RM-77-19

# ANALYSIS OF A NATIONAL MODEL WITH DOMESTIC PRICE POLICIES AND QUOTA ON INTERNATIONAL TRADE

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## Preface

The food problem is to a large extent a local one. Accordingly, the starting point in the Food and Agriculture research of IIASA is the modeling of national food and agricultural systems. After having investigated local, national strategies directed towards specific goals (e.g. introducing new technologies, changing the agricultural structure, etc.) a generalization will be possible and conclusions can be drawn concerning the global outcomes of changing agricultural systems. Thus, the global investigation will be based on national models and their interactions.

To reflect these interactions in a model, a methodological research is required which is concerned with the linkage of national models for food and agriculture. This Memorandum is the second of a series on this topic.

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Previously on this topic: RM-77-2, Linking National Models of Food and Agriculture: An Introduction, January 1977.

Summary

This paper is the second in the series on the linkage of national models for food and agriculture. It develops some of the ideas presented in the first, introductory paper [14].

In Section 1, the model with domestic price policy and quota, is rehearsed and reformulated. A proof is presented for the existence of domestic equilibrium at given world market prices. It is shown that when this equilibrium is unique the national excess demand functions are continuous in world market prices and satisfy Walras' Law so that the requirements for linking, presented in [14], are satisfied. The proof is also valid for an economy with production.

In Section 2, the uniqueness of the domestic equilibrium is investigated on the basis of properties of the Jacobian matrix. Although this analysis does not lead to any useful results for the present model, it gives an indication of the problems one has to face and, moreover, the derivation of the Jacobians is useful for the world market algorithm, which will be discussed in a separate paper.

In Section 3, attention is centered on the actual computation of the domestic equilibrium. The first paragraph deals with the computation of domestic equilibrium prices when the traded quantities are given. Although the case is not very relevant in itself, the simplicity of the problem makes it useful as a starting point. In the second paragraph, a complementary pivoting algorithm is developed which can solve the domestic equilibrium problem in a pure exchange economy with Cobb Douglas utility functions. In the third paragraph, several other cases are discussed which are relatively easy to solve. Stock policy is introduced and a model with lagged production is discussed.

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# SECTION 1: A NATIONAL MODEL WITH DOMESTIC PRICE POLICIES AND QUOTA ON INTERNATIONAL TRADE

## 1.1 Introduction: the main features of the model

- We discuss the pure exchange version of the model, which means that we take supply as given and concentrate on demand by the consumer, at given endowments.
- The consumer is taxed by a government which has to pay subsidies on international trade, or the consumer receives income transfers from tariff receipts. These receipts may also be used in other ways; this will be discussed in 1.3.
- Price differences between world market and domestic market are caused, either by a domestic price policy, or by quota on international trade.
- The government must tax the consumers in such a way that both its budget and the balance of trade are in equilibrium.
- The model presented in [14] is now repeated and then reformulated. In 1.4 an existence proof for the domestic equilibrium is presented.

## 1.2 The model

1) Consumer

$\max \ u^j(x^j)$

S.T. $px^j = \alpha_j tr + py^j$

2) Government

a) $l_i \leqq y_i - x_i \leqq r_i$

b) $\bar{p}_i = \bar{p}_i^*$

c) $tr = (p^w - p)(y - x)$

3) Domestic market equilibrium

$\mu_i(y_i - x_i - l_i) = 0$

$\nu_i(y_i - x_i - r_i) = 0$

$p_i = \bar{p}_i + \mu_i - \nu_i$

$p_i, \mu_i, \nu_i \geqq 0$

4) Equilibrium on the balance of trade

$\Sigma p_i^w (y_i - x_i) = 0$

The model has been discussed in ([14], § 4.3) existence of a domestic price equilibrium will now be proved after some reformulations.

Symbols

| | |
|---|---|
| $u^j$ | utility of the $j^{th}$ income class ($j=1,..,m$) |
| $x^j$ | (vector of) demand of the $j^{th}$ income class |
| $y^j$ | net endowments of the $j^{th}$ income class |
| tr | total tariff receipts by the government |
| $\alpha_j$ | share of $j^{th}$ income class in tr |
| $l$ ,r | minimum resp. maximum export of the $i^{th}$ commodity ($i = 1,..,n$) |
| $\bar{p}_i$ | price target for the $i^{th}$ commodity |
| $p_i^w$ | world market price |
| $p_i$ | domestic price |
| $\mu_i, \nu_i$ | price differential as defined under 3) |

## 1.3 Reformulation of the model

### 1.3.1 The export constraint

The export constraint may lead to an inconsistency as it implies $x \geqq y - r$ . This may be incompatible with nonnegative domestic prices. To solve this problem an extra slack vector s must be introduced.

Define

$$d \equiv x + s \quad .$$

The quota constraint becomes:

$$l \leq y - d \leqq r \quad .$$

The complementarity (market equilibrium) conditions are then:

$$\nu_i (y_i - d_i - r_i) = 0$$

$$p_i \; s_i \qquad\qquad = 0$$

$$\mu_i (y_i - d_i - l_i) = 0 \quad .$$

The tariff receipts are

$$tr = (p^w - p)(y - d) \quad .$$

Balance of trade equilibrium implies

$$p^w (y-d) = 0 \quad .$$

When solving the model we first compute consumer demand, when $p_i = 0$ we compute $s_i = \max\ (0, y_i - r_i - x_i)$.

### 1.3.2 Taxation and distribution of tariff receipts

Up to this point the budget equation of the consumer has merely been specified as:

$$px^j = \alpha_j tr + py^j \quad .$$

It was not said whether $\alpha_j$ was a variable or a parameter. If we would consider it as a parameter we have the following problem: Under balance of trade equilibrium the budget equation is:

$$px^j = \alpha_j\, p(x - y) + py^j$$
$$= p(\alpha_j (x - y) + y^j) \quad .$$

It can be seen from this equation that for any given vector $\alpha$ such that $\Sigma\alpha_j = 1, 0 < \alpha_j \leq 1$ and given $(x - y) \neq 0$ there exists a nonnegative price vector p such that $p\, x^j \leq 0$ . This is not acceptable. The vector $\alpha$ must therefore be considered as a variable. It reflects the tax system in the country. This system may discriminate among production sectors and income classes. A more general formulation would be

taxes government expenditures tariff receipts

$$ta \equiv tg - tr$$

$$\beta_j = f(py^j, p) \quad \text{(function to determine taxation rate)}$$

$$ta_j = \beta_j \cdot py^j$$

$$px^j = py^j\,(1-\beta_j) \quad .$$

We assume that the government expenditures are totally inelastic, and that the tax share $\beta_j$ is homogeneous of degree zero in domestic prices

$$tg = p \cdot \bar{g} \quad , \bar{g} \text{ is given.}$$

1.3.3 The balance of trade

We replace

$$p^w(y - x) = 0 \quad \text{by:} \qquad p^w(y - (x + s)) \leqq 0 \quad .$$

Where s is defined as above.

The inequality is only a slight relaxation because we shall find that in world market equilibrium it becomes again an equality.

1.3.4 The national model reformulated

1) Consumer

$$\max \quad u^j(x^j)$$

$$\text{S.T.} \quad px^j = p[y^j(1-\beta_j) + \delta^j] \quad ; \quad j = 1,..,m \quad ; \quad \delta_j > 0 \quad ;$$
$$= py^j - \alpha_j\, ta \quad .$$

[$\beta_j$: income class specific taxation rate

$\delta_j$: gift in kind (see below)]

2) Government

a) $l \leqq y - d \leqq r$ quota constraints

b) $\bar{p} = \bar{p}^{*}$ [1] domestic price policy

c) $tr = (p^w - p)(y - d)$ net tariff receipts

d) $\bar{g} = \Sigma \delta^j + \bar{g}_o$ ; $tg = p\bar{g}$ ; government expenditures

e) $ta = tg - tr$ taxes

f) $\beta_j = f(py^j)$ ; $\beta_j < 1$ taxation rates

g) $ta = \sum_j \beta_j \cdot py^j$ .

3) Domestic market: definitions

$d = x + s + \bar{g}$ aggregate demand

$p = \bar{p} + \mu - \nu$

4) Domestic market equilibrium

a) $\mu_i (y_i - d_i - l_i) = 0$ $i=1,..,n$

b) $\nu_i (y_i - d_i - r_i) = 0$

c) $p_i s_i = 0$

d) $p, \mu, \nu \geqq 0$

e) $s, x \geqq 0$ .

5) Balance of trade equilibrium

$p^w (d - y) \leqq 0$ .

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1) More precisely $\bar{p} = k\bar{p}^{*}$, $\Sigma p_i^w = k$ , $k = 1$ .

6) Assumptions on the policy variables

a) $p^w\, l \leqq 0$ Quota compatible with balance

b) $p^w\, r \geqq 0$ of trade equilibrium.

c) $l \leqq r$ by definition.

d) $r < y$ less exports than domestic availability

e) $\bar{p}^* > 0$ desired domestic price is positive

f) $p^w\, \bar{g} < p^w\, y$ .

7) Assumptions on endowments

For each $j, \exists i$ such that

$y_i^j > 0$ for some $i$ .

$\sum_j y_i^j > 0$ all $i$ .

1.3.4 The solution of the national model

The national model is a set of equations which is simultaneous on three levels:

1) A utility maximization problem in principle involves the solution of a (simultaneous) set of first order conditions. The simultaneity may be avoided however by making use of duality theory.
2) The utility maximization problems are interdependent through the taxation policy because tariff receipts are influenced by aggregate demand (eq. 2), c).
3) The domestic equilibrium prices are not given but are determined simultaneously with demand.

ad 1) We know from elementary demand theory that problem 1) will have a unique solution, $x^j$ for any positive income and nonnegative prices p, under the appropriate nonsaturation assumptions for the utility function, $u^j$ . We also know that at given $\beta_j$ the demand will be homogeneous to the degree zero in domestic prices. Let $S_n$ be the set of nonnegative domestic prices. We assume that the utility functions are strictly quasi-concave. First set $\delta^j = 0$, then the demand function $x^j = x^j(p)$ can be shown to be continuous for all p such that $p > 0$, $p \varepsilon S_n$ and $py^j > 0$. Some problems of discontinuity however arise when some prices tend to zero, first because the income of certain income groups might be zero, second because the demand for a commodity might be infinite. In order to avoid the first complication we assume that the government offers an infinitely small amount of all commodity endowments to all income classes ($\delta^j$) so that all incomes are positive at all prices in $S_n$ . We know that in this case the demand functions will be upper semicontinuous and $\{x_i^j | x_i^j = x_i^j (p)\}$ is a closed bounded convex set (cf. Lancaster [9] or Arrow and Hahn [1]).

The assignment of a positive $\delta^j$ may seem restrictive from a theoretical point of view and in fact less restrictive solutions are available (cf. Arrow and Hahn [1]), but one can hardly imagine that the error introduced could be of any importance. As mentioned before we may compute the slack variable $s_i$ as follows: $s_i = 0$ if $p_i > 0$ and $s_i = \max(0, y_i - r_i - x_i)$ otherwise.

ad 2) <u>Simultaneous solution of the utility maximization problems;</u>

$$tr = (p^w - p)(y - d)$$

* Assume that 4 c holds everywhere, also out of market equilibrium.

$$tr = (p^w - p)(y - (x + \bar{g} + s))$$

$$= p^w(y - (x + \bar{g} + s)) - p(y - (x + \bar{g}))$$

$$\longrightarrow \quad tg - tr = - p^w(y - (x + \bar{g} + s)) + p(y - x) \quad .$$

This equation is the equilibrium condition for the simultaneous solution of the utility maximization problems. In general the utility maximization problems have to be solved independently given domestic prices and a share $\alpha_j$ of a given total amount of taxes $ta = -t$:

$$\max \quad u^j(x^j)$$

$$\text{S.T.} \quad px^j = py^j + \alpha_j t \quad .$$

* $\alpha_j$ is assumed to satisfy $\frac{d(\alpha_j t)}{dt} \geqq 0$ and $\Sigma\alpha_j = 1$ . Moreover $\alpha_j$ is assumed to be homogeneous of degree zero in domestic prices. Summation of budget equations yields:

$$p(y - x) + t = 0 \quad ,$$

so that the budget equilibrium coincides with equilibrium of the balance of trade.

The equation

$$p^w(y - d(t)) = 0$$

will have a unique solution if

$p^w d(t)$ is a monotonously increasing function of t, such that $\lim_{t \to +\infty} p^w d(t) = +\infty$ .

We know that by Walras' Law (nonsaturation)

$$\lim_{t \to +\infty} pd(t) = \quad \text{and} \quad \frac{d(pd(t))}{dt} = 1 \quad .$$

* If $\frac{d(d_i(t))}{dt} \geq 0$, $\forall i$ (no inferior goods)

and $p_h^w > 0$, $\frac{d(d_h t)}{dt} > 0$ , for some h

then we know that the condition is satisfied.

We assume that this condition holds. Again it is clear that theoretically speaking the balance of trade equilibrium condition is unnecessarily restrictive for the existence of market equilibrium. We shall now relax this condition and discuss domestic equilibrium under quota and domestic price policy.

## 1.4 Domestic price equilibrium

The existence proof for a domestic price equilibrium is not a trivial one. We shall proceed in three stages:

1) First we shall literally reproduce the proof of the excess demand theorem by Debreu [4]. This proof would apply to the national model if $l = r = 0$ .
2) Then we shall open up the economy and formulate an appropriate maximization problem.

The crux of the proof is the extension of the linear programming problem occurring in the proof by Debreu. Several linear programming problems are formulated, first for the case of an import quota only, then for export quota, then for both and finally for a combination of import quota, export quota, and a domestic price policy.

1.4.1 Debreu's excess demand theorem

Consider the set of excess demand function $z = z(p)$, which satisfies $p\, z(p) \leqq 0$ . Does this problem have a solution $z \leqq 0$ ? Let p be the set of normalized prices. This is clearly a compact convex set. Denote by Z the set of all z(p) for $p \in P$[Z is the union of the sets Z(p)]. If Z is not the convex, we replace it by any compact convex set containing Z, which we denote by Z'.
Now define the set S(z) as follows:

$$S(z) = [p|pz \text{ is a maximum for } z \in Z', \ p \in P] \quad .$$

That is, we choose an arbitrary excess demand vector from the set of all excess demand vectors which are attainable at some prices, then find the price vector for which the value of this excess demand is maximized. It is important to note that the price vector is any price vector, not necessarily the particular p which is associated with z through the mapping $p \longrightarrow Z(p)$.

Clearly $z \longrightarrow S(z)$ is a mapping from Z' into a subset of P. Since Z is convex we know this mapping to be upper semi-continuous. S(z) is a convex set since it is the intersection of the hyperplane $[y|yz = \max pz]$ with P.

Consider the set P x Z', that is, the set consisting of normalized price vectors paired with excess demand vectors. If we take some point p, z in p x Z', then Z(p) associates a set of excess demand vectors with p, and S(z) associates a set of price vectors with z. In other words, the mapping $p, z \longrightarrow Z(p), S(z)$ maps a point in P x Z' into a subset of P x Z'.

We have shown the mapping $z \longrightarrow S(z)$ to be upper semi-continuous, and $p \longrightarrow Z(p)$ has been assumed to have the same property, so that the combined mapping is upper semocontinuous also. We have shown that S(z) is convex and Z(p) has been assumed convex, so that S(z) x Z(p) is convex.

Thus we have an upper semicontinuous mapping p, z $\longrightarrow$ S(z) from the set P x Z' into a convex subset of itself. These are the conditions for invoking the Kakutani Fixed Point Theorem. The theorem states that there exists some $p^* \varepsilon$ P, $z^* \varepsilon$ Z' which is a fixed point, that is, for which $p^* \varepsilon S(z^*)$ and $z^* \varepsilon Z(p^*)$.

From the construction of S(z), $p^* \varepsilon S(z^*)$ implies that, for all $p \varepsilon P$,

$$pz^* \leqq p^*z^* \quad .$$

Using the weak budget condition it follows that, since $z^* \varepsilon Z(p^*)$,

$$p^*z^* \leqq 0 \quad .$$

Thus

$$pz^* \leqq 0, \qquad \text{for all} \quad p \varepsilon P \quad .$$

2) Clearly the last inequality is satisfied for all $p \varepsilon P$ only if

$$z^* \leqq 0 \quad ,$$

thus proving the theorem.[1] One important feature of this proof is that it does not require p and z to have the same dimension.

The other important feature of this proof for our purpose is that $S(z) = [p|\max pz \text{ for } z \varepsilon Z', p \varepsilon P']$ represents the solution of a linear programme.

$$\max \quad p\, z$$

$$\text{S.T. } \Sigma p_i = 1$$

$$p_i \geqq 0 \quad .$$

1) Scarf [13] p. 119-129, has derived an algorithm for computing this equilibrium solution.

This programme can be extended without changing the essence of the proof. The budget equations yield Walras' Law for the present case (t = - taxes).

$$pz - t = 0 \quad .$$

From this we can derive the simplex for the present case: z is homogeneous of degree zero in (p,t). If (p,t) is a linear function of another vector, say w, $w \geqq 0$, then z is homogeneous of degree zero in this vector.

We may therefore set the sum of these nonnegative variables to equal 1, and thus constrain them to the simplex.

1.4.2 Existence proof for domestic equilibrium

1.4.2.1 Import quota only:

The essential part of the proof is that we substitute out the variable t from Walras' Law. We set $t = - \mu\, l$ so that $pz + \mu\, l = 0$ .

Define

$$q = z + l$$
$$p = \mu + \phi p^w$$

where

$$\mu_i,\ \phi \geqq 0$$
$$\phi + \Sigma\mu_i = 1 \qquad (\mu,\ \phi \text{ on the simplex}) \quad .$$

We may restrict $(\phi,\ \mu)$ to the simplex because $\Sigma p_i^w = 1$ and because of the substitution $t = - \mu\, l$ .

We can rewrite Walras' Law as

$$0 = \phi p^w z + \mu q \quad .$$

We now set up a linear programme analogous to the one in Debreu's proof

$$\begin{array}{ll} \max & \phi p^w z + \mu\, q \\ \text{S.T.} & \phi + \Sigma\mu_i = 1 \\ & \mu_i,\ \phi \geqq 0 \quad . \end{array}$$

Analogously to the previous case we find that the goal function has zero value in the fixed point (the mapping can be considered just as before to be an upper semi continuous mapping of a compact convex set into itself).

So that we find

$$p^w z^* \leqq 0 \quad (\text{if } \phi^* > 0 \text{ then } p^w z^* = 0; \quad \text{if } p^w 1 > 0 \text{ then } \phi^* = 0)$$
$$q^* \leqq 0$$
$$\mu^* q^* = 0$$

Note that $t \geqq -\mu 1$ implies $t > -py$ because

$$py = \phi p^w y + \mu y$$
$$py - \mu 1 = \phi p^w y + \mu(y - 1)$$
$$y > 1 \longrightarrow py - \mu 1 > 0 \quad .$$

1.4.2.2 Exports quota only

As mentioned earlier export quota present the difficulty that if they are applied to the excess demand ifself infeasibility might arise with the condition of nonnegativity of prices.

We therefore define

$$q = -(z + s + r) = -(d - y + r)$$
$$\phi + \Sigma \nu_i = 1$$
$$\phi = \phi p^w - \nu; \ \phi, \nu \geqq 0 \quad .$$

We restrict the taxation to:

$$t = \nu r$$

Walras' Law is then: $0 = p^w(d - y) + \nu q$

consider now the L.P.:

$$\begin{array}{l|l} \max & \phi p^w(d - y) + \nu q \\ \text{S.T.} & \phi + \Sigma \nu_i = 1 \\ & p = \phi p^w - \nu \\ & p.s = 0 \\ & \phi, p, \nu \geqq 0 \end{array}$$

as before we set $s_i = 0$ if $p_i > 0$ and $s_i = \max(0, y_i - r_i - x_i)$ otherwise.

In the fixed point we find:

$$0 = \phi^* p^w(d^* - y) + \nu^* q^*$$

setting $\phi = 1$ we get $p^w(d^* - y) \leqq 0$ .

We can however not set $\phi = 0$ without leaving the constraint set.

However, because we have assumed $p^w r \geqq 0$ we may write

$$0 \geqq p^* q^* \geqq p\, q^* \quad \text{all } p \text{ on simplex}$$

so that $p^* \leqq 0$

and $\mu^* q^* = 0$

and $p^w(d^*-y) \leqq 0$ (if $\phi^* > 0$ then $p^w(d^*-y) = 0$) .

As before the condition $y > r$, guarantees a positive income.

1.4.2.3 Import and export quota

Combining both previous problems we procees as follows: Define:

$$q_1 = x + s - y - l$$

$$q_2 = (x + s - y - r) \quad .$$

We set

$$t = -\mu\, l + \nu r$$

so that Walras' Law is

$$0 = \phi p^w(d - y) + \mu q_1 + \nu q_2$$

We can prove by combination of both previous problems ($\phi$, $\mu$, on simplex) that

$$q_1^* \leqq 0$$

$$q_2^* \leqq 0$$

$$\mu^* q_1^* = \nu^* q_2^* = 0$$

$$p^w(d^*-y) \leqq 0 \qquad (\text{if } \phi^* > 0 \text{ then } p^w(d^*-y) = 0)$$

1.4.2.4 Import, export quota and domestic price policy

In this case we set:

$$t = -\mu\, l + \nu r + \phi h$$

$$h = (\bar{p} - p^w)z \quad .$$

This yields the complication that the demand functions have to be solved simultaneously:

At given $\phi$, $\mu$, $\nu$ we must iterate over $h$ in order to realize $h = (\bar{p} - p^w)z$. We have however shown before (ad 2) that this problem has a unique solution.

Otherwise the case is identical to the case without domestic price policy.

This completes the existence proof.
We have only assumed on production that $y > r > 1$ and that $p^w r \geqq 0$. In an economy with production 1, r can by assumption be set at this level. We then first solve a profit maximization problem at given prices and when output has been determined adjust (r, 1).

Computation of domestic equilibrium:
As mentioned before Scarf ([13], p. 119-129) has presented an algorithm to compute a fixed point of the mapping in Debreu's proof. The same algorithm would apply for the computation of domestic equilibrium in our model. We are however, mainly interested in unique domestic equilibria as will be explained below. This paper will therefore be oriented towards the development of alternative algorithms which specifically apply to unique equilibria. We return to this matter in section 3.

## 1.5 World market equilibrium under pure exchange

The existence of world market equilibrium can be shown in several ways.

The most direct approach would be to consider all domestic markets simultaneously with the world market and to formulate a linear programme accordingly. The proof would be straightforward but hardly instructive for the linking problem. It is computationally a very hard task to solve all prices for all countries simultaneously, when there are quota. Moreover the interpretation of the model is very difficult when everything is computed in one algorithm. We therefore want to decompose the equilibrium problem into two components.

1) Compute a domestic equilibrium price and excess demand given a world market price: $p^w \rightarrow z_c$, $z_c = d_c - y_c$.
2) Compute a world market equilibrium price.

We have seen that in every country for every world market price the domestic equilibrium excess demand $z_c$, exists, and satisfies balance of trade constraint $p^w z_c \leqq 0$.

The only further prerequisite for decomposition is that $z_c(p^w)$ is sufficiently continuous. We therefore prove two lemma's on continuity. We need for this the following lemma by Arrow & Hahn [1(p. 102)]:

* If the utility function U(x) is strictly quasi concave, if the income is positive for all p then x(p) is continuous in its domain of definition which includes all $p > 0$ and $\Sigma x_i(p)$ is continuous everywhere on the unit simplex, where $\Sigma x_i(p) = \infty$ if $x_i(p)$ is not defined.

The formulation of the theorem is more complex than might appear at first sight.

1. It is not stated that whenever $p_i = 0$ for some i, $x_i(p)$ is not defined. This would imply that in equilibrium all prices must be positive. A commodity, for which this however happens to be the case, is called numeraire.

2. We know that the mapping $p \longrightarrow x$ is uppersemicontinuous on the unit simplex and have used this in the proof of the existence of domestic equilibrium. The theorem is in accordance with this but provides more information.

We now state our lemma's

Lemma(1)- If the domestic market has a unique equilibrium and if there are finite import quota on all commodities then the mapping $p^w \longrightarrow z$ is a continuous point to point mapping.

Proof:

As there are import quota on all commodities

$$z_i \leqq k_i$$

so that $\Sigma z_i \leqq \Sigma k_i$ in equilibrium.

Thus $z_i(p)$ must be defined (uniquely) in equilibrium so that $z_i(p)$ is continuous for all p in equilibrium.

Consider a sequence $p_r^w \longrightarrow p_o^w$ . Since $p^r$ is unique the univalued function $p^r = p(p_r^w)$ is defined. Since $p^r$ is bounded there exists a convergent subsequence.

$$p^r \longrightarrow p^o .$$

Since $z(p^r)$ is a vector of continuous functions

$$\lim_{r \to \infty} z(p^r) = z(p^o) .$$

But by construction $z(p^r) \leq k \;\forall r$ so that

$$z(p^o) \leqq k \quad .$$

Then by the uniqueness hypothesis $p^o = p_o^w$ q.e.d.

Lemma (2): If the domestic market has a unique equilibrium and if the domestic price for the commodities is positive then the mapping $p^w \longrightarrow z$ is a continuous point to point mapping.

Proof:

If the domestic prices p are positive then $z_i(p)$ is continuous in p so that the proof in (1) holds.

It would be possible to generalize the proposition but this may be superfluous for our present purposes: in our agricultural model no government will ever let the domestic price of any commodity be zero.* The world market equilibrium price might well be zero.

We may thus list the theorem.

Theorem

Under either the assumptions of (1) or (2) the national excess demand mappings are continuous functions which satisfy $p^w z \leqq 0$ so that a world market equilibrium exists. The excess demand functions are homogeneous of degree zero in world market prices.

The previous theorem has assumed uniqueness of domestic price equilibrium. The second section of this paper will be centered around this issue.

This however needs some preliminary work such as the derivation of Jacobians. The Jacobians will not be very helpful for our problem but we have to see why. Moreover, the Jacobians will show to be helpful for the computation of the equilibrium solution on the world market, this will be discussed in a separate paper.

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* A stock policy to maintain a positive floor price for a domestically produced commodity would be relevant (see also section 3.3). Therefore in most practical cases the domestic price will be positive and the domestic excess supply zero.

# SECTION 2: THE UNIQUENESS OF DOMESTIC EQUILIBRIUM

## 2.1 The Slutsky equations and the Jacobian

Before we investigate uniqueness, the demand responses of the model as expressed in the well known Slutsky equation will be investigated, because the corresponding Jacobian plays a crucial role both in the analysis of uniqueness, and the computation of equilibrium.

### 2.1.1 Given income and given prices

This is the classical case. The derivation can be found in Lancaster [9]

$$\max \quad u(x)$$

$$\text{S.T.} \quad px = m$$

the F.O.C. are $u_i = \lambda p_i$

$$px = m$$

a) differentiation of F.O.C. to the $n^{th}$ price yields: (budget equation)

$$\lambda \Sigma \frac{\partial x_j}{\partial p_n} = - \lambda x_n$$

$$- p_i \frac{\partial \lambda}{\partial p_h} + \sum_j u_{ij} \frac{\partial x_j}{\partial p_h} = \begin{cases} \lambda & \text{if } i = n \\ 0 & \text{else} \end{cases}$$

b) differentiation to m yields:

$$\lambda \sum_j p_j \frac{\partial x_j}{\partial m} = \lambda$$

$$-p_i \frac{\partial \lambda}{\partial m} + \Sigma u_{ij} \frac{\partial x_j}{\partial m} = 0 \quad .$$

After substitution $u_i = \lambda p_i$ , the equations sub a) can be rewritten in matrix-vector form.

$$\begin{bmatrix} 0 & u_1 & \cdots & u_n \\ u_1 & u_{11} & \cdots & u_{1n} \\ \vdots & & \ddots & \vdots \\ u_n & u_{1n} & \cdots & u_{nn} \end{bmatrix} \begin{bmatrix} -\frac{1}{\lambda}\left(\frac{\partial \lambda}{\partial p_h}\right) \\ \frac{\partial x_1}{\partial p_h} \\ \vdots \\ \frac{\partial x_n}{\partial p_h} \end{bmatrix} = \begin{bmatrix} -\lambda x_h \\ 0 \\ \vdots \\ 0 \\ \lambda \end{bmatrix}$$

The matrix of this equation system is the bordered Hessian of u : $\hat{U}$ .

Solving for $\frac{\partial x_r}{\partial p_h}$ by Cramer's rule one gets

$$\frac{\partial x_r}{\partial p_h} = \frac{1}{\det \hat{U}} \left(- \lambda x_n U_r + \lambda U_{rh}\right)$$

where $U_r$ is the cofactor of $u_r$ in det $\hat{U}$

$U_{rh}$ is the cofactor of $u_{rn}$ in det $\hat{U}$

Analogously one can obtain from the set of equations sub b)

$$\frac{\partial x_r}{\partial m} = \frac{\lambda U_r}{\det \hat{U}}$$

Writing $K_{rn} = \lambda\ \hat{U}_{rn}/\det \hat{U}$ and substituting in the previous equation one gets the well known Slutsky equation.

$$\frac{\partial x_r}{\partial p_h} = -x_n \left(\frac{\partial x_r}{\partial m}\right) + K_{rh}$$

$K_{rn}$ is symmetric.

* $K_{nn} < 0$

* and the matrix $[-\hat{u}]$ has positive principal minors

* both $pK = 0$ and $Kp = 0$

* the sign of $K_{rn}$ can be positive or negative

but the own price effect, $\frac{\partial x_r}{\partial p_r} = - x_r \frac{\partial x_r}{\partial m} + K_{rr}$, is negative if the commodity r is not inferior that is if $\frac{\partial x_r}{\partial m} \geqq 0$ or not negative enough.

### 2.1.2 The pure exchange economy

We now consider the case where m - py and y is given. The only change which then occurs in the derivation of Slutsky's equation is in the differentiation of the budget equation to prices; here one gets:

$$\Sigma p_j \frac{\partial x_j}{\partial p_n} = - \lambda z_n$$

Thus Slutsky's equation becomes:

$$\frac{\partial x_r}{\partial p_n} = - z_n \frac{\partial x_r}{\partial m} + K_{rn}$$

The own price effect is:

$$\frac{\partial x_r}{\partial p_r} = - z_r \frac{\partial x_r}{\partial m} + K_{rr}$$

even if then there are no inferior goods $\frac{\partial x_r}{\partial p_r}$ may be positive for a net producer of commodity r (wealth effect).

### 2.1.3 Pure exchange economy with tariffs and unequal income distribution (cf. II, § 4 in [14]).

2.1.3.1 The consumer's model is (at the level of the income class):

max $\quad u^j(x^j)$

S.T. $\quad m_j = \nabla_j \; tr + py^j$

and $\quad m_j = px^j$

Now one can differentiate to domestic and to world market prices, we thus assume a domestic price policy on all commodities.

Domestic prices

Again only the differentiation of the budget equation to prices yields a change, so that Slutsky's equation now becomes:

$$\frac{\partial x_r^j}{\partial p_h} = - \left(z_h^j - \frac{\partial \alpha_j tr}{\partial p_h}\right) \frac{\partial x_r^j}{\partial m_j} + K_{rn}^j$$

The income effect may dominate the substitution effect so that the own price effect may be positive.

World market prices

World market prices only affect in a direct way the tariffs receipts so that differentiation to prices yields

$$\lambda \sum_h p_h \frac{\partial x_h^j}{\partial p_n^w} = - \lambda \left(z_n - \frac{\partial \alpha_j tr}{\partial p_n^w}\right)$$

$$- p_i \frac{\partial \lambda}{\partial p_n^w} + \sum_h u_{ih} \frac{\partial x_h}{\partial p_n^w} = 0 \quad .$$

The change in the second equation is important as it implies that there is no substitution effect anymore. The Slutsky equation is (the own price effect can be of any sign):

$$\frac{\partial x_r^j}{\partial p_n^w} = - \left(z_n^j - \frac{\partial \alpha_j tr}{\partial p_n^w}\right) \frac{\partial x_r^j}{\partial m_j}$$

2.1.3.2 The Slutsky equations at the national level follow from summation over income classes

Define

$$l_r = \sum_j \alpha_j \frac{\partial x_r^j}{\partial m_j}$$

$$w_{rh} = - \sum_j z_h^j \frac{\partial x_r^j}{\partial m_j}$$

$$\alpha_j = \frac{py^j}{py}$$

then:

$$\frac{\partial x_r}{\partial p_h} = l_r \frac{\partial tr}{\partial p_h} + W_{rh} + K_{rh} + \sum_j \frac{\partial \alpha_j}{\partial p_h} \cdot tr$$

and

$$\frac{\partial x_r}{\partial p_h^w} = l_r \frac{\partial tr}{\partial p_h^w} + W_{rh}$$

We shall now consider the Jacobians when balance of trade equilibrium is satisfied. These are the relevant ones for the external behaviour of the country.

$$tr = (p-p^w)z \quad ; \quad p^w z = 0 \quad \text{in equilibrium} \quad ;$$

thus $\sum_\partial \frac{\partial \alpha_j}{\partial p_h} \cdot tr = 0$ ,

and $\frac{\partial tr}{\partial p_h^w} = \sum_i p_i \frac{\partial x_i}{\partial p_h^w}$

$$\frac{\partial tr}{\partial p_h} = \sum_i p_i \frac{\partial x_i}{\partial p_h} + z_h \quad .$$

Now define[1)]

$$V = (I - [l_i p_h]) \qquad \text{for } i,h = 1..n \quad ,$$

$$\tilde{Q} = \left[\frac{\partial x_i}{\partial p_h^w}\right]$$

$$\tilde{F} = \left[\frac{\partial x_i}{\partial p_h}\right]$$

$$\tilde{G} = \left[K_{ih} + W_{ih} + l_i \; z_h\right]$$

---

1) $[a_{ik}]$ indicates a matrix with elements $a_{ik}$ .

Then previous results may be written as:

$$\tilde{V}\ \tilde{F} = \tilde{F}$$

$$\tilde{V}\ \tilde{Q} = \tilde{W}$$

The matrix $\tilde{V}$ is however singular: we know that

$$\sum_i p_i l_i p_h = p_h$$

by the definition of $l_i$ ,

thus

$$p \quad (I - [l_i p_h]) = 0 \quad \text{for all } p, \text{ so that}$$

$\tilde{V}$ is singular and no explicit formulation for $\tilde{F},\tilde{Q}$ is available

* Note however that

$$\sum_{i=1}^{n-1} p_i l_i p_h < p_h \quad \text{so that}$$

any principal minor of V has diagonal dominance and is non-singular (if there are no inferior goods).

* Note also that if all income classes have the same marginal propensities to consume,

$$\frac{\partial x_i^j}{\partial_j} = \frac{\partial x_i^h}{\partial_h} \quad \text{then} \quad W_{ih} + l_i\ z_h = 0 \quad .$$

We now shall reformulate the equations in order to get an explicit Slutsky equation. (We shall time and again find that the difference between marginal propensities to consume complicates matters).

From balance of trade equilibrium follows

$$\sum_i^{n-1} p_i^w \frac{\partial x_i}{\partial p_h} = - \frac{\partial x_n}{\partial p_h} p_n^w \qquad h=1..n$$

and

$$\sum_{i}^{n-1} p_i^w \frac{\partial x_i}{\partial p_h^w} = -\left(\frac{\partial x_n p_n^w}{\partial p_h^w} + z_h\right) \qquad h=1..n$$

Thus

$$\frac{\partial tr}{\partial p_h^w} = \sum_{i=1}^{n} p_i \frac{\partial x_i}{\partial p_h^w}$$

$$= \sum_{i=1}^{n-1} p_i \frac{\partial x_i}{\partial p_h^w} - \frac{p_n}{p_h^w}\left[\sum_{i=1}^{n-1} p_i^w \frac{\partial x_i}{\partial p_h^w} + z_h\right]$$

$$\boxed{= \sum_{i=1}^{n-1}\left(p_i - \frac{p_n}{p_n^w} p_i^w\right)\frac{\partial x_i}{\partial p_h^w} - \frac{p_n}{p_h^w} z_h \quad ; \qquad h=1,..,n}$$

and

$$\frac{\partial tr}{\partial p_h} = \sum_{l=1}^{n} p_i \frac{\partial x_i}{\partial p_h} + z_h$$

$$\boxed{= \sum_{i=1}^{n-1}\left(p_i - \frac{p_n}{p_n^w} p_i^w\right)\frac{\partial x_i}{\partial p_h} + z_h} \qquad h=1,..,n$$

<u>Lemma</u>

Define $V = I - \left[l_k (p_i - \frac{p_n}{p_n^w} p_i^w)\right]$ $\qquad k=1,..n-1$, $i=1,..n-1$ .

V has diagonal dominance for the prices p if there are no inferior goods.

<u>Proof</u>

From the definition of $l_i$ follows:

$$\sum_{i=1}^{n-1} p_k | l_k (p_i - p_n p_i^w) | < | (p_i - p_n p_i^w) |$$

thus

$$\sum_{\substack{i=1 \\ i \neq k}}^{n-1} p_k \mid l_k (p_i - p_n p_i^w) \mid < \mid (p_i - p_n p_i^w) \mid (1 - p_i l_i)$$

then we prove that

$$\mid (p_i - p_n p_i^w) \mid (1 - p_i l_i) \leq p_i \mid 1 - l_i (p_i - p_n p_i^w) \mid$$

this can be seen as follows:

$$p_i l_i < 1, \quad (1 - p_i (p_i - p_n p_i^w)) > 0 \quad .$$

<u>case 1</u>: assume

$$p_i \geq p_n p_i^w$$

then we have to prove

$$(p_i - p_n p_i^w) \leqq p_i$$

$$- p_n p_i^w \leqq 0 \qquad \text{this is clearly the case}$$

<u>case 2</u>: $p_i - p_n p_i^w < 0$

then we must prove that

$$(p_n p_i^w - p_i)(1 - p_i l_i) \leqq p_i (+ l (p_n p_i^w - p_i))$$

or

$$p_n p^w - 2 p_i (1 + p_n p_i^w l_i) + 2 l_i p_i^2 \leqq 0$$

this is a parabola.

$$p_i^{1,2} = \frac{2(1+p_n p_i^w l_i) \pm 2\sqrt{1+(l_i p_i^w p_n)^2}}{4 l_i}$$

Clearly

$$(1+l_i p_i^w p_n) < \sqrt{1+(l_i p_i^w p_n)^2}$$

so that only one positive root exists.
This root is however larger than $p_i^w p_n$ if:

$$(1+p_n p_i^w l_i) + \sqrt{1+(l_i p_i^w p_n)^2} > 2 l_i p_i^w p_n$$

that is if

$$\sqrt{1+(l_i p_i^w p_n)^2} > l_i p_i^w p_n - 1$$

which is clearly the case.

So the matrix V has diagonal dominance and thus is non-singular. (end of proof.)

$$\frac{\partial x_k}{\partial p_h^w} = l_k \left( \sum_{i=1}^{n-1} (p_i - \frac{p_n}{p_n^w} p_i^w) \frac{\partial x_i}{\partial p_h^w} - \frac{p_n}{p_n^w} z_h \right) + w_{kh}$$

for $\quad k,h = 1,..n \quad .$

consider the reduced system of n-1 commodities

Defining

$$H = \left[ w_{kh} - l_k z_h \cdot \frac{p_n}{p_n^w} \right] \quad , \quad k,\ h = 1\ .\ .\ n-1$$

We may write

$$V \cdot Q = H \quad .$$

thus we get an explicit formulation for the Jacobian:

$$\boxed{Q = V^{-1} H}$$

From this the elements $\frac{\partial x_n}{\partial p_h^w}$ , $\frac{\partial x_k}{\partial p_n^w}$ and $\frac{\partial x_n}{\partial p_n^w}$ can easily be derived.

In a similar way one gets an explicit formulation for domestic prices.

$$\frac{\partial x_k}{\partial p_h} = 1_k \sum_{l=1}^{n-1} \left( p_i - \frac{p_n}{p_n^w} p_i^w \right) \frac{\partial x_i}{\partial p_h} + \left[ 1_k z_h + K_{kh} + W_{kh} \right]$$

$$h, k = 1 \;.\;.\; n-1$$

Define

$$F = \left[ \frac{\partial x_k}{\partial p_h} \right]$$

$$G = \left[ K_{ih} + W_{ih} + 1_i z_h \right] \text{ as before}$$

then

$$\boxed{F = V^{-1} G} \;.$$

Again the $n^{th}$ row and column may be derived from this.

Induced changes in domestic prices:

Changes in world market prices may induce changes in domestic prices. The total effect of a mutation in world market prices then becomes for the first n-1 commodities:

$$E = V^{-1} (H + GP)$$

where

$$E = \left[ \frac{dx_i}{dp_h^w} \right]$$

and

$$P = \frac{dp_i}{dp_h^w}$$

and H and V are defined as before.

The direct price effect of world market prices at the national level (and at world level) can be thought of existing of

a) an income effect due to differences in

1) marginal propensities to consume
2) resource ownership y
3) shares in tariff receipts $\alpha_j$

b) a substitution effect due to different Slutsky matrices $K^j$ . The price effects are thus aggregate effects and can be positive or negative if only because the income effect will be positive for net producers and negative for net consumers.

2.1.4 The Jacobian matrix under (tariffs and) quota

The domestic market equilibrium can at given desired domestic prices $\bar{p}$ and $p^w$ be represented by

$$q = q(\rho)$$

$$\rho \cdot q = 0$$

$$\rho, q \geqq 0 \ .$$

Where

$$q = \begin{bmatrix} q_1(\rho_1, \rho_2) \\ q_2(\rho_1, \rho_2) \end{bmatrix}$$

and

$$q_1 = z(p^w, \bar{p} - \rho_1 + \rho_2) + r$$

$$q_2 = -z(p^w, \bar{p} - \rho_1 + \rho_2) - l$$

Domestic equilibrium has been shown to exist at any world market price.
The Jacobian can be set up in two ways

1) Jacobian of the domestic market at given world market prices;

2) the Jacobian of the total system (domestic and world market equilibrium considered simultaneously).

2.1.4.1 The Jacobian for domestic equilibrium

As the problem has been formulated in "standard" format we may proceed by writing down the matrix

$$J = \frac{\partial q_i}{\partial \rho_j}$$

and investigate its properties.
From the definition of q follows that:

$$J = \begin{bmatrix} -V^{-1} G & V^{-1} G \\ V^{-1} G & -V^{-1} G \end{bmatrix}$$

* The reaction of the domestic market to mutations in world market prices.

When we studied the effects of mutations in world market prices on an economy with tariffs without quota restrictions we allowed for possible "induced" mutations in domestic prices.

The Jacobian matrix was then

$$E = V^{-1} \quad (H + GP)$$

If we now disregard all induced mutations in domestic prices having other causes than quota restrictions and if we only consider the national excess demand function where it is <u>differentiable</u> to world market prices (i.e. where a marginal change does not change the list of effective quota) and if we assume domestic price equilibrium, then we know that either $\frac{dz_i}{dp_n^w} = 0$ or $\frac{dp_i}{dp_n^w} = 0$ (other world market prices remaining constant)

We may now decompose the equation for the Jacobian matrix

define $U = V^{-1}$

$$U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}$$

$$P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$$

$$G = \begin{bmatrix} G_{12} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$$

$$E = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}$$

We may set $E_1$, a matrix with dimensions r x n, equal to zero, indicating that the first r commodities have (and keep!) effective quota constraints.

Complementarily the matrix $p_2$ with dimensions (n - r), n can be set equal to zero.

One thus gets:

$$0 = U_1 H + U_1 G_1 P_1$$

$$E_2 = U_2 H + U_2 G_1 P_1$$

Solving the first set of equations for $P_1$ and substituting in the second one gets:

$$P_1 = -(U_1 G_1)^{-1} U_1 H$$

$$E_2 = U_2(I - G_1(U_1 G_1)^{-1} U_1)H$$

and

$$E = \begin{matrix} 0 \\ E_2 \end{matrix}$$

(Note that $(I - G_1(U_1\ G_1)^{-1}U_1)$ is indempotent.)
We still have not proved that $U_1G_1$ is nonsingular. As U is nonsingular $U_1$ has rank r; now if $G_1$ also has rank r, then $U_1\ G_1$ has rank r and is nonsingular.
We assume here that G has rank n-1. This assumption will be discussed in more detail below, in a note: it illustrates some problems of aggregation of Jacobians.
Under this assumption we may however conclude that $U_1\ G_1$ will be nonsingular if $r \leqq n-1$.

2.1.4.2 The Jacobian matrix of the total system

The Jacobian matrix derived in the previous pages is not very general because of the differentiability requirement. In domestic equilibrium national excess demand functions are not differentiable for all world market prices. In order to restore differentiability one must simultaneously consider the equilibrium conditions for all markets. In order to do this the restriction $q \geqq 0$ of the domestic market must be relaxed and the total Jacobian matrix must be investigated.
So the total matrix can be written out as:

$$E^w = \begin{bmatrix} J_1 & & 0 & R_1 \\ 0 & \ddots & J_m & R_m \\ C_1 & & C_m & S \end{bmatrix}$$

where m indicates the number of countries and i is the country index:

$$J_i = \left[\frac{\partial q_h^i}{\partial \rho_j^i}\right] = \left[\begin{array}{cc|cc} -V_i^{-1} & G_i & V_i^{-1} & G_i \\ \hline V_i^{-1} & G_i & -V_i^{-1} & G_i \end{array}\right],$$

$$C_i = \left[\frac{\partial q_h^i}{\partial p_k^w}\right] = \left[ V_i^{-1}\ H_i - V_i^{-1}\ H_i \right],$$

$$R_i = \left[\frac{-\partial z_h}{\partial \rho_j^i}\right] = \begin{bmatrix} V_i^{-1} & G_i \\ \hline -\bar{V}_i & \bar{G}_i \end{bmatrix},$$

and $S = \frac{-\partial z_h}{\partial p_k^w} = - - \sum_{i=1}^{m} (V_i^{-1} H_i)$

This system is however very large as soon as many countries are considered. We go on considering domestic and world market equilibrium separately.

Note

If the set

$$G\,q = 0$$

has as only nontrivial solution $q = \lambda\, p$ then G has rank n-1. It is not possible to prove that for $q \neq 0$ , $\exists\, q \neq \lambda p$ but there cannot be said more than that the assumption that $\tilde{G}$ has rank n-1 does not seem restrictive. Note that the knowledge from demand theory that $K_j q = 0$ has only the nontrivial solutuion $q = \lambda p$ does not help us because it does not inform us about the rank of K so that neither the matrix of the aggregate income effect nor the matrix of aggregate substitution effect have a definite rank. We shall return to this problem in the next paragraph. We have already seen that nothing can be said with certaintiy about the sign of elements of the Jacobian.

Debreu [5] has actually shown that when there are more consumers than commodities to any continuous excess demand function satisfying Walras'Law corresponds a distribution of endowments and a set of well behaved utility functions.

## 2.2 Uniqueness of equilibrium: conditions on the Jacobian

### 2.2.1 Introduction

Uniqueness of equilibrium becomes especially relevant when an equilibrium model is used in comparative static analysis. In this case the effect of the change in a parameter is investigated by comparing the equilibrium before and after the change. This is only possible if the equilibrium is unique. However, if the model is used in a dynamic context and a descriptive function is accorded to the algorithm used to compute the equilibrium, then whichever new equilibrium is computed by the algorithm is the relevant one. At any rate the model as a whole should be such that after a shift in parameters only one equilibrium is obtained, this is somewhat trivial.

We now are interested in the uniqueness of equilibrium in the "ex ante" sense, so that the algorithm used to compute equilibrium is irrelevant because the algorithm does not select an equilibrium.

### 2.2.2 Some theorems from the literature (cf.Arrow and Hahn [1] and Nikaido [11].

Define $s \equiv - z \equiv$ excess supply.

A. Assume:

1) that the excess supply functions are homogeneous of degree zero in prices; (H)

2) for all $p \in S_n$[2], $ps(p) = 0$ (Walras' Law); (W)

3) $\exists$ R, finite positive such that for all $p \in S_n$, $s_i(p) < R$ (boundedness[1]); (B)

4) s(p) is defined at least for all $p > 0$, $p \in S_n$ and is continuous wherever defined. If s(p) is not defined in $p = p^o$ then $\lim_{p \to p^o} \Sigma s_i(p) = -\infty$. This is the weakened continuity requirement. (C')

---

1) This is trivial if supply is given.

2) $S_n$ is the price simplex.

B. Assume further that

1) $s(p)$ is differentiable wherever defined; (D)

2) in equilibrium there is at least one commodity (say the $n^{th}$), for which $\sum_i s_i(p) = -\infty$ when $p_n = 0$, (the $n^{th}$ good is then called the numeraire). (N)

Consider now the Jacobian of n-1 commodities:

$$J(p) = \frac{\partial s_i}{\partial p_j} \quad , \; i,j = 1,..n-1 \quad ,$$

or consider

$$J(p) = \frac{\partial s_i}{\partial \left(\frac{p_j}{p_n}\right)} \quad i,j = 1,..,n-1 \quad .$$

Without proof the following theorem is stated (cf. Arrow and Hahn [1] for a proof.):

Theorem for uniqueness

Under assumptions A and B, there is only one price vector $p \in S_n$ such that $s(p) \geqq 0$ if $J(p)$ has only principal minors with positive determinants.[1)]

This property of the Jacobian matrix is called the Gale property. It is quite difficult to give any economic interpretation to this property. Moreover it merely indicates a sufficient condition, not a necessary one.

Because of this the discussion in this paragraph will have to be casuistic. We now proceed by discussing certain (sufficient) conditions which garantee uniqueness.

The model under discussion can be considered alternatively as a national model with zero international trade (quota prohibit any trade) or as a world model with continuous national excess demand functions (cf. 1.5).

---

1) A weaker formulation: If under A and B, J (P) has Gale property (GP) for all equilibrium P, then the equilibrium is unique

$$P_i = \left(\frac{p_i}{p_n}\right)$$

### 2.2.3 One household economy

Consider an economy with only one household and let $p^*$ be an equilibrium for that economy:

$$\max \quad u(x)$$

$$\text{S.T.} \quad px = py$$

in equilibrium

$$s(p^*) = y - x(p^*) \geqq 0$$

$$p^* s(p^*) = 0$$

consider

$$p \neq p^* \quad , \; p \in S_n \quad ,$$

then

$$p\, s(p^*) \geqq ps(p) = 0 \quad \text{(if not then } p^* \text{ would not be an equilibrium)}.$$

Thus

$$p(x(p) - x(p^*)) \geqq p(y(p) - y(p^*)) \quad .$$

The right hand side will be nonnegative because of profit maximization (or when $y$ is given because $y(p) = y(p^*)$).

From the weak axiom of revealed preference we know then that $p^*(x(p) - x(p^*)) \geq 0$ .
However, from profit maximization $p^*(y(p)-y(p^*)) \leqq 0$, so that $p^*(y(p^*)-x(p^*)) \geqq p^*(y(p)-x(p))$ .
But at a given $p$ there is only one $s$; $p^*s(p)$ is a scalar; $p^*s(p^*) = p^*s(p)$ would imply that both $s(p)$ and $s(p^*)$ would be chosen at $p^*$ then there would be not only one $s$ at a given $p$. Thus $p^*s(p) < 0$ which implies that $s(p) \ngeq 0$ , so that the equilibrium is unique.

No use was made of the Gale property so that differentiability is not required. The model can however be shown to have the property.

### 2.2.4 Hicksian economy

An economy with m consumers, n commodities and given resources for each consumer is called Hicksian if

a) $$x_h(p, m_h) = {}^1/_k\, x_h\,(p, km_h) \quad , \quad \forall\, k > 0 \quad ,$$

b) $$x_h(p,1) \quad = x_i(p,1) \quad ,$$

where

$$m_h = p\bar{x}_h + \alpha_h\, py \quad . \qquad (\alpha_h \text{ is a parameter})$$

The Hicksian economy behaves as if there is only one household and thus has a unique equilibrium. We shall come back to this matter in § 3.3.
Note however that in the Hicksian economy all income classes spend their income in the same proportions over commodities. This is highly unrealistic.

### 2.2.5 Gross substitutability

Definition: Two commodities are said to be gross substitutable (GS) if $\frac{\partial s_i}{\partial p_j} < 0$ for $i \neq j$ .
Under GS all off-diagonal elements of J(p) are negative and due to Walras' Law the diagonal elements then must be positive.

Again without proof we state:
Under assumptions A and B , if there is GS for all equilibrium prices then there is a unique equilibrium because J(p) then has GP for all equilibrium prices and the equilibrium price vector is strictly positive. Note that the sum of Jacobians with GS also has GS (this is not the case for GP!) GS however implies that

$$\varepsilon_{ii} = \frac{\partial s_i}{\partial p_i}\, \frac{p_i}{s_i} < 1 \quad ,$$

and that if

$$p_i \quad = 0 \quad , \quad \sum_h s_h(p) \; = -\infty \quad \text{for any } i \quad .$$

### 2.2.6 Diagonal dominance

Definition: If J(p) is such that

a) $s_{ii}(p) > 0 \; \forall_i$ ,

b) $\exists$ h(p) such that $h_i . s_{ii}(p) > \sum_{\substack{i=1 \\ i \neq j}}^{n-1} |s_{ij}(p)| \, h_j(p) \quad \forall i < n$

then the economy has diagonal dominance (DD). $(s_{ij} = \frac{\partial s_i}{\partial p_j})$ .

Theorem

If the economy has DD for all equilibrium prices and has a numéraire then the Jacobian has GP and the equilibrium is unique.

### 2.2.7 Other sufficient conditions

1) Theorem: If for all equilibrium P , J(P) is either positive definite or positive quasidefinite then J(P) has GP and P is unique under assumptions A, B.

2) If J(P) is nonsingular and $J^{-1}$ $(\bar{z}(p))$ is continuously differentiable for all P > 0, if $\lim_{p \to p_o} \Sigma s_i(p) = -\infty$ whenever $p_{oh} = 0$ then the economy has a unique strictly positive equilibrium.

### 2.2.8 Consequences of the theorems for the national model with domestic price policy and quota on international trade

Here we do not discuss uniqueness of equilibrium on the world market but concentrate on the uniqueness of domestic equilibrium.

There are 4 cases to consider:

a) Free trade: in this case the uniqueness of domestic equilibrium is trivial if the utility functions are, as we have assumed, strictly quasi concave.

b) Domestic price policy only: the demand can be computed at given domestic prices in the same way as under free trade. The taxation must however be adjusted such that the balance of trade and then also the government budget are in equilibrium. As long as for all possible taxation levels an increase in taxes leads to a decrease in the value of demand (evaluated at world market prices), this equilibrium will be unique.

c) Completely closed national economy: here the theorems mentioned above apply directly.

1) Hicksian economy

If the utility function is homothetic and if all consumers have the same utility function then the economy has a unique equilibrium. We shall return to this matter in section 3.

2) Gross substitutability recalling the Slutsky equation at national level we write:

$$E = V^{-1} (H + GP) \quad .$$

For the definition see § 2.1 .

* Under free trade the equation at the national level would be:

$$E = W + K$$

$$= \Sigma(W^j + K^j)$$

even if $K^j$ is assumed to have GS for all income classes then still $\Sigma W^j$ will not have this property as the sign of it is quite unclear because of aggregation, whether the GS of K is then strong enough is difficult to say. But even the assumption on K is very restrictive.

* In any case when $E = V^{-1} H$ the substitution term is completely dropped from the equation.[1] It is then the income effect after tariff redistribution which decides on the Gross Substitutability.

We know that V has DD with positive elements on the diagonal. This does not imply very much however on the inverse of V . Even if H happens to have GS then the GS property of $V^{-1}$ H is not obvious.

3) Diagonal dominance

Similar reasoning applies to diagonal dominance. The sum of diagonally dominant matrices is not necessarily a diagonally dominant matrix so that even the diagonal dominance in all income classes would be insufficient to prove diagonal

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[1] If there is a domestic price policy for all commodities

dominance at the national level. Moreover the picture is again disturbed by redistribution effects. If nothing can be said at the national level, nothing can be said at world level.

(Quasi) positive definite Jacobian, Non-singular Jacobian
Again the aggregation and redistributive effect made it impossible to derive any conclusions.

d) The model of section 1

* The Jacobians derived before under the assumption $p^w(y - x) = 0$ are not necessarily relevant for the national model with quota because we have seen that in this case only $p^w(x + s - y) \leqq 0$ could be proved.
* For testing the uniqueness of domestic equilibrium one would need a reformulation of the Jacobians under the assumption $t = - \mu l + \nu r + \phi(\bar{p}-p^w)z$ instead of $p^w(y - x) = 0$, $(z = x + s - y)$ .

We would however again be confronted with the problem of aggregation discussed above. Moreover, the Jacobian derived under this specific taxation rule would not lend itself to very much economic interpretation as the construction from which it stems was purely technical. We shall therefore leave now the analysis of Jacobians and treat the matter of uniqueness in a more constructive way:

If an algorithm designed to compute domestic equilibrium converges to one and only one solution, if the excess demand in this solution is a continuous function of world market prices, we conclude that there is uniqueness of equilibrium in a restricted, but sufficient sense.

# SECTION 3: THE COMPUTATION OF DOMESTIC EQUILIBRIUM

## 3.1 Computation of domestic prices at given domestic availability

The previous paragraph has been disappointing. Because of aggregation and redistribution effects not even a non-singularity property could be attributed to the Jacobian. This leads us to an even more casuistic approach where on the basis of specific features of the utility function the uniqueness of equilibrium can be investigated.

For reasons which will become clear in 3.2 we are mainly interested in the uniqueness of domestic equilibrium in the restricted sense that we investigate whether to any given world market price and to any given total domestic availability (which satisfies the balance of trade equilibrium and the quota constraints) corresponds a unique domestic equilibrium price. This condition is not sufficient for the uniqueness of domestic equilibrium in the sense of § 2.2. The model of § 1 then becomes

$$\max \quad u_j(x^j)$$

$$\text{S.T.} \quad px^j = p(\alpha_j \bar{z} + y^j)$$

$$\text{while} \quad \sum_j z^j \leq \bar{z} \quad .$$

We assume further that $\alpha_j = \dfrac{py^j}{py}$ .

The corresponding Jacobian is $F = V^{-1}G$ .

Again nothing in general can be concluded about the structure of F.

We shall not only be interested in uniqueness (U) but also in direct computability (DC) of domestic prices. By this we mean that domestic prices can be computed without making use of a procedure in which equilibrium is reached in an iterative way.

No general theory will be presented. Only a few examples will be discussed.

3.1.1 Utility maximization; uniqueness and direct computability of domestic equilibrium price

We assume as before that the utility function is continuous, strictly quasi-concave and has continuous first and second derivatives and a positive first derivative.

3.1.1.1 General case:

From the first order condition follows:

$$\frac{p_i}{p_h} = \frac{\frac{\partial u_j}{\partial x_{ij}}}{\frac{\partial u_j}{\partial x_{hj}}} \quad .$$

We know that

$$\sum_j (y^j - x^j) \geq \bar{z} \quad ,$$

and that

$$p(x^j - y^j) = p(\alpha_j z + y^j) \quad .$$

The uniqueness of the solution is not clear and it is obvious that we cannot decide whether the solution is directly computable (DC).

3.1.1.2 One income class: one household economy

We know already from § 2.2 that the equilibrium will be unique in this case.

$$\frac{p_i}{p_h} = \frac{\frac{\partial u}{\partial \bar{x}_i}}{\frac{\partial u}{\partial \bar{x}_h}} \quad .$$

The right hand side is given so that domestic prices follow directly.

Note that if there had been more income classes with the same utility function, neither U or DC could have been established in general.

3.1.1.3 One common utility function, more than one income class

a) Duality theory:

Before we discuss cases with more than one income class a very brief introduction into duality theory will be provided.

The problem Ia: max $u(x)$
$py = m$

is equivalent with

Ib: max $u(x)$
$wx = 1$

where

$$w_i = \frac{p_i}{m} \quad .$$

Under fairly general conditions on the utility function the following results from duality theory emerge. (cf. Diewert [6]).

* $m = m(p,u)$ is concave and linear homogeneous and monotone in p .
* $m(y,u) = 1$ defines implicitly[1] a function $\frac{u}{h(y)} = 1$ .
  $u = h(y)$ is called the indirect utility function.
  $\frac{1}{u} = g(y)$ is called reciprocal indirect utility function.

We thus get

$$\Sigma y_i x_i = u(x) \cdot g(y)$$

from this follows

$$x_i = u(x) \frac{\partial g}{\partial y_i} \quad ,$$

so that

$$\Sigma y_i x_i = u \cdot \Sigma y_i \frac{\partial g}{\partial y_i} \quad .$$

So that

$$u = \frac{1}{\Sigma y_i \frac{\partial g}{\partial y_i}} \text{ for } \Sigma y_i x_i = 1 \quad ,$$

1) if $\frac{\partial u}{\partial m} > 0 \; \forall \; m > 0$, this is a nonsaturation assumption.

thus

$$p_i y_i = \frac{\frac{y_i}{g} \frac{\partial g}{\partial y_i}}{\sum_h \frac{y_h}{g} \frac{\partial g}{\partial y_h}} .$$

This relation is called Roy's identity.
Defining

$$s_i \equiv p_i y_i \quad \text{(the income share allocated to commodity i).}$$

$$e_i \equiv \frac{y_i}{g} \frac{\partial g}{\partial y_i} .$$

Roy's identity is

$$s_i = \frac{e_i}{\Sigma e_h} .$$

Note that we know already from the primal problem Ib that

$$x_i \frac{\partial u}{\partial x_i} = \lambda y_i x_i ,$$

$$\Sigma x_i \frac{\partial u}{\partial x_i} = \lambda .$$

Defining

$$d_i = \frac{\partial u}{\partial x_i} \frac{x_i}{u}$$

one gets

$$s_i = \frac{d_i}{\Sigma d_h} .$$

b) <u>Homothetic utility function</u>

* It can be proven (Diewert [6]) that if the utility function is homothetic, both the indirect utility function and the reciprocal indirect utility are homothetic.
* A utility (or production) function $u = F(x)$ is said to be homothetic, if it can be written as
$u = F(f(x))$ ,

where

$$\frac{dF}{df} > 0 \quad , F(0) = 0 \quad , \lim_{f \to \infty} F(f) = \infty \qquad \text{and}$$

$f(x)$ is positively linear homogeneous and concave. A homothetic function has the following properties:

$$u = l\ (\frac{1}{\lambda}) \cdot F(\lambda x) \quad , \quad l \text{ is a monotonous function}$$

and

$$u.\ l(1) = \Sigma x_i \frac{\partial F}{\partial x_i} \quad .$$

If $g = n(y)$ is a homothetic function then it can be written as

$$g = n\ (k(y)) \quad .$$

Multiplying y by m yields:

$$g = n\ (k(p)) \cdot l(m) \quad ,$$

thus

$$\frac{\partial g}{\partial p_i} = \frac{dn}{dk} \cdot \frac{\partial k}{\partial p_i} \cdot l(m) \ ,$$

and

$$\Sigma p_i \frac{\partial g}{\partial p_i} = \frac{dn}{dk} \cdot l(m) \cdot \sum_i p_i \frac{\partial k}{\partial p_i}$$

$$= \frac{dn}{dk} \cdot k.l(m) \quad ,$$

so that

$$s_i = \frac{p_i}{k} \frac{\partial k}{\partial p_i} \quad .$$

k is only a function of prices so that

$$\frac{\partial s_i}{\partial m} = 0 \quad \text{and} \quad \frac{\partial \ln x_i}{\partial \ln m} = 1 \quad .$$

c) <u>One common homothetic utility function</u>

After these preparations we are now in a position to discuss uniqueness and direct computability.
By definition

$$x_{ij} = s_{ij} \frac{m_j}{p_i} \quad .$$

We have seen that when the utility function is homothetic $s_{ij} = s_{ij}(p)$ so that all income classes have the same budget proportions (because they are confronted with the same price by assumption).
We thus get

$$\bar{x}_i = \frac{s_i}{p_i} \sum_j m_j$$

and

$$\frac{x_{ij}}{\bar{x}_i} = \frac{m_j}{\Sigma m_j} \quad .$$

The economy can thus be considered to be Hicksian, so that uniqueness of domestic prices can be established.
We know that

$$\frac{\partial u(x)}{\partial \bar{x}_i} = \frac{\partial u(x_j)}{\partial x_{ij}} \frac{\partial x_{ij}(p/m_j)}{\partial \bar{x}_i}$$

$$= \frac{\partial u(x_j)}{\partial x_{ij}} \frac{m_j}{\Sigma m_j} \quad .$$

Moreover:

$$\frac{\frac{\partial u}{\partial x_{ij}}}{\frac{\partial u}{\partial x_{nj}}} = \frac{p_i}{p_n} \quad ,$$

so that

1) $$\frac{p_i}{p_n} = \frac{\frac{\partial u}{\partial \bar{x}_i}}{\frac{\partial u}{\partial \bar{x}_n}} ,$$

so that the price ratios can be directly computed.

and

2) the demand proportions by income class are given $\frac{\bar{x}_i}{\bar{x}_n}$ .

Define

$$\beta_i = \frac{\bar{x}_i}{\bar{x}_n} \qquad i=1,..n$$

and

$$d^j = y^j + \alpha_j \bar{z}$$

then

$$p\beta \; x_{nj} = pd^j$$ so that the demand is

$$x_{ij} = \frac{pd^j}{p\beta} \cdot \frac{\bar{x}_i}{\bar{x}_n} \quad ; \quad i=1,..,n \quad .$$

The direct computability has thus been obtained because

$$\frac{\frac{\partial u}{\partial \bar{x}_i}}{\frac{\partial u}{\partial \bar{x}_n}} = \frac{\frac{\partial u}{\partial x_{ij}}}{\frac{\partial u}{\partial x_{nj}}} = \frac{p_i}{p_n} \quad .$$

d) Cobb Douglas utility function functions with different coefficients between income classes

$$u_j = \prod_{i=1}^{n} x_{ij}^{\alpha_{ij}} \quad ; \quad \alpha_{ij} > 0 \quad ; \quad u \text{ concave} \quad .$$

We know that in this case the utility function is homothetic so that

$$s_{ij} = \frac{\alpha_{ij}}{\Sigma\alpha_{ij}} \quad .$$

$$p_i \overline{x}_i = \phi \cdot \sum_h p_h \sum_j y_{hj} \, e_i^j \quad , \quad \text{where } e_i^j = \frac{\alpha_{ij}}{\Sigma \alpha_{ij}} \quad \phi = \frac{px}{py}$$

$e_i^j$ is constant so that the system is linear at given $\phi$. Writing $\overline{x}_i$ as a diagonal matrix one gets

$$p(\overline{X} - \phi A) = 0 \tag{1}$$

where

$$A = [A_{hi}] \quad (m x n) \quad ,$$

$$= \sum_j Y_{hj} \, e_i^j \quad .$$

If we assume indecomposability of A the uniqueness follows from the Perron-Frobenius theorem. We then also may conclude that the equilibrium price will be strictly positive. $^1/\phi^*$, $p^*$ will be dominant eigenvalue and eigenvector of $A\overline{X}^{-1}$ . This case will play the central role in § 3.2.

e) Reconsidering the problem of direct computability we may say that it amounts to finding a "simple" solution for the prices from the primal and dual equations:

a) primal equations

$$\frac{\dfrac{\partial u^j}{\partial x_{ij}}}{\dfrac{\partial u^j}{\partial x_{nj}}} = \frac{p_i}{p_n} \qquad \begin{array}{l} i = 1 \ldots n \\ j = 1, \ldots m \end{array}$$

b) dual equations

$$x_{ij} = \frac{\dfrac{p_i}{g_j} \dfrac{\partial g_j}{\partial p_i}}{\sum_h \dfrac{p_h}{g_j} \dfrac{\partial g_j}{\partial p_h}} \cdot {}^{mj}/p_i$$

c) budget equations

$$m_j = \Sigma p_\iota \; d_{ij}$$

where $\alpha_{ij} = \alpha_j \, \overline{z}_i + y_{ij}$ .

d) domestic availability

$$\sum_j x_{ij} = \bar{x}_i$$

We can see from b) that at given prices a solution $x_{ij}$ can be calculated easily.

Summation of b) over income classes yields

$$p_i \bar{x}_i = \sum_j \left( \frac{e_i^j}{\sum_h e_h^j} m_j \right)$$

where

$$e_i^j = \frac{\partial \ln g_j\left(\frac{p}{m_j}\right)}{\partial \ln p_i}$$

After substitution of c) the implicit nature of the equation and the possibility of non-unique solutions are obvious.

As a matter of fact none of the well known non-homothetic functions seem to yield direct computability (CRES, CRESH, non-homothetic CES, addilog, translog). See e.g. Hanoch [7]. The unfruitful attempts to do this will not be presented here. This does not imply that none of them has a unique solution but only that the uniqueness is not proved and seems to be more of an exception than a rule. Anyhow the Cobb Douglas case is at this moment the only one which can be solved by linear methods, is derivable from either direct or indirect utility theory and yields different expenditure shares for different income classes.

### 3.1.2 Some summarizing remarks

#### 3.1.2.1 Dynamic demand models

We have seen that in static demand models direct computability can only be derived in very simple cases while uniqueness is very difficult to prove in the other cases.

As a consequence direct computability must be bought at the expense of the flexibility of the econometric specification. These constraints are of course serious but one should keep in mind that they only need to be imposed on the short run demand functions. We take the Cobb Douglas case as an example. In the Cobb Douglas case every income class allocates expenditures to commodities in a predetermined way.

$$p_{it}x_{ij} = \alpha_{ijt} \cdot m_{jt}$$

$$s_{ijt} = \alpha_{ijt} = f(\alpha_{ijt-1}, p_{t-1}, m_{jt-1}) \quad .$$

We know from duality theory that

$$s_{ij} = \frac{\frac{\partial g_j(p/m_j)}{\partial pi}}{\sum_h \frac{\partial g_j}{\partial p_h}} \quad .$$

Where $g_j = g_j(p/m)$ is the indirect utility function of the $j^{th}$ income class. Any econometrically estimable indirect utility function can be used to generate the income shares, but this may be supplemented by all kinds of non-economic variables. Although the practical advantages of the approach are obvious, the short run specification is still a Cobb Douglas specification with all its limitations.

The same applied to the other approaches discussed before: the long run demand functions ($s_{ijt} = s_{ijt-1}$) need not be influenced by the limitations on the short run functions.

3.1.2.2 Committed expenditures

Up to this point we have assumed that the demand within the period is determined in one step. If one however assumes that the utility maximization only applies to the so called "uncommitted" expenditures while the committed demand is totally inelastic (subsistence levels), the model can be extended:

$$\max \quad u_j(x^j)$$

$$\text{S.T.} \quad px^j = p(\alpha_j\bar{z} + y^j - q^j) \quad .$$

The total demand is then

$$x^j = \tilde{x}^j + q^j \quad .$$

All the previous derivations apply after redefination of the income and the demand. The income elasticity of demand is not unitary anymore when the utility function is homothetic.

$$\frac{\tilde{m}_j}{\tilde{x}_{ij}} \frac{\partial \tilde{x}_{ij}}{\partial \tilde{m}_j} = 1 \quad .$$

$$m_j = \tilde{m}_j + p\, q^j \quad .$$

$$x_{ij} = \tilde{x}_{ij} + q_{ij} \quad ,$$

now

$$\frac{\partial x_{ij}}{\partial m_j} = \frac{\partial \tilde{x}_{ij}}{\partial m_j} \quad ,$$

so that

$$\eta_i = \frac{\partial \tilde{x}_{ij}}{\partial m_j} \cdot \frac{m_j}{x_{ij}} \quad ,$$

$$= \frac{m_j}{\tilde{m}_j} \Big/ \frac{x_{ij}}{\tilde{x}_{ij}} \quad ,$$

$$= \left(1 + \frac{pq_j}{\tilde{m}_j}\right) \Big/ \left(1 + \frac{q_{ij}}{\tilde{x}_{ij}}\right) \quad ,$$

the income elasticity for luxury good is likely to drop with rising income while the income elasticity for necessities will increase.

Committed expenditures however present the problem that if the quantities committed by an income class exceed its own endowment the positiveness constraint on the uncommitted income might be violated at certain prices.

3.1.2.3 Inputs for production: Even when outputs are considered to be lagged, inputs are not so that the pure exchange economy does not depict this case. When the income classes are assumed to own their production factors, their (gross) savings must be equal to their (gross) investments. If the demand for inputs is determined in a production submodel, the cost of these inputs has to be considered as committed expenditures for the consumer. The production model must however take into account that certain limits have to be imposed on the savings capacity of the class (sector).

Under the present assumptions it would be theoretically more acceptable to determine investment and consumption plans simultaneously by maximization of a (multiperiod) utility function within the constraints of the factor resource endowments given expected prices and given a technology.

This would however be quite complex so that the decentralizing assumption seems preferable by which producer and consumer decisions are taken separately. We shall return to this matter in 3.3.

3.1.2.4 The necessity of direct computability

The direct computability requirement lacks any theoretical basis. If one however wants to compute a domestic equilibrium under quota it is highly expedient if one is able to perform "complementary pivots" in a simple way. By a pivot is meant that up to a certain switch point the quantities of certain commodities are kept constant while their prices are allowed to vary (and vice versa for the other commodities) and after the switch point some other list of commoditiies with constant prices prevails. This will be explained in more detail in the next paragraph.

Another argument in favor of direct computability is that it allows a simple solution for a quasi-equilibrium (in a quasi equilibrium all equilibrium conditions are met, except that the prices are not equalized between markets).

## 3.2 Domestic equilibrium under Cobb Douglas utility functions

### 3.2.1 Introduction

We shall now extensively study the case in which all income classes have Cobb Douglas utility functions with different coefficients. This case was already discussed before in § 3.1.3.4.

We recall that the set of demand equations could at given domestic availability be written as an eigenvalue system:

$$p(X - \phi A) = 0$$

where

| | | |
|---|---|---|
| $p$ : | domestic price vector | (dominant eigenvector of $AX^{-1}$) |
| $\phi$ : | taxation level | (reciprocal of dominant eigenvalue of $AX^{-1}$) |
| $A =$ | $[a_{ih}]$ ; $a_{ih} = \sum_j (Y_{ij} \cdot e_{jh})$ | |
| $y_{ij}$ | endowment of $i^{th}$ good in $j^{th}$ income class | |
| $e_{jh}$ | income share allocated to $h^{th}$ commodity by $j^{th}$ income class. | |

When X is not given the computation of domestic equilibrium is more complex. We shall now derive a procedure to solve this problem.

Before we discuss the computation of domestic equilibrium two theorems by O.L. Mangasarian [10] will be presented.

1. Consider the linear complementarity problem of finding a $z$ in $R^n$ such that $Mz + q \geq 0$ , $z \geq 0$ , $z^T (Mz + q) = 0$ where $M$ is a given real n x n matrix and $q$ as a given vector in $R^n$ .

A Z-matrix is a matrix with nonpositive off diagonal elements. If $M$ is a Z-matrix, then for any $p > 0$ the solution $z$ of the linear programme

$$\max \quad p^T z$$
$$\text{S.T.} \quad Mz + q \geqq 0$$
$$z \geqq 0$$

solves the linear complementarity problem.

2. If $M$ is a Z-matrix, then for each $q$ for which the polyhedral set $S = \{z \mid Mz + q \geq 0 \quad , \quad z \geq 0\}$ is nonempty, $S$ contains a unique least element, which is the solution of the linear programme for any $p > 0$ .

$\bar{z}$ is a least element in S if $\bar{z} \leq z , \forall \; z \in S$ .

Note that nothing is said about the uniqueness of the solution of the linear complementarity problem.

There are three cases which must be distinguished from the outset

1. the case with import quota only;
2. the case with export quota only;
3. the case with both import and export quota.

3.2.2 Import quota only

Define the diagonal matrices $X = [ x_i ]$

$H = [ y_i - l_i ]$

Define $a_{ih} = \sum_j y_{ij} e_{jh}$ , $A = [ a_{ih} ]$ .

The model may be written as follows, if we assume taxation proportional to wealth ($\alpha_j = py^j/py$)

1. Consumer demand

$$pX = \phi pA$$

2. Definitions

$\mu = p - p^w, \ \mu \geqq 0$ .

3. Market equilibrium

$\mu \ (H - X) = 0$

4. Policies

$X \leqq H$

5. Balance of trade equilibrium

$p^w X \iota = p^w A \iota$ .

6. Assumption on instruments

$p^w l \leqq 0$

$y > 1$ .

* Under free trade, that is when no quota are effective ($p^w(H-A) \geqq 0$ [1]) the system reduces to:

$p^w X = p^w A$ (Clearly balance of trade equilibrium is satisfied)

* Under domestic price policy 2 - 3 and 4 may be left out and replaced by

$p = \bar{p}$ .

Assuming $\bar{p} > 0$ and defining

$$\bar{P} = \begin{bmatrix} \bar{p}_1 & 0 \\ 0 & \bar{p}_n \end{bmatrix} ,$$

one gets

$$x = \bar{P}^{-1} \ A^T \bar{p}_\iota$$ [2]

and

$$p^w X = p^w A \ .$$

We now discuss the more general case with import quota. We assume for expository purposes that no price policy is pursued. Consider first the solution for given $\phi = \phi^*$ then if $\mu H = \mu X$ holds $p^w X + \mu H = \phi^* (p^w A + \mu A)$ . We can formulate the linear complementarity problem:

---

1) $p^w > 0$ , we shall maintain this as an assumption until 3.2.4.4.

2) We have assumed throughout this paper that $\bar{p} > 0$ .

$$\left[\begin{array}{l} q^T P^w - \mu H = -\phi^{\bullet}\ (p\ A + \mu A) + \iota^T H P^w \\ q \geq 0 \\ \mu \geqq 0 \\ \mu q = 0 \quad . \end{array}\right.$$

Where

$$q \equiv (H-X)\iota$$

again

$(H - \phi^* A)$ has nonpositive off diagonal elements so that the L.P. solution is available. This solution does however not guarantee that $p^w x = p^w y$ , because $\phi^{\bullet}$ was fixed. One must iterate over $\phi$ in a so called parametic linear programming procedure. For this we must investigate what the effect on $p^w x$ is of a change in $\phi$ .

The linear programme is:

$$\min \quad \mu \ \iota$$

$$\text{S.T.} \quad \mu(H-\phi^{\bullet}A) + p^w(H-\phi^{\bullet}A) \geqq 0$$

and $\mu \geqq 0$ .

Denote by $\mu^*$ the optimum of the original problem.

a) When $\phi^{\bullet}$ decreases no element of $\mu^*$ increases: $\mu^*$ is the least element of the polyhedral set. When $\phi^{\bullet}$ decreases all constraints become ineffective so that in the new optimum the goal function will have a smaller value (if it was non-zero). This implies that some elements of $\mu$ must decrease. Assume that the first h elements decrease and the other increase. This is impossible because the original optimum is contained in the new set so a vector some elements of which are larger than in the original optimum cannot be a new optimum because it cannot be the least element (Theorem 2). q.e.d.

b) *if $\mu_i + \Delta\mu_i > 0$ then $\dfrac{\Delta X_i}{\Delta\phi} = 0$

*if $\mu_i + \Delta\mu_i = 0,\ \mu_i > 0$ then $\dfrac{\Delta X_i}{\Delta\phi} \geqq 0$

* $\mu_i = 0$ (then by a) also $\Delta\mu_i = 0$); now:

$$p_i^w \quad q_i = -\phi^{\bullet}\,\mu\,A_i + p^w(H- \phi^{\bullet}A)_i$$

From this we can derive

$$p_i^w \, \Delta x_i = (\Delta\phi)(\mu A_i + p^w A_i) + \phi \cdot \Delta\mu \, . A_i$$

$\Delta\phi$ is negative, $\Delta\mu$ is seminegative so that

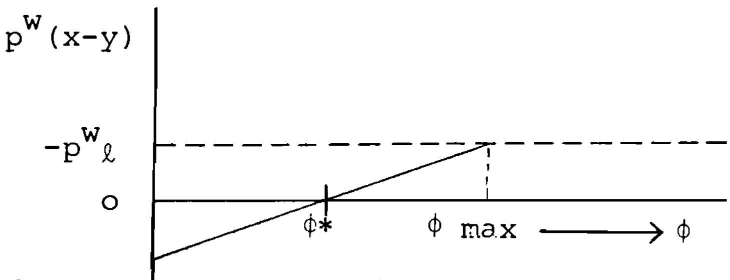
$p_i^w \; \Delta x_i$ is negative (for $p_i^w > 0$) .

A decrease in $\phi$ thus generates a decrease in $p^w x$. Two conclusions may be drawn:

a) Because we know that an equilibrium solution exists and because $p^w x$ is monotonously decreasing when $\phi$ decreases, only one equilibrium solution exists, which can be obtained from the linear programme. The linear complementarity problem could have had more solutions.[1)] The solution obtained is very attractive because it has the smallest value for all the components of the vector $\mu$ .

b) The equilibrium solution can quickly be computed by iterating over decreasing $\phi$ , starting at every step the linear programme from the optimal solution of the previous step, which is known to be feasible for the current step. The montonicity property also makes it possible to efficiently adapt $\phi$ .

c) Graphically the situation is as follows




for $\phi > \phi$ max the linear programme is infeasible.

$\phi^*$ is the equilibrium level. It is not clear whether it is below or above 1.

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1) This is not the case: $(H - \phi_{max} A^T)p^* = 0$ , $p^* > 0$ then $(H - \phi A^T)p^* > 0$ , $H_i > A_{ii}$ for $\phi < \phi$ max thus $(H - \phi A^T)$ has nonpositive off diagonal elements and positive diagonal elements and a positive inverse, so that it has Gale property (it is a Minkowsky matrix, see [3]). Following Samelson, Thrall, Wesler [12] the solution is then unique.

If for $\phi = 1$ a solution is feasible it is not necessarily the free trade solution, nor does it imply balance of trade equilibrium.

$$\phi = 1 \text{ implies } p^W (x-y) + \mu l = 0 \quad .$$

Clearly if $\mu = 0$ we have a free trade solution and balance of trade equilibrium is fulfilled.

If $l = 0$ we have the autarkic solution.

We finally note that $\phi$ max is the reciprocal of dominant eigenvalue of the matrix $AH^{-1}$, a semipositive indecomposable matrix.

3.2.3 Export quota and domestic price policy:
the model is now

1) consumer

$pX = \phi pA$

2) normalizations and definitions

$\nu = \bar{p} - p, \quad \bar{p} \geqq \nu \geqq 0$

3) policies

$X + S \geq K$

$\bar{p} = \bar{p}^*$ (given)

4) market equilibrium

$\nu\ (X+S) = \nu\ K$

$p.S = 0$

$S \geq 0$

5) balance of trade equilibrium

$p^W\ (X + S - A)\ \iota = 0$

6) assumptions on instruments

$p^W r \geq 0$

$y > r$

$\bar{p}^* > 0$

. S is the diagonal matrix of excess supply

. $K = \begin{bmatrix} y_i - r_i & & 0 \\ & \ddots & \\ 0 & & y_n - r_n \end{bmatrix}$

We first write down the linear complementary problem

define $q \equiv (X+S-K)\iota$ ; $\bar{P} = \begin{bmatrix} \bar{p}_1 & 0 \\ o & \bar{p}_n \end{bmatrix}$ , $s = S\iota$ .

Then the equations may be rewritten as

1. $q^T\bar{P} - \nu K = (\bar{p}A - \nu A) - \bar{p}K$

2. $\nu q = 0$

3.(a) $\nu \geqq 0$ , $q \geqq 0$

(b) $\nu \geqq \bar{p}$

4. $(\bar{p} - \nu) . S = 0$ , $S \geqq 0$ ,

5. $p^w q = p^w r$ .

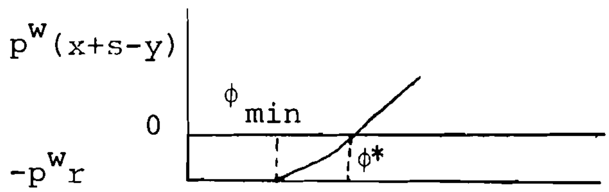
We note that s plays no role in the determination of $\nu$, $q$ . Consider 1, 2, and 3a, for a given value of $\phi$ . The matrix $(K - \phi A)$ again is a Z-matrix.

Consider

$$\begin{aligned} &\text{Min} \quad \nu\iota \\ &\text{S.T.} \quad \nu(K-\phi A) + \bar{p}A - \bar{p}K \geqq 0 \\ &\qquad\quad \nu \geqq 0 \ . \end{aligned}$$

This programme solves the linear complementary problem 1, 2, 3a. Does the solution satisfy 3 b? It does. To see this we first note that $\nu = \bar{p}$ is a feasible basis for the linear programme.

By Mangasarian's second theorem we know that the solution of the linear programme is the least element of the polyheder, so that $\nu^* \leqq \nu$ for all feasible $\nu$; thus $\nu^* \leqq \bar{p}$ .
The linear programme thus solves the equations 1 - 3 .
We can show in a similar way as was done for import quota that $p^w$ (S+X-Y) increases as increases.




We note that $\phi$ min is the reciprocal of the dominant eigenvalue of $AK^{-1}$ .

### 3.2.4 Import and export quota, domestic price policy

3.2.4.1 The model may be written as

1) consumer

$pX = \phi pA$

2) normalizations and definitions

$p = \bar{p} + \mu - \nu$

$p, \mu, \nu \geqq 0 \quad ; \qquad D = X + S$

3) policies

$D \leqq H$

$D \geqq K$

$\bar{p} = \bar{p}^*$

4) market equilibrium

$\mu D = \mu H$

$\nu D = \nu K$

$p.S = 0$

$S \geqq 0$

5) balance of trade equilibrium

$p^w (D-A) \iota = 0$

6) assumptions on the instruments

$p^w r \geqq 0, \quad p^w l \leq 0$

$y > r \geq l$

$\bar{p}^* > 0$

all the variables are defined as before.

Define $q_1 = (D - K)\iota$

$q_2 = (H - D)\iota$

$q_3 = (\bar{p} - \nu)^T$

we may write

$$q_1^T \bar{P} + \mu H - \nu K = \phi\bar{p}A + \phi(\mu - \nu)A - \bar{p}\, K$$

$$q_2^T \bar{p} - \mu H + \nu K = -\phi\bar{p}A - \phi(\mu - \nu)A + \bar{p}\, H$$

$$q_3^T = \bar{p} - \nu$$

$$\nu q_1 + \mu q_2 + s q_3 = 0$$

$$q_1, q_2, q_3, \mu, \nu, s \geqq 0 \quad .$$

The matrix is now:

$$M = \begin{bmatrix} K - \phi A & , & -K + \phi A & , & -I \\ -H + \phi A & , & H - \phi A & , & 0 \\ 0 & , & 0 & , & 0 \end{bmatrix}$$

Note that this is <u>not</u> a Z-matrix.

Define
$$\rho^T = (\nu, \mu, s)$$
$$n^T = \left[\left(\frac{\bar{P}0}{0I}\right)\right] [-\bar{p}(K-\phi A) \quad , \bar{p}(H-\phi A) \quad , \bar{p}]$$
$$N = \left[\left(\frac{\bar{P}0}{0I}\right)\right]^{-1} M^T$$
$$q^T = (q_1^T \quad , \quad q_2^T \quad , \quad q_3^T)$$

then the standard linear complementarity problem is

$$q = N\rho + n$$

$$\rho q = 0 \quad q, \rho \geqq 0 \quad .$$

We know that $\phi \min \leq \phi \leq \phi \max$.
Where $\phi \min$ is reciprocal of the dominant eigenvalue of $A K^{-1}$ ;
$\phi \max$ is reciprocal of the dominant eigenvalue of $A H^{-1}$ .

The present case cannot be reduced to any of the linear programming situations mentioned in Mangasarian's article. As a matter of fact we cannot formulate it as a quadratic programming case either.

We shall however show that Lemke's algorithm will converge for the present problem and develop an alternative algorithm. Before we do so some possible simplifications will be discussed.

3.2.4.2 <u>Economic interpretation of the domestic equilibrium with quota:</u> The cases with quota are characterised by the feature that if one quota constraint becomes effective a price change arises which may make other constraints effective. For this to occur at least one constraint must be effective at the original domestic prices. As soon as this is the case complex "resonance" patterns may arise. We thus do not know a prior which constraint will be effective and which will not. This is the reason for the complexity of the problem. In the linear complementarity case under consideration some cases however yield useful information.

1) It is trivial to remark that when $n \geq 0$ the L.C.P. has the solution $\rho = 0$. We further disregard this case.

2) When $-\bar{p}\,(K - \phi\,A) \geqq 0$ , the L.P. solution with import quota only also solves the problem with both import and export quota.

<u>Proof</u>

* Under import quota $p \geqq \bar{p}$ .

* $pX = \phi\ pA$ so that $\frac{\partial x_i}{\partial p_j} > 0$ for $i \neq j$ .

If an import quota is effective the export quota is not so that if no export quota was effective before imposition of import quota no one will be after q.e.d.

3) When $\bar{p}\,(H - \phi\,A) \geqq 0$ the L.P. solution with export quota only also solves the problem with both import and export quota.

4) An export constraint cannot become effective if $(K_i - \phi A_i) \leq 0$. As a consequence such constraints may be dropped from the linear complementarity problem.

<u>Proof</u>: the $i^{th}$ constraint is

$q_i = (K_i - \phi A_i^T)(\nu^T - \bar{p}^T) + \phi\,A_i^T\,\mu^T$ .

If $(K_i - \phi A_i) \leq 0$ then for all $0 \leqq \nu \leqq \bar{p}$, $\mu \geqq 0$ we get $q_i > 0$ , because $A_i$ is a semipositive vector. We shall assume that the L.C.P. only contains export quota which may become effective.

We rewrite the L.C.P. after deleting n-m constraints:

$$M^T = \begin{bmatrix} (\bar{K} - \phi\bar{A}^T) & , -(\bar{H} - \phi\bar{A}^T) & ,-I \\ (-\tilde{K} - \phi\tilde{A}^T) & (H - \phi A^T) & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \bar{K}, \bar{A}: \; m \times m, \quad m \leq n$$

$$\text{dimensions:} \begin{bmatrix} m \times m & m \times n & m \times m \\ n \times m & n \times n & n \times m \\ m \times m & m \times n & n \times m \end{bmatrix}$$

The matrices $\bar{A}$, $\bar{K}$ are principal minors of A, K respectively (relevant rows and columns deleted) while in the matrices $\tilde{A}$ and $\tilde{K}$ the relevant columns have been deleted but the relevant rows have been replaced by zero's.

3.2.4.3 Convergence proof of Lemke's algorithm

We shall not discuss here Lemke's algorithm. For reference the reader should consult Cottle [2] or Cottle and Dantzig [3] . We essentially follow the development by Cottle and Dantzig. We know that the L.C.P. has an equilibrium solution for $\phi\,\text{min} \leqq \phi \leqq \phi\,\text{max}$. If Lemke's algorithm does not converge it then must terminate in in a ray. (Cottle-Dantzig theorem 1, corollary.)

For this to occur there must exist a non-negative vector $\rho$ such that $\rho_i(M\rho)_i \leqq 0$ (Cottle-Dantzig theorem 4) with $M\rho_i < 0$ if $\rho_i > 0$ .

To proof that this cannot occur we proceed in three stages:

1) In a ray we must have $\mu_i\nu_i = 0$ when defined. We drop transpose signs for $\nu$ and $\mu$ .

$$M\rho = \begin{bmatrix} (\bar{K} - \phi\bar{A}^T)\nu - (\bar{H} - \phi\bar{A}^T)\mu \\ -(\tilde{K} - \phi\tilde{A}^T)\nu + (H - \phi A^T)\mu \\ -\nu \end{bmatrix}$$

assume $\mu_i \cdot \nu_i \neq 0$

then for a ray we must have

$$(K_i - \phi A_i^T)\nu - (H_i - \phi A_i^T)\mu < 0$$

and

$$-(K_i - \phi A_i^T)\nu + (H_i - \phi A_i^T)\mu < 0$$

this is impossible.

2) Assume that (only) the first h import constraints have positive $\mu_i$ and that the algorithm ends in a ray. Then:

$$\mu_i[(H_i - \phi A_i^T)\mu + \phi A_i^T \nu] \leqq 0 \quad , \quad i = 1..h$$

In matrix form for $\mu_i > 0$

$$(\bar{\bar{H}} - \phi \bar{\bar{A}}^T)\mu + c \leqq 0 \qquad \text{with } c \geqq 0 \quad .$$

The matrix $\bar{\bar{H}} - \phi\bar{\bar{A}}^T$ however has negative off diagonal elements and positive inverse so that it has the so called Gale property. (It is a so called Minkowsky matrix.) Therefore the system has no other solution than $\mu = 0$ .

3) Given the fact that in a ray $\mu = 0$ we must have in a ray:

$$\nu_i[K_i - \phi A_i^T)\nu \leqq 0 \quad .$$

This would imply that a principal minor of $(K - \phi A)$ ; $(\bar{K} - \phi\bar{A}^T)$ should have a solution.

$$(\bar{K} - \phi\bar{A}^T)\bar{\nu} \leqq 0 \quad , \quad \bar{\nu} > 0$$

but

$$(\bar{K} - \phi\bar{A}^T)\bar{\nu} = (\bar{K}' - \phi\bar{A}^T)\nu$$

Where $\bar{K}'$ is formed by the full rows of K corresponding to $\bar{K}$ and $\bar{A}'^T$ is formed by the full rows of $A^T$ corresponding to $\bar{A}^T$
The quota constraints imply:

$$(\bar{K}' - \phi\bar{A}'^T)\nu \geqq (\bar{K}' - \phi\bar{A}'^T)\bar{p}$$

so that

$$-(\bar{K}' - \phi\bar{A}'^T)\bar{p} \geqq 0 \quad .$$

For the other rows, that is for the rows for which $\nu_i = 0$ the constraints imply

$$-(K_i - \phi A_i^T)\bar{p} \geqq \phi A_i^T \nu \geqq 0 \quad .$$

Hence a ray would imply

$-(K - \phi A^T)\bar{p} \geqq 0$ this is in contradiction with our assumptions so that Lemke's algorithm must converge.

### 3.2.4.4 The parametric complementarity problem: from an almost - complementarity algorithm to a complementarity algorithm.

Up to this point we have assumed that $\phi$ is given. This parameter must however be adapted in such a way that the balance of trade equation is satisfied. One way to do this would be to solve a series of complementarity problems through Lemke's algorithm. This is however quite expensive if only because Lemke's algorithm cannot take very much a priori information into account. Moreover we did not yet prove the uniqueness of the solution so that a parametric approach might cause serious problems in terms of continuity.

We shall therefore develop a new algorithm very strongly inspired by Lemke's algorithm which operates along a fully complementary path and which has only one driving variable: $\phi$. In this paper only a sketch of the algorithm will be presented. It is based on two features.

1) To $X = K$ (all export quota effective) and to $X = H$ (all import quota effective) there corresponds only one nonzero price vector (which is determined up to a scalar) and only one value $\phi$, the dominant eigenvector and receprocal eigenvalue of the matrices $AK^{-1}$ and $AH^{-1}$ respectively.

Moreover to that given value of $\phi$ corresponds only one X and one p, this may be seen as follows. We know that the only semipositive solution of $\phi^* pAH^{-1} \geqq p$ is $p = p^*$ (dominant eigenvector) and $p^* > 0$ ;
thus the only nonnegative solution of $\phi^* pA \leqq pH$ is $p^*$
but $p^* X = \phi^* p^*A = p^*H$ . (The same holds for K.)

2) The L.C.P. has thus a unique[1] solution for $\phi = \phi$ max and $\phi = \phi$ min which is relatively simple to compute. The idea to be sketched below is that starting for example from $X = H$, $\phi = \phi$ max , a pivoting algorithm along a complementary path with decreasing $\phi$, will yield a decreasing value for $p^w x$ until $p^w x = p^w y$ i.e. until balance of trade equilibrium is reached. Before we further develop the algorithm we give an economic interpretation.

---

[1] If we disregard the trivial solution $p = 0$ .

* <u>Economic interpretation:</u> An increase in $\phi$ implies that the ratio of the value of expenditures to the value of receipts (in domestic prices) increases through a decrease in the level of taxation. At constant prices this certainly implies an increase in all demands so that the balance of trade reacts accordingly. However, prices change through the changed pressures of demand. Can they "overreact" in such a way that demand decreases? An intuitive answer would be negative. We shall show that intuition is right.

There are three types of commodities to be distinguished

1) Commodities, which have the same active quota constraints before and after the change in $\phi$ . These commodities do not directly influence the balance of trade as their demand is constant. They, however, exert an influence on the demand for other commodities, through possible price changes.

2) Commodities which were unconstrained by quota and remain so.

3) Other Commodities

* <u>The non-switching case</u>.

We shall first neglect the third category and assume that a change in $\phi$ does not generate any "switches". The demand system may then be written as:

$$(p_1, p_2)\left\{\begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} - \phi \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\right\} = 0$$

At a given $(X_1, X_2)$ , $(p_1 , p_2)$ is the dominant eigenvector, $1/\phi$ the dominant eigenvalue. Suppose $p_2$ and $X_1$ do not change when $\phi$ changes.

<u>Consider a decrease in $\phi$ from $\phi$ to $\phi^*$</u>

The demand system after the change may then be written as

$$(p_1^* , p_2 ) \begin{bmatrix} 0 & 0 \\ 0 & \Delta X_2 \end{bmatrix} + (p_1^* , p_2) \left\{\begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} - \phi^* \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\right\} = 0$$

Define: $M = [X - \phi^* A]$

$T = [X - \phi^* A]^{-1}$ .

$\phi^* < \phi$ so that T is known, by the Perron-Frobenius theorem to be strictly positive.

We can now write

$$(p_1^*\ ,\ p_2)\begin{bmatrix}0 & 0\\ 0 & \Delta x_2\end{bmatrix}\begin{bmatrix}T_{11} & T_{12}\\ T_{21} & T_{22}\end{bmatrix} + (p_1^*\ ,\ p_2) = 0\ .$$

This may be written as

$$p_2\ \Delta X_2\ T_{21} + p_1^* = 0$$

$$p_2\ \Delta X_2\ T_{22} + p_2 = 0$$

$$-p_2\ \Delta X_2 = p_2\ T_{22}^{-1} = p_2[M_{22} - M_{21}M_{11}^{-1}\ M_{12}] > 0$$

(by Frobenius)

We may conclude that the diagonal matrix $\Delta\ X_2$ has negative diagonal, so that the demand for the commodities of the second category will in the present (non-switching) situation decrease. Thus in the non-switching case

$$\frac{\Delta X_i}{\Delta\phi}\begin{cases}> 0 & \text{for } i\ \varepsilon \text{ group 1}\\ = 0 & \text{for } i\ \varepsilon \text{ group 2.}\end{cases}$$

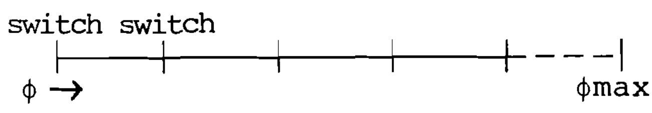
As long as not all the commodities of group 1 happen to have a zero price on the world market we have

$$\frac{\Delta p_X^W}{\Delta\phi} > 0 \text{ as expected}\ .$$

* Switches: we need to prove that in the third group of commodities:

$$\frac{\Delta X_i}{\Delta\phi} \geqq 0\ .$$

If we decompose a change in $\phi$ into small components we get segments along which no switch occurs and nodes at which switches occur.




- Because along the segments we have

$$\frac{\Delta X_i}{\Delta\phi} \geq 0\ ,$$

we can eliminate the case that with decreasing $\phi$ new import quota become effective.

- The only possibility for $\frac{\Delta X_i}{\Delta\phi} < 0$ is then that an existing export quota becomes ineffective. This could have resonance effects on the other constraints. But if we can exclude this possibility the resulting resonance will not occur either.

We show this by proving that along the segments no price increase occurs when $\phi$ is decreasing.
Define $M = T^{-1}$ with corresponding decomposition. We know that $p_2 \Delta X_2 = - p_2 T_{22}^{-1}$ .

Substitution in the first set of equations yields:

$$- p_2 T_{22}^{-1} T_{21} + p_1 = - \Delta p_1 \quad .$$

From the partitioned inverse we know that

$$- T_{22}^{-1} T_{21} = M_{21} M_{11}^{-1} \quad .$$

We know that

$$p_1 M_{11} + p_2 M_{21} > 0 \quad , \quad M_{11}^{-1} > 0$$

so that $\quad p_1 + p_2 M_{21} M_{11}^{-1} > 0$

$$\rightarrow \Delta p_1 < 0 \quad \text{q.e.d.}$$

We therefore know that $p^w(x - y)$ will react monotonously to a change in $\phi$, although $\frac{\Delta p^w(x - y)}{\Delta \phi} = 0$ is possible (see below).

Our argument however started along a segment and not in the node $\phi$ max. We therefore need to produce the first segment on the left and side of $\phi$ max.

This is easy because the price vector in $\phi$ max is only determined up to a scalar.

Let the desired domestic price be $\bar{p}$ and the eigenvector in $\phi_{max}$ $p^* = \lambda a^*$ . Let $\lambda$ be such that $p^* \geq \bar{p}$ with one equality (no special complications occur if there are more equalities).
Let the $i^{th}$ component have $p_i^* = \bar{p}_i$ . Any decrease in $\phi$ will then make $p_i < \bar{p}_i$ for $X = H$ [1]. Perform a complementarity pivot for the $i^{th}$ import constraint: Set the price $p_i = \bar{p}_i$ and let $X_i$ "free". We then have the first segment; decrease $\phi$ until it is "blocked" by $p_j < \bar{p}_j$ or $X_h < K_h$ perform a complementary pivot etc.
Stop when $|p^w(x - y)| < \varepsilon$ .

Because of the monotonicity result proved above the algorithm will converge. We also have proved the uniqueness of the equilibrium (disregarding the trivial one with p = 0), because under a change in $\phi$ there was a unique path from a unique solution.

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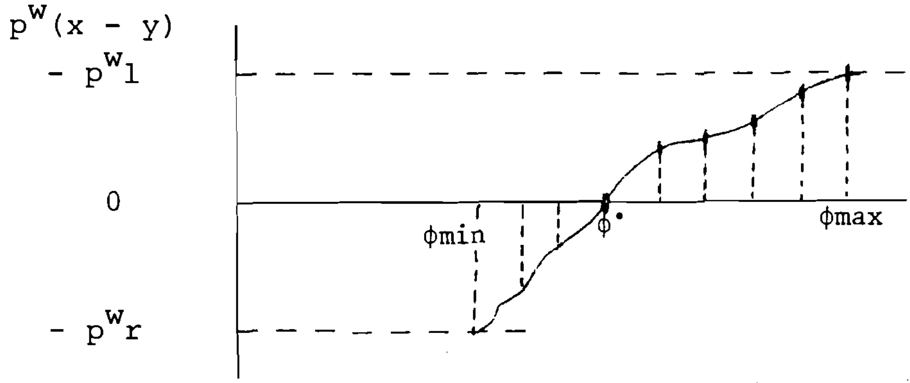
1) This is independent from the choice of $\lambda$

In terms of the graph we have




The vertical lines indicate switches.[1)]

* The algorithm has not yet been programmed.
* It is not clear whether it will be more efficient in the one sided case than the L.P. method. It anyhow also applies to that case. The algorithm then can be started without the computation of an eigenvalue. To see this consider the case of export quota only. Start the algorithm for a large value of $\phi$ with all prices set at $p = \bar{p}$ and all quantities variable, then decrease $\phi$ etc.
* When the world market prices are adapted in order to reach world market equilibrium the algorithm can be used in a parametric way:

a) If all commodities are subject to a domestic price policy the change in world market prices only affects $\phi^{*}$ so that the old equilibrium is a feasible starting point.

b) If some commodities are "unprotected" it may be necessary to repeat the whole complementary pivoting part. The starting eigenvalues and eigenvectors need however not to be computed again.

### 3.2.5 Alternative taxation policies

#### 3.2.5.1 Constant share in taxes by income class

##### (i) No price policy, import quota only

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1) The line might have "flat" parts. This will be the case if the commodities which are not fixed in quantity would have zero price on the world market. If such a flat would occur at $p^w(x-y)=0$ there would be multiple equilibria. In this very improbable case we choose the largest possible value of $\phi$ so that the algorithm remains unchanged.

The consumer model for this case is:

$$p_i \, x_{ij} = e_{ij} \, m_j$$

$$m_j = py^j + \alpha_j t \quad .$$

In equilibrium $\mu H = \mu X$

$$t = - \mu 1 \quad .$$

Substituting this in the demand equations one gets

$$\mu H + p^w X = \mu (Y^T - 1\alpha^T) E^T + p^w Y^T E^T$$

where $Y = [Y_{ji}]$ matrix of endowments i owned by j; $E = [e_{ij}]$

define $M = H - (Y^T - 1\alpha^T) E^T$

$$P^W = \begin{bmatrix} p_1^w & & 0 \\ & \ddots & \\ 0 & & p_n^w \end{bmatrix}$$

$$N^T = MP^{w-1}$$

$$n^T = p^w (H - Y^T E^T) \, P^{w-1}$$

$$q = (H - X) \iota$$

$$\rho = \mu^T \quad .$$

Then we may write the model as a linear complementarity problem

$$q = N\rho + n$$

$$\rho^T q = 0 \quad ; \quad \rho, q \geqq 0 \quad .$$

If we assume that $(Y^T - 1\alpha^T) E^T$ is nonnegative then N has the useful property that it has nonpositive off diagonal elements so that it is a so called Z-matrix.

Note that $M\iota = 0$ so that the balance of trade equilibrium is automatically satisfied by the demand equations. We now make use of the theorem by Mangasarian [10] according to which the present linear complementarity problem can be solved by a single linear programme.

$$\min \quad c^T \rho$$

$$N\rho + n \geqq 0$$

$$\rho \geqq 0 \quad .$$

Note that this problem can be solved in one step, without iterations over a parameter $\phi$ .

(ii) Export quota

The same holds for export quota except that now the matrix $(Y^T - r\alpha^T)E^T$ must be nonnegative.

(iii) Domestic price policy, import and export quota

For this case we develop our own algorithm, as the convergence of Lemke's algorithm is doubtful and a simple alternative procedure is available.

The central equation is

$$(I - E\alpha\iota^T)\ Xp = E(I - \alpha\iota^T)Yp + \lambda e \quad .$$

Where $p$ is the column vector of prices.

$e = E\alpha$

$\lambda = p^w(x - y)$

$X$ is the diagonal matrix of aggregate demand $(n \times n)$

$Y$ is the matrix of endowments by income class $(m \times n)$.

Define $A = (I - E\ \alpha\iota^T)X - E(I - \alpha\iota^T)Y$

$b = \lambda e$ .

Take $X$ as given.

The set of equations $Ap = b$ is consistent if and only if $y^Tb = 0$ where $y$ is the solution of $y^TA = 0$ .

Note that $\iota^TA = 0$ so that the set of equations will only be consistent for $\lambda = 0$ .

The algorithm proceeds as follows:

1) Set $X = H$
2) Determine the solution $Ap = 0$ . This solution is positive and has only one degree of freedom if a reasonable assumption is made, as will be shown below.

   It may thus be written as $p = \rho p^*$ .
3) Determine $\rho$ such that $p \geqq \bar{p}$ with one equality say the th commodity.
4) Effectuate a pivot for the $i^{th}$ commodity.

   This is done as follows

define

$$C = E(I - \alpha\iota^T)Y$$
$$D = I - E\ \alpha\iota^T \quad .$$

Write the central equation in partitioned form

$$(- C_{11} + D_{11}\ X_1)p_1 + (- C_{12} + D_{12}\ X_2)p_2 = 0$$
$$(- C_{21} + D_{21}\ X_1)p_1 + (- C_{22} + D_{22}\ X_2)p_2 = 0 \quad .$$

Let commodity group 1 have effective quota constraints ($X_1$ is given, $p_1$ unknown) and group 2 ineffective constraints ($p_2$ is given $X_2$ unknown). Indicating the constant variables by bars and writing again matrices in capital letters, the system can be rewritten as

$$(- C_{11} + D_{11} \bar{X}_1) p_1 + D_{12} \bar{P}_2 x_2 = C_{12} \bar{P}_2$$
$$(- C_{21} + D_{21} \bar{X}_1) p_1 + D_{22} \bar{P}_2 x_2 = C_{22} \bar{P}_2 \quad .$$

Define

$$C^k = [C_1, 0] \quad , \; y^k = \begin{pmatrix} p_1 \\ x_2 \end{pmatrix} \; , \quad A^k = D \begin{pmatrix} \bar{X}_1 & 0 \\ 0 & \bar{P}_2 \end{pmatrix} - C^k \quad ;$$

$$b^k = C_2 \bar{P}_2 \quad .$$

We have the nonhomogeneous system:

$$A^k y^k = b^k \quad .$$

Note that $\iota^T A^k = \iota^T b^k = 0$ .

The system has a known particular solution $y_p^k$ , where for $k = 0$: $y_{ph}^o = p_h$ , $h = 1, .. n-1$

$$y_{pn}^o = H_n$$

more in general $y_p^k$ is the blocked value of the previous step.

5) Compute the general solution of $A^k y^k = b^k$ ;

$$y^k = \rho y_*^k$$

The solution space is then

$$y^k = y_p^k + \rho y_*^k$$

with $y_p^k > 0$ and $y_*^k > 0$ (to be proved below).
Let $Y_b^k$ be vector of blocking values for $y^k$ ($y^k \geq y_b^k$) .

6) Decrease $\rho$ , starting from $\rho = 0$ . If balance of trade equilibrium is reached stop, else decrease $\rho$ until $y^k$ is blocked by $y_b^k$ .
7) Go to 3) .

The convergence of the algorithm is obvious, as

$$\frac{\Delta x}{\Delta \rho} \geqq 0 \quad , \; \frac{\Delta p}{\Delta \rho} \geqq 0 \quad , \text{ by construction.}$$

Again $p^W(x - y)$ as a function of $\rho$ may show "flat" segments.

We still have to prove, that under a certain assumption $A^k y^k = b^k$ has a positive general solution determined up to one scalar.

Define

$$F^k = - A^k \begin{bmatrix} \bar{X}_1^k & 0 \\ 0 & \bar{P}_2^k \end{bmatrix}^{-1} + I$$

$$= - (D - C^k \begin{pmatrix} X_1^k & 0 \\ 0 & \bar{P}_2^k \end{pmatrix}^{-1} ) + I$$

if $F^k > 0$ then by the Frobenius theorem there is a dominant eigenvalue and corresponding unique and positive eigenvector. (that is a vector determined up to one scalar). Moreover $\iota^T F^k = \iota^T$ so that this dominant eigenvector is the solution of $(I - F^k)z = 0$ and the positiveness of $y^k$ follows directly. $F^k > 0$ means that

$$C_1 \bar{X}_1^{-1} + D_1 - I > 0 \quad .$$

A sufficient condition for this is that

$$E(I - \alpha\iota^T)Y + E\alpha\iota^T\bar{X} > 0$$

$$\bar{X} \geqq K \quad ,$$

so that the condition is satisfied if

$$E(I - \alpha\iota^T)Y + E\alpha\iota^T K > 0$$

which is the condition we have already found in the one side case $\quad E\ (Y - \alpha r^T) > 0 \quad .$

The condition is sufficient, not necessary, on the other hand $E(Y - \alpha l^T) > 0$ would be necessary but not sufficient. The convergence conditions thus lie in between the conditions for the one-sided problems when L.P. is applied.

3.2.5.2 Constant distribution of income

(i) Import quota

$$m_j = \beta_j(py + t)$$

$$= \beta_j(\mu H\iota + p^w y) \quad .$$

Define

$$\gamma^T = \beta^T E$$

$$\delta^T = \gamma^T \cdot p^w y \quad .$$

Then we may write

$$\mu H + p^w X = \mu H \iota\ \gamma^T + \delta^T \quad ;$$

define

$$M = H(I - \iota\gamma^T)$$

$$n^T = p^w (H - y \cdot \gamma^T) p^{w-1}$$

Again the corresponding matrix $N = p^{w-1}\ M^T$ has nonpositive diagonal elements so that the solution with the linear programme is available.
Again $M\iota = H\ \ (\iota(1 - \gamma^T\iota)) = 0$ as $\gamma^T\iota = 1$ so that balance of trade equilibrium is realized. The export case is analogous.

(ii) The case of domestic price policy

$$x^T \cdot \bar{P}[(I - \iota\gamma^T) + \bar{P}^{-1} P^w \iota\gamma^T] = p^w y \cdot \gamma^T \quad .$$

The matrix on the left hand side will always be singular.

<u>Proof</u>

Define $\quad \pi^T = (\frac{p_1^w}{\bar{p}_1}, \ldots, \frac{p_n^w}{\bar{p}_n})$

and

$$Q = [I - (\iota - \pi)\gamma^T] \quad .$$

If Q is singular then $\exists\ \lambda \neq 0$ such that

$$\lambda - (\iota - \pi)\gamma^T\lambda = 0 \quad .$$

Premultiply by $\gamma^T$, then remembering that $\gamma^T\iota = 1$ :
$\gamma^T\pi \cdot \gamma^T\lambda = 0$ which is possible q.e.d.
Thus the linear complementarity methods cannot be applied in a straightforward way when domestic price policy is involved. Moreover, when both import and export quota are involved the usual convergence conditions for Lemke's algorithm are not satisfied. We are not able to prove that it cannot end in a ray.

(iii) A pivoting algorithm for the case with import and export quota and domestic price policy

We therefore develop an alternative algorithm of the same type as sketched before.

We know that in equilibrium of the balance of trade

$$p^T X \ (I - \iota\gamma^T) = 0$$

the matrix $(I - \iota\gamma^T)$ is singular, so $p^T X$ is determined up to a scalar and is positive.

$$\begin{cases} p^T X = \lambda b \\ K \leqq X \leqq H \end{cases} .$$

Find $\lambda, p$ such that $p^w X\iota = p^w y$ , and such that the domestic market is in equilibrium.
The algorithm proceeds as follows (the convergence is obvious)

1) set $X = H$
2) determine $\lambda$, p such that
$p^T H = \lambda b$ , and $p \geq \bar{p}$ with one equality say for the $i^{th}$ commodity.
3) Set $p_i = \bar{p}_i$
4) Decrease $\lambda$ until it is blocked by an export constraint $X_j \geqq K_j$ or by $p_h \geqq \bar{p}_h$ .

* In the first case set $X_j = K_j$ and do not further consider this commodity (except in (5)).
* In the second case set $p_k = \bar{p}_k$ .

5) If $|p^w(y - x)| > \varepsilon$ go to (4) .

The algorithm has the advantage that the pivoting does not involve any matrix inversion, and only one solution of a simultaneous set of equations.

## 3.3 Some further results on the computation of domestic equilibrium under tariffs and quota, in a pure exchange economy

In this section no new algorithm will be developed. It will only be investigated in which cases the algorithms of section 3.2, or a convex programming algorithm can be used to compute domestic equilibrium.

### 3.3.1 Hicksian pure exchange economy with quota

Consider the optimization:

$$\max \quad u(x^j)$$

$$\text{S.T.} \quad px^j = py^j + \alpha_j tr$$

The first order conditions are:

$$\frac{\partial u}{\partial x_i} = \lambda_j p_i$$

$$px^j = py^j + \alpha_j tr$$

$u(x^j)$ is assumed to be homothetic then (cf p. 46);

$$\frac{p_i}{p_n} = \frac{\frac{\partial u}{\partial x_{ij}}}{\frac{\partial u}{\partial x_{nj}}} = \frac{\frac{\partial u}{\partial x_i}}{\frac{\partial u}{\partial x_n}}$$

We may thus consider the case

1) Consumer:

$$\max \quad u(x)$$

$$\text{S.T.} \quad px = py + tr$$

2) Government:

$$tr = (p^w - p)(y - x)$$

$$l \leqq y - x \leqq r$$

3) Market equilibrium

$$p = p^w + \mu - \nu$$

$$\mu(y - x) = \mu l$$

$$\nu(y - x) = \nu r$$

$$\mu, \nu, p \geqq 0$$

4) Balance of trade:

$$p^w x = p^w y$$

We show that this model is equivalent with:

max $\quad u(x)$

S.T. $\quad p^w y = p^w x$

and $\quad l \leqq y - x \leqq r$ .

Define again $h = y - l$ ; $k = y - r$

The Lagrangean is:

$$L = u(x) + \lambda\left[p^w(y - x) + \mu(h - x) + \nu(x - k)\right]$$

$\lambda$ has been taken out of brackets because we assume that it is positive. This will be the case as long as

$\frac{\partial u}{\partial x_i} > 0$ for all commodities entering the utility function. This condition also guarantees $p_i^w - \nu_i > 0$ for these commodities so that no slack variable for the export constraint needs to be introduced. Weaker assumptions would be possible but will be investigated when needed.

Note that the uniqueness of equilibrium is obvious as long as $y > 0$ .

Among the first order condition we find:

$$\frac{\partial u}{\partial x_i} = \lambda p_i$$

$$l \leqq y - x \leqq r$$

$$p^w y = p^w x$$

$$\mu(y - x) = \mu l$$

$$\nu(y - x) = \nu r$$

We know that $\frac{\partial u}{\partial x_i} > 0$ , $\forall x$ so that $p \geqq 0$ will also be satisfied q.e.d.

A gradient algorithm can solve the maximization problem. It has however to be considered in which case a complementary pivoting algorithm of the type derived before can also be used.

### 3.3.2 Domestic equilibrium in a Hicksian economy with domestic price policy, quota and a CES utility function

$$u = \left(\sum \delta_i x_i^{-\rho}\right)^{-\frac{1}{\rho}}$$

then

$$\frac{\partial u}{\partial q_i} = -\rho \frac{\partial u}{\partial u^{-\rho}} \delta_i x_i^{-(\rho+1)}$$

$$= u^{\rho + 1} \delta_i x_i^{-(\rho + 1)}$$

$$\ln \frac{p_i}{p_n} = -(\rho + 1) \ln\left(\frac{x_i}{x_n}\right) + \ln\left(\frac{\delta_i}{\delta_n}\right)$$

Due to the Hicksian character of the economy the taxation system is irrelevant for the determination of prices and aggregate demand. We therefore set up the following algorithm:

1) Set $x = h$ (all import quota effective)
2) Determine the relative prices from:

$$\frac{\partial u}{\partial x_i} = \lambda p_i \qquad (x = h)$$

3) Set p in such a way that $p \geqq \bar{p}$ with one equality say for the $h^{th}$ commodity. Consider the set of equations:

for $i = 1,\ldots,h-1,h+1,\ldots,n$

$$\ln p_i - \ln p_h = -(\rho + 1)(\ln x_i - \ln x_h) + \ln\left(\frac{\delta_i}{\delta_h}\right)$$

$\bar{p}_h$ is constant, $\ln(x_h)$ is variable.

For a given $\ln(x_h)$ we have $(n - 1)$ linear equations in $n - 1$ unknows.

4) $\ln(x_h)$ is the driving variable. (1) Decrease it until $p_i$ or $x_i$ blocked.

5) Pivot (this only involves setting $p_i = \bar{p}_i$ and $x_i$ free, no matrix inversion or the like is involved).

6) Stop the algorithm when $p^w x = p^w y$ .

Convergence conditions:

if $x_i$ constant: $\ln p_i = \left[-(\rho + 1)\ln\bar{x}_i + \ln\bar{p}_h + \ln\frac{\delta_i}{\delta_h}\right] + (\rho + 1)\ln x_h$

if $p_i$ constant $\ln x_i = \left[\ln\bar{p}_h - \ln\bar{p}_i + \ln\frac{\delta_i}{\delta_h}\right]/(\rho + 1) + \ln x_h$

As long as $\rho + 1 > 0$ the algorithm will certainly converge. This is equivalent to the condition that the elasticity of substitution

$$\sigma = \frac{-1}{1+\rho}$$

must be negative, a totally acceptable assumption.

3.3.3 <u>Generalized CES</u>

The algorithm is obviously unchanged if the Mukerji-Dhrymes-Kurz function [ 7 ] is used, provided the convergence conditions are satisfied.

$$u = \Sigma\left(\alpha_i x_i^{\,b_i}\right)\frac{1}{\rho}$$

(1) Any free quantity or price may be chosen.

$$\frac{\partial u}{\partial x_i} = a_i b_i x_i^{b_i - 1} \cdot \frac{du}{du\frac{1}{\rho}} = \lambda p_i$$

thus $$\frac{a_i\ b_i\ x_i^{b_i-1}}{a_h\ b_h\ x_h^{b_h-1}} = \frac{p_i}{p_h}$$

convergence conditions:

$$\ln p_i = \ln \frac{a_i b_i}{a_h b_h} + \ln p_h + (b_i - 1) \ln x_i - (b_h - 1) \ln x_h$$

$$\ln x_i = \ln\left(\frac{a_i b_i}{a_h b_h} + \ln\bar{p}_h - \ln\bar{p}_i\right) / (b_i - 1) + \left(\frac{b_h - 1}{b_i - 1} \ln x_h\right)$$

It is sufficient for convergence that $\frac{b_h - 1}{b_i - 1} > 0$ .

Further generalizations may be possible but will be looked for when they are needed. Actually it seems that the principle of the algorithm would apply in most cases where we can show direct computability. The algorithm is however the most useful when not only domestic prices can be computed easily at given domestic demand, but when also complementarity pivots can easily be computed, essentially a sort of combined primal dual demand functions is needed, otherwise the maximization problem would seem more efficient.

### 3.3.4 Pure exchange economy where the commodities with quota form a linear expenditure subsystem

Consider:

max $\quad u_j(x^j)$

S.T. $\quad px^j = \phi p y^j$

$$u_j = u_{j1}^{\beta_j}\ u_{j2}^{1-\beta_j}$$

and $$u_{j1} = \prod_{i=1}^{k} x_{ij}^{\alpha_{ij}}$$

$$u_{j2} = u_{j2}\ (x_{ij}),\ i = k + 1, \ldots n$$

The first order conditions are:

$$\frac{\partial u}{\partial x_{ij}} = \lambda_j p_i$$

$$= \frac{\partial u}{\partial u_{j1}} \frac{\partial u_{j1}}{\partial x_{ij}} = \lambda_j p_i \quad ; \quad i = 1,\ldots,k$$

$$= \beta_j \alpha_{ij} \frac{u^j}{x_{ij}} = \lambda_j p_i$$

$$\phi \; \beta_j \alpha_{ij} \sum_{h=1}^{n} p_h \, y_{hj} = p_i x_{ij}$$

define $A_{hi} = \sum_j \beta_j y_{hj} \alpha_{ij} \quad ; \; h,i = 1,..,k$

$$X_1 = \left[\sum_j x_{ij}\right]$$

$$p_1 = (p_1,\ldots,p_k)$$

$$b = \sum_j \sum_{h=k+1}^{n} \bar{p}_h \, y_{hj} \alpha_{ij} \beta_j \quad ( > 0)$$

then;

$$p_1 X_1 = \phi p_1 A + \phi b \quad .$$

The complementary pivoting algorithm for the L.E.S. case applies are. The initial $p_1$ must however be calculated as a generalized eigenvector. This yields a procedure to compute domestic equilibrium for a pure exchange economy in which the commodities with quota form a linear expenditure subsystem.

3.3.5 <u>Hicksian pure exchange economy with domestic price policy and quota</u>

The complementary pivoting algorithm is only applicable for a specific type of utility function. We now describe a parametric convex programming algorithm for the general case. Consider the following maximization problem

$$\begin{array}{l|l} \max & u(x) \\ \text{S.T.} & \bar{p}\, x = \bar{p} y + \bar{t} \\ & l \leqq y - x \leqq r \\ & x \geqq 0 \end{array}$$

For all $\bar{t} \varepsilon T = \left\{ \bar{t} \mid \bar{p}\, l \leqq - \bar{t} \leqq \bar{p}\, r \right\}$ , l, r finite, and for any strictly quasi concave utility function this problem is a feasible convex programming problem so that it has a unique solution.

Moreover, provided $p^w r \geqq 0$ , we know from section 1 that there will exist a $\bar{t} = \bar{t}^*$ such that $p^w(x - y) \leqq 0$ . The situation is analogous to the one occurring in the complementary pivoting scheme: $\bar{t}$ is the driving variable which is adjusted until the balance of trade constraint is satisfied (preferably with equality). It is again the monotonicity of $p^w(x - y)$ as a function of $\bar{t}$ which would (at positive world market prices) guarantee the uniqueness of the domestic equilibrium. Whether this condition is fulfilled depends however on the specification of the utility function.

Anyhow, in equilibrium we find for this algorithm.

$$\frac{\partial u}{\partial x_i} = \lambda \bar{p}_i + \mu'_i - \nu'_i$$

$$\bar{p}\, x = \bar{p}\, y + \bar{t}$$

$$p^w(x - y) \leqq 0$$

$$x,\ \lambda,\ \mu,\ \nu \geqq 0$$

$$l \leqq y - x \leqq r$$

$$\mu(y - x - l) = 0$$

$$\nu(y - x - r) = 0$$

this coincides with the market equilibrium conditions. Introduction of a technology: $g(y) \leqq 0$ within the constraints of the maximization problem yields the solution for a Hicksian economy with production.

### 3.3.6 Domestic price policy, quota and stock policy in a pure exchange economy with L.E.S.

When the government operates a stock policy it tries to maintain a certain desired stock $(\bar{s})$.

If however the government also subjects the $i^{th}$ commodity to a domestic price policy and to import and export quota, the stock policy may also be used in order to more or less maintain a desired domestic price $(\bar{p}_i)$ while quota constraints are binding. In this case the domestic price policy may overrule the stock policy in a similar way as the quota policy overrules the domestic price policy. We present the model for this case.

1) Consumer: $pX = \phi pA$

2) Definitions:
$$p = \bar{p} + \mu - \nu, \quad p \geqq 0$$
$$s = \bar{s} + q$$
$$y = y^g + \sum_{j=1}^{m} y^j$$
$$A = \left[a_{ih}\right] = \left[\sum_{j=1}^{m} y_{hj} \quad e_{ji}\right]$$

3) Government:
$$l \leqq y - x - s \leqq r$$
$$0 \leqq s_{min} \leqq s_{max}$$
$$\bar{p} = \bar{p}^*$$
$$\bar{s} = \bar{s}^*$$

4) Equilibrium conditions:
$$\mu(y - x - s - l) = 0$$
$$\nu(y - x - s - r) = 0$$
$$\mu(s - s_{min}) = 0$$
$$\nu(s - s_{max}) = 0$$
$$\mu, \nu \geqq 0$$

5) Balance of trade: $p^w(y - x - s) = 0$ .

The boundaries s min and s max need not be physical, they can be considered as limits outside of which the stock policy cannot be overruled. The previously derived complementary pivoting algorithm solves this case after some minor modifications. This is easily seen after substitution of the third and fourth equilibrium condition into the first and second respectively.

We again can write $\mu H = \mu X$ and $\nu K = \nu X$ ,

where
$$H = \left[y_i - l_i - s_{min,i}\right]$$
$$K = \left[y_i - r_i - s_{max,i}\right]$$

Negative elements of K represent export quota which cannot be active and thus can be discarded.

Considering these equations together with the consumer demand equation, we have the basic elements for the pivoting algorithm. It remains to be seen what the influence of a change in $\phi$ is on the balance of trade.

The variable which needs explicit solution is s ;

$$s_i = \bar{s}_i + q_i$$

$$y_i = \max (y_i - x_i - \bar{s}_i - r_i, 0) + \min (y_i - x_i - \bar{s}_i - l_i, 0)$$

This allows to compute the balance of trade deficit.

The introduction of stock policy as an extra instrument implies that the net export of a certain commodity remains unchanged when a quota is effective but that the effectiveness of a quota constraint does not automatically imply overruling of the domestic price policy. The pivoting algorithm proceeds as before. Note that when one knows that the domestic price policy will not be overruled, a direct iteration over taxation rates can be applied. The balance of trade equilibrium implies again equilibrium of the government budget, by the consumer's budget equilibrium:

$$\begin{aligned} \text{income tax} &= - \{\text{tariff receipts} + \text{receipts on sales of stocks}\} \\ &= - \{(p^w - p)(y - x - s) + p(y^g - s)\} \end{aligned}$$

3.3.7 Stock policy in Hicksian pure exchange with quota

The consumer equation in the previous paragraph is changed and the domestic price policy is omitted.

The constraints are now:

$$k \leqq x = s \leqq h \quad ; \qquad k = y - h \quad ; \quad h = y - l$$

$$0 \leqq s_{min} \leqq s \leqq s_{max}$$

$$p^w(x + s) = p^w y$$

We consider the utility maximization under these constraints (and $x, s \geqq 0$) .

The Lagrangean is (taking again $\lambda$ out of brackets).

$$L = u(x) - \lambda[(p^w(x + s) - p^w y) + \mu(x + s - h) - \nu(x + s - k)$$

$$- \rho(s - s_{min} + \phi(s - s_{max})] .$$

The Kuhn-Tucker equilibrium conditions are

$$\frac{\partial L}{\partial x_i} = \frac{\partial u}{\partial x_i} - \lambda(p_i^w + \mu_i - \nu_i) \leqq 0 \quad ; \quad \frac{\partial L}{\partial x_i} \cdot x_i = 0$$

$$\frac{\partial L}{\partial s_i} = \lambda(p_i^w + \mu_i - \nu_i - \rho_i + \phi_i) \leqq 0 \quad ; \quad \frac{\partial L}{\partial s_i} s_i = 0 \quad .$$

The constraints

$$x_i + s_i - h_i \leqq 0 \quad ; \qquad \mu_i(x_i + s_i - h_i) = 0$$

$$x_i + s_i - k_i \geqq 0 \quad ; \qquad \nu_i(x_i + s_i - k_i) = 0$$

$$s_i - s_{min,i} \leqq 0 \quad ; \qquad \rho_i(s_i - s_{min,i}) = 0$$

$$s_i - s_{max,i} \leqq 0 \quad ; \qquad \phi_i(s_i - s_{max,i}) = 0$$

$$x, s, \mu, \nu, \rho, \phi, \lambda \geqq 0$$

We assume that $s_{min} > 0$
then $p^w + \mu - \nu - \rho + \phi = 0$

$$\frac{\partial u}{\partial x_i} = \lambda p_i \quad .$$

This policy is not one in which stock policy is always overruled by quota policy: the price at which the domestic consumer can buy does not always get priority over the policy of adjusting stocks. It may be much "better" to have the consumer price go up and sell stocks on the world market. We shall however not go into welfare theoretical arguments in this paper.

### 3.4 Domestic equilibrium in an economy with lagged production quota, domestic price policy and/or more than one consumer

#### 3.4.1 Linear technology, no intermediate inputs

In an economy qith production, factor ownership is the primary income distributing mechanism.

* When all producers own the factors they use and no intermediate inputs are involved, the lagged output is also fully owned by the producer and the pure exchange case follows.

* As long as there is a lag in production and no intermediate inputs are involved a pure exchange model can be obtained if specific assumptions are made on the ownership of output.
* We consider a linear technology:

$$y = Dq$$

$$Aq \leqq b \quad ; \quad q \geqq 0$$

| | |
|---|---|
| $y = (y_i)$ | final output |
| $q = (q_{h_i})$ | level at which the $k^{th}$ activity to produce the $i^{th}$ product is operated |
| $b = (b_k)$ | factor availability |
| $A = A_{k,h_i}$ | technology matrix: requirement on the $k^{th}$ factor when the $h_i{}^{th}$ activity is operated at unit level |
| $D = D_{i,h_i}$ | output of i when $h_i$ is operated at unit level. |

The variables are defined in such a way that the nonzero elements of D have unit value.

* The factors b are owned by the income groups.

$$b = \sum_j b^j$$

* The producer is assumed to maximize the value of this output this yields the primal and dual linear programme,

| | |
|---|---|
| max $p\,D\,q$ | min $w\,b$ |
| S.T. $A\,q \leqq b$ | S.T. $wA \geqq pD$ |
| $q \geqq 0$ | $w \geqq 0$ |

Various distributive assumptions can be made:

a) If we assume that the optimal output $Dq^*$ is divided over factor owners so that

$$\lambda_j\, p\, Dq^* = w^* b^j \quad ,$$

this yields the endowment $y^j = \lambda_j\, Dq^*$ which implies that the factors are paid in kind before the exchange.

b) An alternative assumption would be that the income distribution between factors is set on the basis of the ratio

$$\frac{w^* b^j}{w^* b} \quad .$$

c) There is however also a possibility to evaluate the factor rentals on the basis of current prices. Suppose the optimal basis is $A_B$ and that all factors are fully employed then

$$w^* A_B = p_B D_B \quad .$$

As long as price fluctuations between periods are small, the current value of w computed under changing current prices will be positive; this implies that the production decision taken at the expected prices would be unchanged under current prices. As soon as this is not the case some elements of w become negative: factor owners are penalized for wrong decisions. Combinations of the distributive rules a), b) and c) are also possible and may be desirable in order to show different types of price risk to which owners of different factors are exposed. (The approach is still deterministic, allocations are made on the basis of expected prices; no future markets).

The rules a) and b) can obviously be introduced without leaving algorithm developed within the pure exchange framework. This is not so certain for rule c) as will now be shown for the L.E.S. case.

Consider $$p_i x_{ij} = \phi m_j e_{ji}$$

$$= \phi \sum_h w_h b_{hj} e_{ji}$$

define $$F = \left[\sum_j b_{hj} e_{ji}\right] \; .$$

This yields:

$$pX = \phi w F \quad .$$

If the technology selected is $A_B$ with output $Y_B$ (we assume $Y_B$ to be unique even if it is produced by a combination of techniques, because of the production lag) and if all factors are fully employed we get

$$A_B Y_B = b$$

$$w A_B = p_B D_B \qquad \text{(assume } A_B \text{ is nonsingular , square)}$$

$$w = p_B D_B A_B^{-1} \qquad \text{(note that for all } p_B\text{: } p_B D_B Y_B = w\, b)$$

defining $C = \begin{bmatrix} D_B & A_B^{-1} F \\ 0 & \end{bmatrix}$ we get $p X = \phi p C$

If C is semipositive the previously derived algorithm will converge otherwise problems arise. This is a matter of distribution of endowments and differences in tastes between income groups; $Y_B = A_B^{-1} F_1$ is known to be positive. If all income groups were equal this would imply the positiveness of C.

### 3.4.2 Economy with production and domestic price policy

As soon as intermediate inputs are involved one has to abandon the pure exchange view. Even if outputs are given input demands are not. As long as prices are given that is as long as a domestic price policy is effective the problems are not too serious because a recursive approach is feasible:

1. Calculate input demand at given prices;
2. Consider these as committed expenditures for the investing class;
3. Calculate consumer demand at given income, given endowment or the like (see 3.4.1), given the committed expenditures.

There may however be savings constraints on these investment plans. We shall now discuss these problems.

Assumptions:

1. To any domestic price vector corresponds a unique net supply;
2. Outputs have a one year production lag (or more);
3. All desired domestic prices are positive.

ad 1 - This assumption is made in order to maintain the uniqueness of excess demand at given world market prices. The assumption is not unreasonable when many factors are fixed so that diminishing returns to variable inputs are likely to occur.

ad 2 - This assumption is not unrealistic: on the one hand it permits to show the financing problem for the farmer who has to buy current inputs, while on the other hand computation is simplified as the outputs can be taken as given endowment.

ad 3 - This assumption is realistic.

1. $p_i = \bar{p}_i$ for all i . No savings constraint

When all commodities have a desired domestic prices the production plan under profit maximization is fully predetermined: world market prices do not directly indluence supply.

The producer might however be unable to finance the investment needed.

2. $p_i = \bar{p}_i$, all i. Sectoral savings constraints as a side condition of net revenue maximization

Suppose there is a taxation proportional to wealth, then;

$$\bar{p}x^j = \phi\bar{p}y^j \quad .$$

Where

$x^j$ is the demand by the $j^{th}$ income class

$y^j$ endowment of the income class

$1-\phi$ is the rate of taxation $\frac{py - px}{py}$

it is set in order to meet $p^w x = p^w y$ at the national level.

$m_j = \phi p y^j$ is the disposable income of the sector.

Suppose a savings constraint is set:

$$s_j \leqq \beta_j \, m_j \qquad\qquad \beta_j \text{ is given}$$

define the demand for inputs by the $j^{th}$ class as $q^j$.
The savings constraint is a side condition of the maximization of net revenue:

$$\bar{p}q^j \leqq \phi\beta_j\bar{p}y^j \quad .$$

The world market prices can thus influence the investment through the taxation policy. The simultaneity introduced in this way would be avoided if the savings constraint is formulated as a function of earned instead of disposable income:

$$s_j \leqq \beta_j^j \, \bar{p}y^j$$

3. $p_i = \bar{p}_i$, $i = 1,\ldots, h$ ; $p_i = p_i^w$, $i = h + 1,\ldots,n$ , savings constraint

In this case the production plan is dependent on world market prices. As the output is lagged the receipts have only an expected value based on expected yields and expected prices. This introduces price risk, a feature which needs special attention but which will not be discussed in this paper.

Anyhow, the demand for inputs needs to be computed again at all world market prices. This is cumbersome as the production model will usually be quite complex. It may be advisable to use simplified demand functions for inputs (e.g. based on constant price elasticity) for use during the world market iterations.

4. $p_i = \max\ (p_i^w , \bar{p}_i)$ , $i = 1,\ldots,n$ ; $p_i = p_i^w$ , $1 = h + 1,\ldots n$

This case is similar to the previous one.

3.4.3 Economy with production and quota on inputs which are not consumer goods

3.4.3.1 Linear technology

* Consider a producer with a linear technology, and a one period lag in production,
* no savings constraint is considered.
* The government confronts this producer with a minimum and a maximum on the import of current inputs (the assumption of a minimum import may be unrealistic but is maintained for generality).
* The commodity may be produced domestically but it is a net input and is not used by the consumer.

The model now would be:

1) Producer

$$\max \quad \bar{p}_1 x_1 - p_2 x_2$$

$$\text{S.T.} \quad A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leqq b$$

$$x_1, x_2 \geqq 0$$

2) government policy.

$$\bar{p}_1 = \bar{p}_1^* \quad , \qquad \bar{p}_2 = \bar{p}_2^*$$

$$0 \leqq k \leqq x_2 \leqq h$$

3) market equilibrium.

$$\mu_i h_i = \mu_i x_i$$

$$\nu_i k_i = \nu_i x_i$$

$$p_i = \bar{p}_i + \mu_i - \nu_i$$

$$p_i, \mu_i, \nu_i \geqq 0$$

The equilibrium conditions of the linear programme are:

$$\lambda(b - Ax) = 0$$

$$\lambda A \geqq (\bar{p}_1', - p_2)$$

$$\left[(\bar{p}_1', - p_2) - \lambda A\right] x = 0$$

$$Ax \leqq b$$

$$x, \lambda \geqq 0$$

The model can be seen as a linear complementarity problem.

Define:

$$q_1 = h - x_2$$

$$q_2 = x_2 - k$$

$$q_3 = b - (A_1 x_1 + A_2 x_2)$$

$$q_4 = A_1^T \lambda^T - \bar{p}_1^T$$

$$q_5 = A_2^T \lambda^T + \bar{p}_2^T + \mu^T - \nu^T$$

$$\rho = \begin{pmatrix} \mu \\ \nu \\ \lambda \\ x_1 \\ x_2 \end{pmatrix} ; \quad p = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{pmatrix}$$

$$B = \begin{bmatrix} O & I \\ O & -I \\ A_1 & A_2 \end{bmatrix} ; \quad M = \begin{bmatrix} O & -B \\ B^T & O \end{bmatrix} ; \quad n = \begin{pmatrix} h \\ -k \\ b \\ -\bar{p}_1 \\ \bar{p}_2 \end{pmatrix}$$

the L.C.P would be:

$$q = M\rho + n$$

$$\rho^T q = 0 ; \rho, q \geqq 0$$

However, as M has the structure shown above, the solution is identical to the solution of the L.P.

$$\max \quad \bar{p}_1 x_1 - \bar{p}_2 x_2$$

$$\text{S.T.} \quad Bx \leqq \begin{pmatrix} h \\ -k \\ b \end{pmatrix}$$

$$x \geqq 0$$

this implies that the producer would react in the same way as if he was directly exposed to the desired domestic prices. This implies that the demand for inputs can be considered to be predetermined and thus treated as a committed expenditure for the "investor" independedly whether the economy is Hicksian or not. This assumes however that $\bar{p}_1$ is given for the producer.

If we assume quota on outputs also and assume that the expected price for next period equals the realized price for the current period $\bar{p}_1$ is not given anymore. The previous approach still holds when many producers are considered.

* If every producer has a linear goal function with the same coefficient and a convex technology, the maximization of the aggregate goal function under the technology yields the same result as individual profit maximization. As expressed by Koopmans [8], this mathematically trivial property has important economic consequences in terms of decentralization of decision making. Mathematically speaking it only says that when the goal function is linear and the constraints have a block diagonal structure the blockwise solution of the programme is equivalent with the total solution.
* In the linear programming case introduction of quota on inputs however, introduces interdependence (a row in the matrix). Special algorithms are available which make use of this special structure of the aggregate technology matrix.

3.4.3.2 Convex technology

* The convexity guarantees a unique optimum.

The problem now is:

$$\begin{aligned} \max \quad & \bar{p}_1 x_1 - \bar{p}_2 x_2 \\ \text{S.T.} \quad & g(x) \leqq 0 \\ \text{and} \quad & k \leqq x_2 \leqq h \\ & x \geqq 0 \quad . \end{aligned}$$

The Lagrangean is

$$L = \bar{p}_1 x_1 - (\bar{p}_2 + \mu - \nu) x_2 - \lambda \cdot g(x) + \mu k - \nu h$$

the equivalence is again obvious.

### 3.4.4 A Hicksian economy with production and quota

The model for this case is:

1) Consumer

$$\max \quad u(x^j) \qquad \text{(u homothetic)}$$

$$\text{S.T.} \quad px^j = py^j + \alpha_j \; tr$$

2) Producer

$$\max \quad py^j$$

$$\text{S.T.} \quad g^j(y) \leqq 0 \qquad \text{(convex)}$$

3) Government

$$tr = (p^w - p)(y - x)$$

$$1 \leqq y - x \leqq r$$

4) Market equilibrium

$$\mu(y - x) = \mu 1$$

$$\nu(y - x) = \nu r$$

$$p = p^w + \mu - \nu$$

$$p, \mu, \nu \geqq 0$$

5) Balance of trade

$$p^w y = p^w x$$

We prove that this case is equivalent to

$$\max \quad u(x)$$

$$\text{S.T.} \quad p^w x = p^w y$$

$$\text{and} \quad 1 \leqq y - x \leqq r$$

and $g^j(y^j) \leqq 0$

Now $L = u(x) + \left[\lambda(p^w + \mu - \nu)(y - x) - \Sigma\phi^j g^j(y^j)\right] + \lambda \cdot \text{constant}$

$$\left.\begin{array}{l} \dfrac{\partial L}{\partial x_i} = \left[\dfrac{\partial u}{\partial x_i} - \lambda p_i\right] x_i = 0 \\ \dfrac{\partial u}{\partial x_i} - \lambda p_i \leqq 0 \end{array}\right\} \text{utility maximizing conditions}$$

$$\left.\begin{array}{l} \dfrac{\partial L}{\partial y_i^j} = \left[\lambda\, p_i - \phi^j \dfrac{\partial g^j}{\partial y_i^j}\right] y_i^j = 0 \\ \lambda\, p_i - \left(\phi^j \dfrac{\partial g^j}{\partial y_i^j}\right) \leqq 0 \end{array}\right\} \text{profit maximizing conditions}$$

$$\left.\begin{array}{l} p^w x = p^w y \\ l \leqq y - x \leqq r \\ g^j(y^j) \leqq 0 \\ y = \Sigma y^j \end{array}\right\} \text{constraints}$$

$$\left.\begin{array}{l} \mu(y - x) = \mu l \\ \nu(y - x) = \nu r \\ \mu, \nu \geqq 0 \end{array}\right\} \text{market equilibrium}$$

The previous cases suggest situations in which the optimization problem can easily be decomposed. We can summarize the previous discussion by saying that as long as the economy is Hicksian, no domestic price policy is introduced, and no savings constraint is imposed on the production plan, the equilibrium problem is a convex optimization problem. As soon as the economy is non-Hicksian the conflict of interests between income classes destroys the optimality property of equilibrium. The introduction of domestic price policies also causes problems:

In the model the change is minor:

$$p = \bar{p} + \mu - \nu \quad \text{replaces } p = p^w + \mu - \nu \quad .$$

But the consequences are significant: the problem is not longer an optimum problem as the dual variables are constrained it is a nonlinear complementarity problem only, which can be solved as a parametric convex programming problem. We discussed this under 3.3.5.

The computation of domestic equilibrium; summary

To summarize the situation at this moment, the algorithms presented to solve domestic equilibrium problems are listed below. Scarf's fixed point algorithms as mentioned in § 1, is disregarded. It could solve all the cases mentioned below, specific policies like food aid, asset redistribution, stock policy are not explicitly listed:

| | free trade | domestic price policy | quota | domestic price policy and quota |
|---|---|---|---|---|
| One consumer* no production | 1 | 1 | 3 | 5,6 |
| More consumers no production | 1 | 2 | 4 | 4 |
| One consumer* production | 1 | 2 | 3 | 6 |
| More consumers production | 1 | 2 | (7) | (7) |

1. Direct computation, for the consumer through dual, for the producer either through dual or primal;
2. iteration over taxation rate (assumption: no inferior goods);
3. convex programming problem; 4,5 can solve special cases of this;
4. complementary pivoting algorithm. The commodities with quota form a linear expenditure subsystem; several taxation policies are possible;
5. 4 but also valid for generalized C.E.S. utility function;
6. parametric convex programming;
7. only solved for cases with quota on inputs which are not consumer goods.

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* A Hicksian economy is considered as an economy with one consumer.

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