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I. Introduction

As has often been pointed out in the scientific literature, a basic requirement for the successful operation of any system is stability, i.e. that the system be insensitive to small perturbations away from its desired or equilibrium positions. Violations of this requirement lead to "catastrophes" in the sense of Zeeman and Thom [1-3] which generally indicate unsatisfactory system performance or, at least, some type of extreme behavior. In a controlled system, where the controlling action is generated upon the basis of measurements made upon the state of the system, so-called "feedback control", once it is established that it is possible to stabilize the system by <u>some</u> control law, the next question to ask is what kind of measurements are necessary. In other words, how many components of the state need be measured to generate a stabilizing feedback law. The objective of this report is to answer this question in the case of a linear, constant coefficent system utilizing linear feedback laws. As will be seen, this version of the problem will turn out to be sufficiently complex to require some new results in linear control theory for the solution, the primary obstacle being, of course, that the problem

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solution is not invariant under coordinate transformations.

Apparently the first formal statement of the general "minimal measurement" problem was in the article [4], although various versions of the problem have been treated in [5-7]. The results of the current report comprise a substantial extension of those presented in [8-10], although the results of [10] are not included due to the linearity assumptions on the feedback laws. In spirit, the current work is most closely related to that of [11-12], the basic (and important) difference being that only stability and not pre-assignment of the closed-loop system characteristic values is required. As would be expected, the weakened assumptions of this paper drastically alter the nature of the solution in that, in general, less information about the system is necessary for stabilization than that required for pole assignment.

II. Problem Statement

We begin with the linear system

$$\dot{x} = Fx + Gu \quad , \qquad (\Sigma)$$

where x is an n-dimensional vector, F an nxn constant matrix, and G is an nxm constant matrix. The basic problem is to find a constant mxn matrix (control law) K possessing the following properties:

i) the matrix (F - GK) has its characteristic values in the left half-plane (closed loop asymptotic stability) and

ii) the matrix K has as many identically zero columns as possible (the minimal number of components of x appear in the feedback law -Kx).

To avoid complicating the exposition, in this paper we treat only the single-input case (m = 1), deferring discussion of the multiple-input problem to a future work. Basically the same results are obtained, but under somewhat more restrictive algebraic assumptions. As it stands, the foregoing statement of the minimal measurement problem is a difficult question of linear algebra due to the lack of any computationally "clean" linear algorithms for characterizing a stability matrix. To make progress, it is necessary to reformulate the problem in a more tractable form. We accomplish this task by stating an equivalent linear regulator problem.

Consider minimizing the functional

$$J = \int_0^\infty [(y,y) + (u,u)]\, dt \tag{1}$$

over all u where u and y are connected by the relations

$$\dot{x} = Fx + gu \quad , \tag{2}$$
$$y = Sx \quad ,$$

F, g, and S being constant matrices of sizes, nxn, nx1, and pxn, respectively. It is well known that the minimizing u is given by the expression

$$\begin{aligned} u_{min}(t) &= -g'Px(t) \\ &= -Kx(t) \end{aligned}$$

where P is the positive semi-definite solution of the algebraic Riccati equation

$$S'S + PG + G'P - Pgg'P = 0 \quad . \tag{3}$$

To see the equivalence between the above regulator problem and the minimal measurement problem, we note that given any stable law K, if we can find a positive semi-definite P satisfying the relation $g'P = K$, then we may use P in Eq.(3) to generate the matrix $S'S$ which, in turn, gives the matrix S. This will always be possible without further assumptions for single-input systems. The necessary and sufficient conditions for solvability of the P, K relation in the multi-input case are given in [13]. Thus, if we can characterize the number of zero components in the law $g'P$ for the regulator problem, then by imposing the additional assumptions of stability of (F,g) and detectability of (F,S), standard results will insure that the law will be stable. The key issue will be to select a measurement matrix S which has the dual properties of detectability of the unstable modes of $\sum$ and possession of as many zero columns as possible.

III. Diagonal Systems

First of all, we give the solution to the minimal measurement problem in the case of a diagonal system, then show that this solution suffices to answer the general case. To solve the diagonal problem, we shall employ some new results first given in [14]. The results for the diagonal case validate one's intuitive feeling that the number of components of the state which must be measured equals, in general, the number of characteristic roots of the system matrix having non-negative real parts.

Consider the system

$$\dot{x} = Fx + gu \quad , \qquad (\Sigma)$$

where we now assume F is a normal matrix, g and x being as defined in section II. Make the change of state coordinates z = Tx, where T is the nonsingular matrix diagonalizing F. We must now investigate the system

$$\dot{z} = \Lambda z + bu \quad , \qquad (\Sigma')$$

where Λ is the diagonal matrix $\Lambda = TFT^{-1}$ and b = Tg. Recalling the discussion of section II, we form the equivalent regulator problem for (Σ') which leads to the algebraic Riccati equation

$$Q + P\Lambda + \Lambda P - Pbb'P = 0 \quad , \qquad (4)$$

where Q = S'S is as yet undetermined. To simplify Eq.(4), we utilitze the following result from [14]:

<u>Theorem 1. Consider the algebraic Riccati equation</u>

$$Q + F'P + PF - PGG'P = 0 \quad ,$$

<u>where F is a matrix having no purely imaginary complex roots. Then the quantity H = PG is characterized by the equation</u>

$$\mathscr{A}(H) = (G'\otimes I)(I\otimes F' + F'\otimes I)^{-1}\mathscr{A}(HH' - Q) \quad , \qquad (5)$$

where $\mathcal{S}$ is the "stacking" operator whose action is to stack the columns of an nxm matrix into a single nmx1 vector.

Applying Theorem 1 to the algebraic Riccati equation (4) (after imposing the additional constraint that F have no purely imaginary roots), it is seen that the optimal feedback law h = Pb is characterized by the equation

$$h = (b'\otimes I)(\Lambda\otimes I + I\otimes\Lambda)^{-1}\mathcal{S}(hh' - Q) \quad . \tag{6}$$

Next, we note that the nmxn^2 matrix

$$A = (b'\otimes I)(\Lambda\otimes I + I\otimes\Lambda)^{-1} \tag{7}$$

has the structure

$$A = \left[\begin{array}{ccccccccc}
\frac{b_1}{2\lambda_{11}} & & \frac{b_2}{\lambda_{11}+\lambda_{22}} & & \frac{b_n}{\lambda_{11}+\lambda_{nn}} & & & & \\
 & \frac{b_1}{\lambda_{11}+\lambda_{22}} & & \frac{b_2}{2\lambda_{22}} & \cdots & \frac{b_n}{\lambda_{22}+\lambda_{nn}} & & & \\
 & & \ddots & & \ddots & & \ddots & & \\
 & & \frac{b_1}{\lambda_{11}+\lambda_{nn}} & & \frac{b_2}{\lambda_{22}+\lambda_{nn}} & & & & \frac{b_n}{2\lambda_{nn}} \\
\hline
 & & & & & & & & \\
 & & & & 0 & & & & \\
 & & & & & & & &
\end{array}\right]$$

n (m-1) rows (lower block)

It will be convenient to compress the n^2 non-zero elements of A into a new nxn matrix $\mathscr{A} = [\alpha_{ij}]$, where

$$\alpha_{ij} = \frac{b_j}{\lambda_{ii}+\lambda_{jj}} , \qquad i,j = 1,2,\ldots,n \quad . \tag{8}$$

We may now state the first basic result:

Theorem 2. Let Λ and b be as above. Then a necessary condition for Eq.(6) to have a solution h whose ith component $h_i = 0$ is

$$\alpha_{1i}q_{i1} + \alpha_{2i}q_{i2} + \cdots + \alpha_{ni}q_{in} = 0 \quad . \tag{9}$$

Proof. Let $h_i = 0$. Forming the column vector $\mathscr{S}(hh' - Q)$, it is easily verified that the equation

$$h = A\mathscr{S}(hh' - Q)$$

may only be satisfied if the foregoing orthogonality relation holds.

Remark. Theorem 2 imposes constraints on the choice of the measurement matrix S which will be utilized below.

The question which now arises is to what extent condition (9) is sufficient for Eq.(6) to have a solution whose ith component is zero. The following result shows that condition (9) is "generically sufficient" in that it is sufficient for "almost every" system:

Theorem 3. The condition (9) is sufficient for the ith component of a solution to Eq. (6) to be zero for almost every linear system, i.e. the set of systems for which it fails to suffice form a null set in the space of all linear systems.

Proof. If the condition holds, we must solve the set of nonlinear equations

$$h_1[\alpha_{11}h_1 + \alpha_{21}h_2 + \cdots + \alpha_{n1}h_n - 1] = \alpha_{11}q_{11} + \alpha_{21}q_{21} + \cdots + \alpha_{11}q_{n1}$$

$$h_2[\alpha_{12}h_1 + \alpha_{22}h_2 + \cdots + \alpha_{n2}h_n - 1] = \alpha_{12}q_{12} + \alpha_{22}q_{22} + \cdots + \alpha_{n2}q_{n2}$$

$$\vdots$$

$$h_{i-1}[\alpha_{1,i-1}h_1 + \alpha_{2,i-1}h_2 + \cdots + \alpha_{n,i-1}h_n - 1]$$

$$= \alpha_{1,i-1}q_{1,i-1} + \alpha_{2,i-1}q_{2,i-1} + \cdots + \alpha_{n,i-1}q_{n,i-1}$$

$$h_i[\alpha_{1i}h_1 + \alpha_{2i}h_2 + \cdots + \alpha_{ni}h_n - 1] = 0$$

$$\vdots \qquad\qquad \vdots$$

$$h_n[\alpha_{in}h_1 + \alpha_{2n}h_2 + \cdots + \alpha_{nn}h_n - 1] = \alpha_{1n}q_{1n} + \alpha_{2n}q_{2n} + \cdots + \alpha_{nn}q_{nn}.$$

From the ith equation, we see that either $h_i = 0$ or there exists a solution vector h lying on the hyperplane $\alpha_{1i}h_1 + \alpha_{2i}h_2 + \cdots + \alpha_{ni}h_n = 1$. Since the system has only a finite number of solutions, should the second case hold, an arbitrarily small perturbation of the matrix Q, Λ, or b will insure that it fails to hold without changing the number of unstable modes of the system (since the characteristic roots are continuous functions of the matrix elements). Hence, generically condition (9) implies $h_i = 0$.

Theorems 2 and 3 now allow us to resolve the measurement problem for almost every diagonal system. The task is to find a measurement matrix S with the following properties:

i) The pair (S,Λ) is detectable (the unstable modes of Σ' are contained in the space generated by the columns of the observability matrix $\Theta = \left(S', \Lambda S', \Lambda^2 S', \ldots, \Lambda^{n-1} S'\right)$), and

ii) row k of the matrix $Q = S'S$ is orthogonal to column k of the matrix $\mathscr{A}$ for as many indices k as possible, $1 \leq k \leq n$ (condition(9)). The resolution of this question is given by

<u>Theorem 4. Let the system Σ' be stabilizable (i.e. the columns of the matrix $(b, \Lambda b, \Lambda^2 b, \ldots, \Lambda^{n-1} b)$ span the space generated by the unstable modes of Σ') and let every unstable mode contain zeros in the same m components. Then a law h stabilizing Σ' will measure n-m components of the state for almost every system Σ'.</u>

<u>Proof.</u> First of all, note that the unstable modes of Σ' comprise a subset of the usual basis vectors $e_1, e_2, \ldots, e_n$ since Λ is diagonal. Thus, m equals the number of characteristic values of Λ with negative real parts. Assume that the unstable modes of Σ' contain common zeros in rows $i_1, i_2, \ldots i_m$. We choose the same rows of S' equal zero, the remaining rows being chosen to satisfy the detectability requirement. Since Λ is diagonal, it is clear that no further rows of S' may be chosen zero and still have the pair (Λ, S) be detectable. But, the above choice of S' implies that m rows of $Q = S'S$ are

identically zero, thereby satisfying condition (9) for the m indices $i_1, i_2, \ldots, i_m$. Hence, by Theorems 2 and 3, m components of h are zero and, by stabilizability and detectability of Σ', such an h will be a stable feedback law for almost every Σ'.

Theorem 4 characterizes the solution of the measurement problem for almost every diagonal system. Let us recapitulate the assumptions and the steps of the solution:

Assumptions: (1) Λ is diagonal, i.e. F is normal,

(2) Λ has no purely imaginary entries on the diagonal,

(3) the pair (Λ, b) is stabilizable.

Under these assumptions, we construct a minimal measurement matrix by the procedure:

i) determine the unstable modes of Λ,

ii) let $i_1, i_2, \ldots, i_m$ be the indices where the unstable modes all have zero entries and select the matrix S such that

a) row k of S is zero, $k = i_1, i_2, \ldots, i_m$,

b) the non-zero elements of S are chosen so that (Λ, S) is detectable and generic.

IV. An Example

To illustrate the foregoing results, let the system be given by

$$F = \begin{bmatrix} 6 & 4 & 4 & 1 \\ 4 & 6 & 1 & 4 \\ 4 & 1 & 6 & 4 \\ 1 & 4 & 4 & 6 \end{bmatrix}, \quad g = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The relevant diagonalizing transformation is

$$T = \tfrac{1}{2}\begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

giving the diagonal system

$$\dot{z} = \Lambda z + bu \quad , \qquad (\Sigma')$$

where

$$\Lambda = \text{diag}\ (15\ 5\ 5\ -1)$$

$$b = \tfrac{1}{2}(1\ 1\ 1\ 1)' \quad .$$

The system Σ' is stabilizable (in fact, controllable) since the unstable modes are

$$e_1 = (1\ 0\ 0\ 0)' \quad , \quad e_2 = (0\ 1\ 0\ 0)' \quad , \quad e_3 = (0\ 0\ 1\ 0)'$$

which are contained in the space generated by the matrix $[b\ \Lambda b\ \Lambda^2 b\ \Lambda^3 b]$, and Σ' satisfies the other assumptions (1) and (2). Since the three unstable modes have only the fourth entry as a common zero, we have m = 1 and i = 4. Let us assume that Σ has two output terminals. Then we choose an S of the form

$$S' = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \\ s_{31} & s_{32} \\ 0 & 0 \end{bmatrix} .$$

The six independent constants are to be chosen such that (Λ,S) is detectable and generic. It's easy to see that the choice (one of many possible) $s_{11} = s_{21} = s_{31} = s_{12} = s_{32} = 1$, $s_{22} = 2$ satisfies all conditions.

The above example is interesting since it illustrates the fact that the number of output terminals (p) may be critical since, for example, a single output channel ($p = 1$) will not suffice in the above example as in that case no choice of S will make $\sum'$ detectable. This is due to the multiple root $\lambda = 5$. However, if Λ has distinct roots, then a single output will always suffice.

V. The General Single-Input Case

Let us now return to the original system $\sum$:

$$x = Fx + gu \quad . \qquad (\textstyle\sum)$$

Recall that $\sum$ and $\sum'$ are related through the coordinate transformation $z = Tx$. Thus, if we know which components of z appear in a stabilizing law, then for almost every $\sum$ we will also know which components of x occur since

$$z_i = \sum_{j=1}^{n} t_{ij} x_j \quad , \qquad \tau = 1,2,\ldots,n \quad . \qquad (10)$$

However, it should be noted that we may have cancellation in some particular control law. That is, if we have a law

$$u = \gamma_1 z_{i_1} + \gamma_2 z_{i_2} + \cdots + \gamma_m z_{i_m} \quad ,$$

then the choice of the γ's may result in cancellations when substituted back into (10). But, since the γ's are determined by the choice of S, a slight perturbation of the components of S will eliminate cancellations while still preserving the other requirements. Thus, generically the number of components of x which appear is determined by which components of z appear and the zeros which appear in the transformation T.

For example, in the problem of the previous section, even though z_4 did not appear in the diagonal system, since T has no zeros in rows 1-3, all components of x occur in the generic control law generated by the diagonal system.

VI. Discussion

In this work, we have given conditions for solubility of the minimal measurement problem for linear, single-input, constant coefficient systems. The results have relied on various assumptions which are often met in practice. Unfortunately, as is often the case in mathematics, the result could only be established for almost every system which is satisfactory as long as one isn't in one of the singular cases. However, for practical purposes, it is sufficient since no physical system is known precisely enough that it could not be perturbed by a small amount to make it generic.

In subsequent articles, various extensions and modifications of the above results will be investigated, among them the multiple-input case, the case of an infinite-dimensional state vector, and some numerical aspects.

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