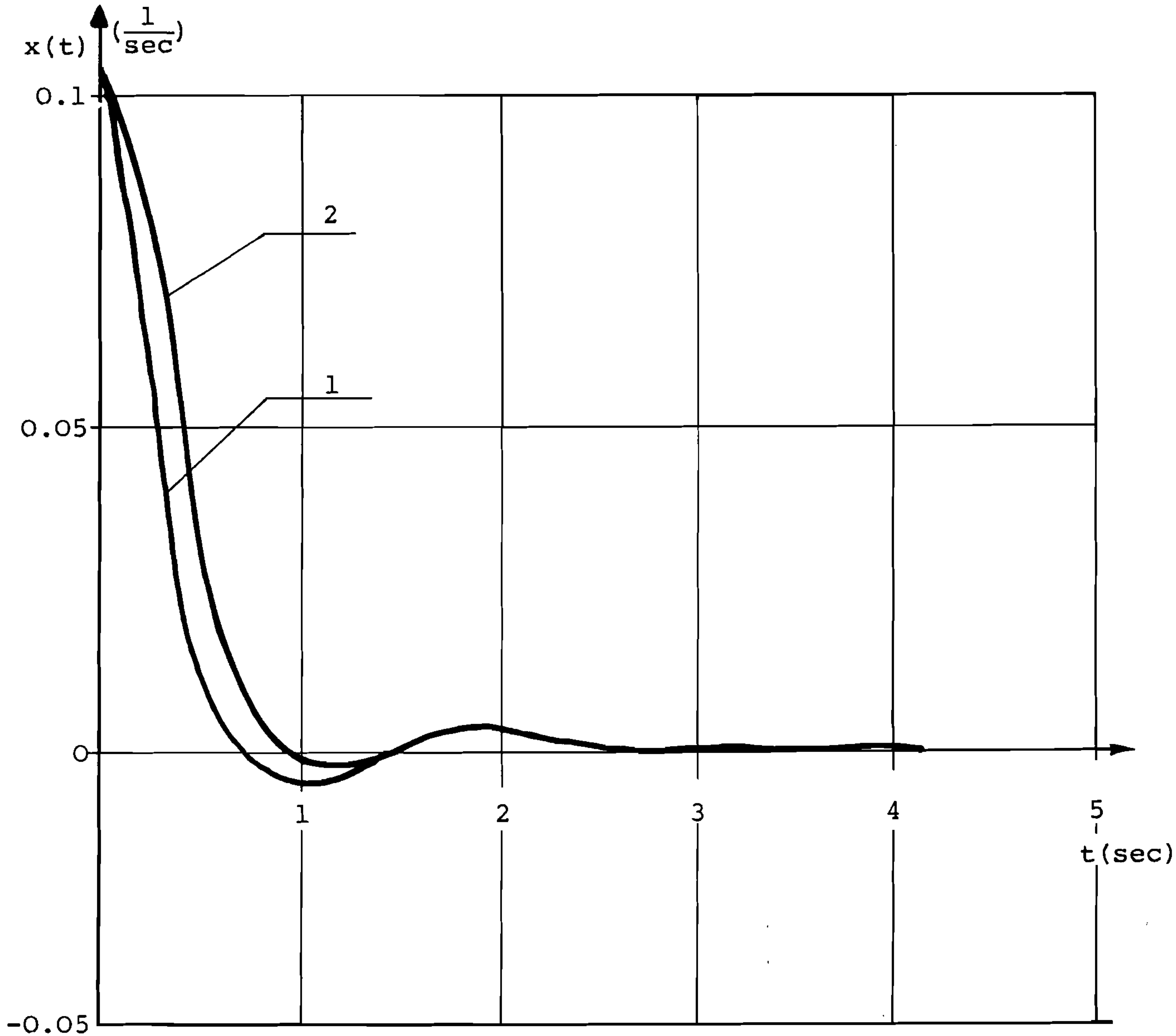

Figure 4. The Response Functions of the System.

Table 1. Dependence of Variables $\sigma(t)$ and $w(t)$ from Time.

t(sec)	$\sigma(t)$	w(t)	t(sec)	$\sigma(t)$	w(t)
0	-3,5836	2,9768	2.2	-3,2827	3,2290
0.2	-3,5674	2,9909	2.4	-3,2304	3,2685
0.4	-3,5502	3,0070	2.6	-3,1697	3,3119
0.6	-3,5311	3,0241	2.8	-3,0979	3,3604
0.8	-3,5104	3,0428	3.0	-3,0117	3,4173
1.0	-3,4878	3,0631	3.2	-2,9064	3,4820
1.2	-3,4624	3,0849	3.4	-2,7753	3,5570
1.4	-3,4345	3,1073	3.6	-2,6074	3,6439
1.6	-3,4031	3,1332	3.8	-2,3860	3,7426
1.8	-3,3681	3,1621	3.9	-2,2454	3,7979
2.0	-3,3284	3,1940	4.0	-2,0772	3,8560

References

[1] Krylov, N.M., and N.N. Bogoliubov, *Introduction to Non-Linear Mechanics*, Academy of Sciences of the Ukranian SSR, Kiev, 1937.

[2] Huston, R.L., and L.F. Doty, Note on the Krylov-Bogoliubov Method Applied to Linear Differential Equations, *SIAM Journal of Applied Mathematics*, 14, No. 3, 1966.

[3] Bittel, R., and D. Silijak, An Application of the Krylov-Bogoliubov Method to Linear Time-Varying Systems, *International Journal of Control*, Series 1, 11, No. 3, 1970.

[4] Hsu, J.C., and A.W. Meyer, *Modern Control Principles and Applications*, McGraw-Hill, New York, 1968.