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Assessment of the Impact of Aggregated Economic Factors on Optimal Consumption in Models of Economic Growth

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Abstract

The problem of consumption-optimal economic growth is considered. In the model there are three factors of production: capital, labor and useful work, that interact in the production of homogenous output. At any instant of time a fraction of this homogeneous output can be allocated to investment in accumulation of capital and useful work. The gross domestic product (GDP) of a country is presented by the linear-exponential (LINEX) production function.

The general goals of the research can be formulated as follows:

- analysis of properties of the LINEX production function and identification of parameters under restrictions on elasticity coefficients;
- analysis of the Hamiltonian system of differential equations for the Pontryagin maximum principle in the optimal control problem;
- elaboration of an algorithm for constructing synthetic trajectories of optimal economic growth;
- development of software for numerical simulation and sensitivity analysis;
- comparison of real and synthetic trajectories of economic growth, and simulation of future scenarios.

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Assessment of the Impact of Aggregated Economic Factors on Optimal Consumption in Models of Economic Growth

Andrey A. Krasovskii

1. Introduction

The paper addresses the problem of dynamic optimization of investment in economic growth. The research is based on classical models of economic growth by K. Arrow [1-2], R. Solow [14] and K. Shell [13]. Unlike the classical approach we consider the model with a linear-exponential (LINEX) production function which reflects specific features of increasing and decreasing returns in economic growth (see R. Ayres [5]). We use optimal control theory, namely, an appropriate version of the Pontryagin maximum principle [11] to construct optimal levels of investment. Technically, the research focuses on the analysis of the Hamiltonian system of the maximum principle. On the basis of this analysis and using methods of the theory of differential games [6, 9, 15] we elaborate an algorithm for constructing optimal trajectories of economic growth and optimal levels of investment, and also investigate the optimal balance between economic factors. Another important part of our research is the econometric analysis of the model. We calibrate macroeconomic parameters using the real data on growth factors for the US economy [5]. The calibrated model shows a good fit with the data. In particular, it explains the appearance of periods of increasing and decreasing returns in the process of economic growth, and also indicates saturation levels for the optimal ratios of production factors.

For the model with two economic factors: capital and useful work per worker, it is shown that the steady state has the saddle character. Comparison results for components of the steady state provide the possibility to describe potential scenarios of the balanced economic growth.

1.1. Interdisciplinary Character of Research

The present research uses methods and instruments from different disciplines: theory of economic growth, optimal control theory, statistics, econometric analysis, and numerical methods.

To construct our economic model we use models of economic growth theory due to K. Arrow, R. Solow and K. Shell. Three production factors in our model are capital, labor and useful work. The linear-exponential (LINEX) production function shows how these factors interact in the production of the homogenous output. Based on the homogeneity of the LINEX production function, we change the initial variables and consider a model with two variables: capital per worker and useful work per worker.

We pose a corresponding optimal control problem with the infinite horizon and solve it using an appropriate version of the Pontryagin maximum principle. In this problem, the control variables are investments in the accumulation of the production factors, and the utility is the integral discounted consumption index. Applying the Pontryagin maximum principle, we find the optimal investment policy and the corresponding trajectories of the accumulated production factors, which maximize the utility function.

Numerical methods are used to design an algorithm for the construction of the synthetic trajectories of optimal economic growth. The elaborated software is developed for the realization of this algorithm. The model is simulated with parameters calibrated on data for the US economy.

The identification of the coefficients of the LINEX production function and of the parameters of the model is fulfilled using tools of econometric analysis including the unit roots and co-integration analysis. A statistical software (SPSS Sigma Stat 3.0, Statistica 6.0, MS Excel) is used to carry out nonlinear regressions under constraints on the parameters and on the elasticity coefficients.

1.2. Methodological Scheme

The methodological scheme of the research is presented in Fig. 1.1.

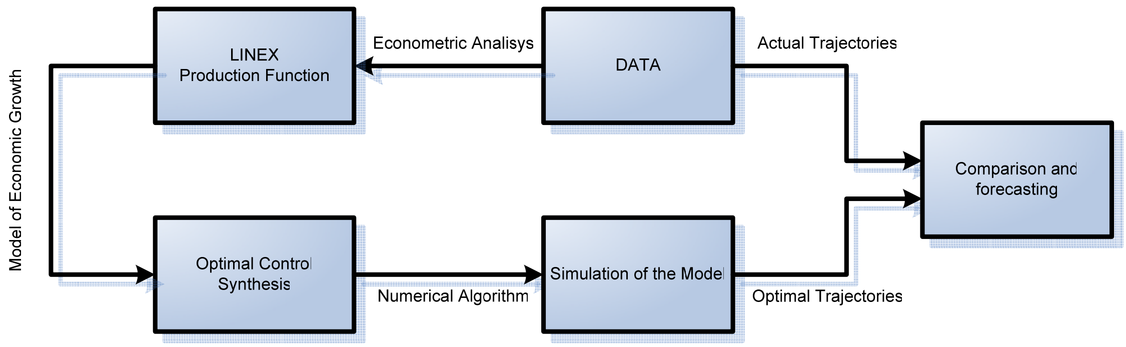

Fig. 1.1. Methodological Scheme

Let us explain this scheme. We start with the “Data” box in the upper row. Originally we have data on the economy of a country. In our case this data is presented by time series of the country’s GDP and three production factors: capital, labor and useful work. Using methods of econometric analysis, we calibrate this data to identify the coefficients of the LINEX production function and parameters of the model: the rate of depreciation of capital stock, rate of growth of labor stock and time discount. On the next step, we consider a model of economic growth with calibrated parameters. We refer to models by R. Solow and C. Shell and modify them to implement the LINEX production function with the additional production factor – useful work. We consider differential equations describing the process of growth of the production factors. The utility function is the integral consumption index discounted in time. The control parameters are investments in the accumulation of the production factors. In other words, investments are tools of a central planner, used to maximize the utility function [8, 12]. As a result we have a dynamical model for the economy growth.

Based on the constructed model of economic growth, one poses an optimal control problem. The model equations describing growth trends of the state variables reflect interaction of the production factors. The goal is to maximize the discounted utility function under given constraints on the control variables and initial values for the state variables. In this block we face a purely mathematical problem. The only connection with the real world is the LINEX production function, but on this stage we are only interested in the structure and properties of this production function as a mathematical object. We solve the optimal control problem in the framework of the Pontryagin maximum principle, finding the optimal investment and corresponding trajectories of optimal growth of the production factors and the GDP. The research is developed in the framework of necessary conditions and sufficient conditions of optimality for control problems with infinite horizon, specification of concavity properties of Hamiltonian functions and qualitative analysis of the vector field of a Hamiltonian system [3, 4, 10, 16].

On the next step, we construct a numerical algorithm for simulation of theoretically designed trajectories and controls. Based on this algorithm, we develop a special software package for simulation of the model. As a result of numerical simulation, we obtain synthetic trajectories of optimal economic growth.

Finally, we come back to the real data and compare the actual economic growth trajectories with the simulated optimal trajectories of the model. Depending on the degree of the agreement between the simulated trajectories of the model and the real economic growth trajectories, one can make judgments about the adequacy of both the model and the optimization approach employed. If the degree of that agreement is satisfactory, then a central planner can use the model to simulate and assess future scenarios of economic growth. Let us stress that a good approximation of the real trajectories by the theoretically optimal trajectories of the model is not evident from the very beginning. On the contrary, it should be surprising if the model's simulated trajectories obtained through the optimization of the utility function but not through a direct approximation to the real trajectories would, nevertheless, match well with the real trajectories. Our results demonstrate such good fitness. We treat it as an indication of an adequate choice of the key components of our study: the LINEX production function, the model of economic growth, and the optimized utility function.

2. Analysis of the LINEX Production Function

This section is devoted to assessment of the impact of production factors on the growth of gross domestic product (GDP) of a country. The model assumes that GDP is produced according to an aggregate production function. The production function is used to express the relationship between factors of production and the quantity of output produced. In our case there are three inputs into production: capital, useful work and labor. Useful work is a recently added production factor which represents the input of energy or available energy into production of GDP.

The research is fulfilled for the linear-exponential (LINEX) production function [5], which is presented by the following expression:

$$F(K,L,U)=K^{\alpha}L^{\beta}U^{(1-\alpha-\beta)}\exp\left\{\gamma\frac{K}{L}+\mu\frac{L}{U}+\xi\frac{U}{K}\right\} \quad \textbf{(2.1)}$$

Here K denotes capital, symbol L stands for labor, and useful work is denoted by U. One can see that unlike classical production functions, i.e. Cobb-Douglas production function, production function with constant elasticity of substitution, etc., the LINEX production function together with a log-linear part has an exponential multiplier in which combinations of ratios of production factors are presented.

The LINEX production function is homogenous of degree one. In other words, if we multiply the quantities of each input by some factor, the quantity of output will increase by the same factor. This property is used in an economic model when we consider per worker (per capita) quantities. Coefficients of elasticity of the LINEX production function to production factors can be calculated as follows:

$$\begin{aligned}
\varepsilon_{Y,K}&=\frac{K}{Y}\frac{dY}{dK}=\alpha+\gamma\frac{K}{L}-\xi\frac{U}{K},\\
\varepsilon_{Y,L}&=\frac{L}{Y}\frac{dY}{dL}=\beta-\gamma\frac{K}{L}+\mu\frac{L}{U},\\
\varepsilon_{Y,U}&=\frac{U}{Y}\frac{dY}{dU}=(1-\alpha-\beta)-\mu\frac{L}{U}+\xi\frac{U}{K}.
\end{aligned} \quad \textbf{(2.2)}$$

One can note that coefficients of elasticity of the LINEX production function are varying in time due to change of ratios of production factors, but their sum is constant and is equal to one. The graphs of elasticity coefficients are presented on Fig. 2.1.

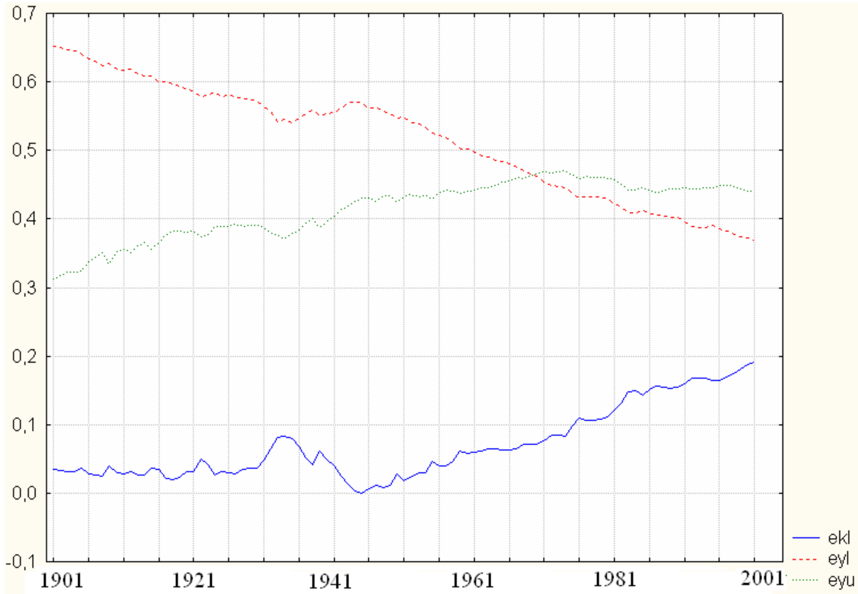

Fig. 2.1. Coefficients of elasticity of the LINEX production function

The econometric analysis is fulfilled for the data on the US economy. The data is presented by time series for GDP and production factors: capital, labor and useful work. Time series of each variable is a sequence of data points measured for each year in interval of 101 years (1900-2001). One can see the graphical illustration of the US data on Fig. 2.2. On this figure the values of all variables are normalized to 1900. The useful work is presented by violet column and is measured in Exajoules (EJ, 10^{18} Joules); the level of the useful work in 1900 constituted 0.64 EJ. The labor is depicted in blue and is measured in Index of Hours Worked (IHW). The capital is shown in the red color and is measured in money equivalent (billion dollars); the level of capital in 1900 was $ 2021 billion. The symbol Y stands for GDP which is denoted in the green color; it is measured in billions of US dollars; the level of GDP in 1900 was $ 354 billion.

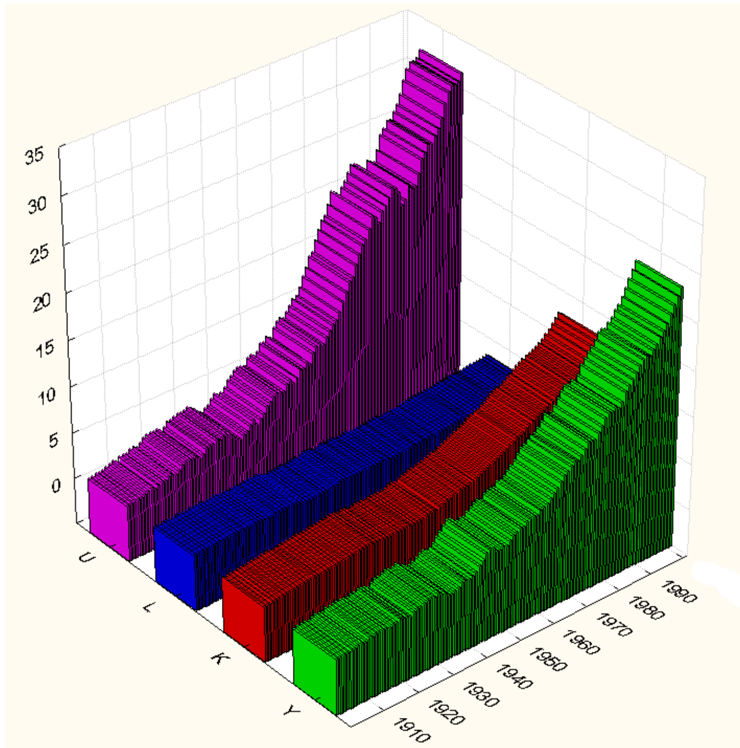

Fig. 2.2. Data for the US economy (1900-2001)

Various statistical software packages (SPSS Sigma Stat 3.0, Statistica 6.0) are used to carry out nonlinear regressions for identification of parameters of the production function. The task was complicated by the necessity to put constraints on elasticity coefficients (2.2) of the LINEX function. In experiments the values of these coefficients with respect to production factors are supposed to be positive. Unit roots and cointegration analysis of the data has indicated that the best fitness is achieved when logarithmic difference method is applied. This method can be presented by the following expression:

$$\ln Y_{i+1} - \ln Y_i = \alpha\left(\ln K_{i+1} - \ln K_i\right) + \beta\left(\ln L_{i+1} - \ln L_i\right) + (1-\alpha-\beta)\left(\ln U_{i+1} - \ln U_i\right) + \\ + \gamma\left(\frac{K_{i+1}}{L_{i+1}} - \frac{K_i}{L_i}\right) + \mu\left(\frac{L_{i+1}}{U_{i+1}} - \frac{L_i}{U_i}\right) + \xi\left(\frac{U_{i+1}}{K_{i+1}} - \frac{U_i}{K_i}\right) \quad \textbf{(2.3)}$$

The graph of fitness of the LINEX production function to the data is shown on Fig. 2.3.

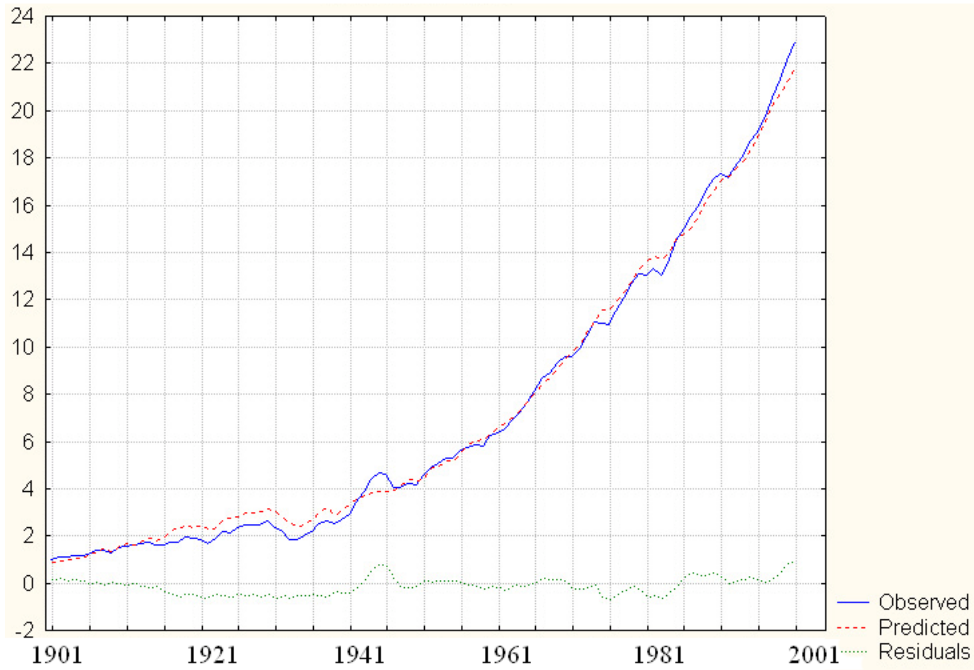

Fig. 2.3. Fitness of the LINEX production function

3. Model of Economic Growth

In our model we focus on analysis of GDP of a country. A region's gross domestic product, or GDP, is one of several measures of the size of its economy. GDP of a country is defined as the market value of all final goods and services produced within a country in a year. GDP can be calculated as either the value of the output produced in a country or equivalently as the total income, in the form of wages, rents, interest, and profits, earned in a country. Thus, GDP is also known as output or national income. In our model GDP is a homogenous output.

In the model there are three inputs into production: capital, useful work and labor. If symbols $K(t)$, $U(t)$ and $L(t)$ denote stocks of capital, useful work and labor, respectively, at time t, then the output at time t, $Y(t)$, is given by

$$Y(t) = F\left[K(t), U(t), L(t)\right] \tag{3.1}$$

Here the symbol $F\left[K(t), U(t), L(t)\right]$ denotes production function. In our model we operate with the LINEX production function (2.1).

Instead of examining the quantity of total output in a country, it is more reasonable to consider relative quantities: the quantity of output per worker. Using the fact that the production function is homogenous of degree one it is possible to establish connection between the quantity of output per worker and quantities of capital per worker and useful work per worker

$$\left(\frac{1}{L}\right)Y = \left(\frac{1}{L}\right)F[K, U, L] = F\left[\frac{K}{L}, \frac{U}{L}, 1\right] = \left(\frac{K}{L}\right)^{\alpha}\left(\frac{U}{L}\right)^{(1-\alpha-\beta)} \exp\left\{\gamma\frac{K}{L} + \mu\frac{L}{U} + \xi\frac{U}{K}\right\} \quad \textbf{(3.2)}$$

Let us denote per worker quantities by lower case letters:

$$y = Y / L \text{ is output per worker,} \quad \textbf{(3.3)}$$

$$k = K / L \text{ is capital per worker,} \quad \textbf{(3.4)}$$

$$u = U / L \text{ is useful work per worker.} \quad \textbf{(3.5)}$$

Then one can introduce per worker LINEX production function of two variables:

$$y = f(k,u) = k^{\alpha}\, u^{(1-\alpha-\beta)} \exp\left\{\gamma\, k + \mu\frac{1}{u} + \xi\frac{u}{k}\right\}. \quad \textbf{(3.6)}$$

Let us analyze the role of production factors in the production output. We deal with two accumulated aggregated production factors capital and useful work. In the first problem we fix the useful work per worker on the average value and analyze the impact of capital per worker on optimization of GDP per worker. In this part we refer to classical models of economic growth by R. Solow and K. Shell.

Let symbols $C(t) \geq 0$ and $I(t) \geq 0$ denote the respective rates at time t of consumption and investment, and the symbol $s(t)$, $0 \leq s(t) \leq 1$, denote the fraction of output at time t which is saved and invested. Then we have the simple national income identities

$$Y(t) = C(t) + I(t) = (1 - s(t))Y(t) + s(t)\; Y(t) \quad \textbf{(3.7)}$$

This is a closed-economy model, in which savings equals investment. Someone who had control over resources and could have spent them on consumption today has instead used them to build a piece of capital that would be employed in future production.

There are two resources of change in capital: investment (the building of new capital) and depreciation (the wearing out of old capital). At any point in time, the change in the capital stock is the difference between the amount of investment and the amount of depreciation. Let us assume that the constant fraction of capital stock depreciates each period. Then capital stock accumulates according to equation

$$\dot{K}(t) = s(t)Y(t) - \mu K(t) \tag{3.8}$$

Here parameter $\mu > 0$ is the rate of depreciation of capital stock. Here and further, we denote derivative of variable with respect to time by a symbol with a point above. For example, notation $\dot{K}(t)$ means the derivative of capital with respect to time t: $\dot{K}(t) = \dfrac{d\,K(t)}{d\,t}$.

We assume that the labor input grows according to equation

$$\frac{\dot{L}(t)}{L(t)} = n \tag{3.9}$$

Here $n > 0$ is a constant growth rate. In this model we assume that the growth rate of the labor force is the same as the growth rate of the population.

Let us consider the process of capital accumulation in per worker terms (3.3)-(3.5). One can differentiate variable k standing for the relative capital (3.4) with respect to time t using the quotient rule:

$$\dot{k}(t) = \frac{d\left(\dfrac{K(t)}{L(t)}\right)}{d\,t} = \frac{\dot{K}(t)L(t) - \dot{L}(t)K(t)}{\left(L(t)\right)^2} = \frac{\dot{K}(t)}{L(t)} - \frac{\dot{L}(t)}{L(t)}\frac{K(t)}{L(t)}. \tag{3.10}$$

Substituting expressions for the growth of capital stock (3.8) and labor (3.9) to (3.10), we obtain:

$$\dot{k}(t) = \frac{s(t)Y(t) - \mu K(t)}{L(t)} + n\frac{K(t)}{L(t)} = s(t)\frac{Y(t)}{L(t)} - (\mu + n)\frac{K(t)}{L(t)} \tag{3.11}$$

Let us rewrite differential equation (3.11) in per worker quantities of output y (3.3) and capital stock k (3.4). Then the growth of per worker capital stock is subject to dynamics:

$$\dot{k}(t) = s(t)\,y(t) - \lambda\,k(t)\,. \tag{3.12}$$

Here parameter $\lambda = \mu + n$ is the sum of the rate of depreciation μ of the capital stock and the rate of capital dilution n (3.9) due to arrival of new workers.

Further in the paper we construct the model which includes the impact of useful work on growth of GDP of a country. This model assumes that one part s_1 of savings s is invested into building of capital and another part s_2 is invested into accumulation of useful work. Similar to expression (3.12) describing growth of capital per worker k we introduce a differential equation for description of dynamics of useful work per worker u.

Let us focus on a slightly simplified, but not trivial, model of economic growth in which we fix variable corresponding to useful work per worker. It is worth to fix it on the average level:

$$u(t) = \tilde{u} \,. \quad \textbf{(3.13)}$$

Here $\tilde{u} > 0$ is a constant average value of useful work per worker. For example, one can calculate $\tilde{u}$ from the given time series.

Following assumption (3.13) we consider function $f(k,u)$ (3.6) of two variables as a per worker LINEX production function $f(k)$ of one variable k

$$f(k) = f(k,\tilde{u}) = k^{\alpha}\, \tilde{u}^{(1-\alpha-\beta)} \exp\left\{\gamma\, k + \mu \frac{1}{\tilde{u}} + \xi \frac{\tilde{u}}{k}\right\}. \quad \textbf{(3.14)}$$

Let us indicate some properties of function $f(k)$ (3.14). It is assumed that the marginal product of capital is positive but declining. Mathematically, this property implies that

$$f'(k) > 0 \ \textbf{for} \ k \in K^0,\ K^0 \subset (0, +\infty)\textbf{, and} \ f''(k) < 0 \ \textbf{for} \ k \in K^1 \subset K^0 \quad \textbf{(3.15)}$$

Here marginal product of capital per worker is calculated as the first derivative of function $f(k)$ with respect to variable k - $f'(k) = \dfrac{\partial\, f(k)}{\partial\, k}$. The second derivative of function $f(k)$ is denoted by the symbol $f''(k) = \dfrac{\partial^2\, f(k)}{\partial\, k^2}$. The symbol K^0 stands for a nonempty set which is called economic domain, and the symbol K^1 stands for a nonempty convex which is called relevant domain.

The assumption of diminishing marginal product means that if we keep adding units of a single input (holding the quantities of any other inputs fixed), then the quantity of new output that each new unit of input produces will be smaller than that added by the previous unit of the input.

In the models of economic growth it is usually assumed that the production function satisfies the so-called “Inada’s limit conditions”. These conditions are presented by the following expressions:

$$\begin{cases} \lim\limits_{k\downarrow 0} f(k) = 0, & \lim\limits_{k\uparrow\infty} f(k) = \infty \\ \lim\limits_{k\downarrow 0} f'(k) = \infty, & \lim\limits_{k\uparrow\infty} f'(k) = 0 \end{cases} \quad \textbf{(3.16)}$$

We take into account conditions (3.15)-(3.16) in econometric analysis of the LINEX production function by introducing additional inequalities on the econometric parameters in regression equations performed in SPSS software.

4. Optimal Control Problem

Let us consider the optimal control problem for growth of the capital stock. Our goal is to maximize the utility function, which represents the discounted consumption per worker of output of a country. One can present the utility function as the integral of the logarithmic consumption index discounted on the infinite horizon:

$$J = \int_0^{+\infty} \left[\ln f(k(t)) + \ln(1 - s(t))\right] e^{-\delta t} dt \ . \quad \textbf{(4.1)}$$

Here the symbol $\delta > 0$ denotes the constant rate of discount, control parameter s is a fraction of output that is invested into the capital accumulation, and function $f(k)$ is the LINEX per worker production function defined by expression (3.14). Logarithmic form of utility function is usually used in the optimal consumption problems in macroeconomic models.

The capital per worker stock is subject to the following differential equation of growth

$$\dot{k}(t) = s(t)\, f\big(k(t)\big) - \lambda\, k(t) \, . \quad \textbf{(4.2)}$$

Here parameter λ is the sum of the rate of depreciation of the capital stock and the rate of capital dilution.

A central planner starts his investment process with the initial level $k(0)=k^0$ of capital per worker and he desires to maximize the integral of discounted consumption per worker on the infinite horizon.

Stated specifically, the problem is to maximize the functional

$$J=\int_0^{+\infty}\left[\ln f(k(t))+\ln(1-s(t))\right]e^{-\delta t}dt \to \max \qquad \textbf{(4.3)}$$

subject to the following constraints:

$$\dot{k}=s\,f(k)-\lambda k\,,$$

$$k(0)=k^0\,,\ s\in[0,a]\,,\ a<1$$

where parameters δ, $\lambda=n+\mu$, k^0 are given positive scalars and $s(t)$ is some measurable control or policy variable to be chosen by a planner. Parameter $0<a<1$ is a positive scalar which separates the right boundary of control parameter $s(t)$ from the unit value.

Remark 4.1. The condition of compactness of control restrictions $s\in[0,a]$ is important for accurate application of the Pontryagin maximum principle. If this condition is fulfilled then one can prove the existence result for the optimal control problem [3].

The problem is to find the optimal investment level $s^0(\cdot)$ and the corresponding trajectory $k^0(\cdot)$ of the capital per worker stock k subject to dynamics (4.2) for maximizing the consumption per worker functional (4.3).

5. The Necessary Conditions of Optimality

In this section we give the necessary conditions of optimality from the paper [3]. To do this, we introduce the standard notations for phase and control variables. Assume that the symbol x stands for the phase variable of the control system and the symbol u denotes the control parameter. For our dynamics of per worker capital (4.2) it means that the phase variable x is the per worker capital k, and the control parameter u is the investment level s. So, we deal with the following infinite-horizon optimal control problem:

$$\dot{x}(t)=f(x(t),u(t))\,,\ u(t)\in U\,; \qquad \textbf{(5.1)}$$

$$x(0) = x_0\,; \tag{5.2}$$

$$\text{maximize } J(x,u) = \int_0^\infty e^{-\delta t} g(x(t),u(t))dt \tag{5.3}$$

Here $x(t) = (x^1(t),...,x^n(t)) \in R^n$ and $u(t) = (u^1(t),...,u^m(t)) \in R^m$ are the current values of the system's states and controls; U is a nonempty convex compactum in R^m; x_0 is a given initial state; and $\delta \geq 0$ is a discount parameter. The functions $f : G \times U \mapsto R^n$, $g : G \times U \mapsto R^1$, the matrix $\partial f / \partial x = \left(\partial f^i / \partial x^j\right)_{i,\, j=1,...,n}$, and the gradient $\partial g / \partial x = \left(\partial g / \partial x^j\right)_{j=1,...,n}$ are assumed to be continuous on $G \times U$. Here G is an open set in R^n such that $x_0 \in G$. An admissible control is identified with an arbitrary measurable unction $u : [0,\infty) \mapsto U$. A trajectory corresponding to a control u is a Carathreodory solution *x*, which satisfies the initial condition.

The basic assumptions are the following.

(A1) There exists a $C \geq 0$ such that

$$\langle x, f(x,u) \rangle \leq C\left(1 + \|x\|^2\right) \text{ for all } x \in G \text{ and } u \in U.$$

(A2) For each $x \in G$, the function $u \mapsto f(x,u)$ is affine, i.e.,

$$f(x,u) = f^0(x) + \sum_{i=1}^m f_i(x) u^i \text{ for all } x \in G \text{ and } u \in U,$$

where $f_i : G \mapsto R^n$, $i = 0,1,...,m$, are continuously differentiable.

(A3) For each $x \in G$, the function $u \mapsto g(x,u)$ is concave.

(A4) There exist positive-valued functions μ and ω on $[0,\infty)$ such that $\mu(t) \to 0$,

$\omega(t) \to 0$ as $t \to \infty$, and for any admissible pair (u,x),

$$e^{-\delta t} \max_{u \in U} |g(x(t),u)| \leq \mu(t) \text{ for all } t > 0;$$

$$\int_T^\infty e^{-\delta t} |g(x(t),u)| dt \leq \omega(T) \text{ for all } T > 0.$$

(A5) For every admissible pair (u,x) and for almost all (a.a.) $t \geq 0$, one has

$$\frac{\partial g(x(t),u(t))}{\partial x} > 0 \text{ and } \frac{\partial f^i(x(t),u(t))}{\partial x^j} \geq 0 \text{ for all } i, j : i \neq j .$$

Let us define the normal-form Hamilton–Pontryagin function $\widetilde{\mathrm{H}} : G \times [0,\infty) \times U \times R^n \mapsto R^1$ and the normal-form Hamiltonian $\overline{H} : G \times [0,\infty) \times R^n \mapsto R^1$ as follows:

$$\widetilde{\mathrm{H}}(x,t,u,\psi) = \langle f(x,u),\psi \rangle + e^{-\delta t} g(x,u); \tag{5.4}$$

$$\overline{H}(x,t,\psi) = \sup_{u \in U} \widetilde{\mathrm{H}}(x,t,u,\psi) \tag{5.5}$$

Given an admissible pair (u^*, x^*), we introduce the normal-form adjoint equation

$$\dot{\psi}(t) = -\left[\frac{\partial f(x^*(t),u^*(t))}{\partial x}\right]^* \psi(t) - e^{-\delta t} \frac{\partial g(x^*(t),u^*(t))}{\partial x} . \tag{5.6}$$

Any solution ψ to (5.6) on $[0,\infty)$ will be called an adjoint variable associated with (u^*, x^*). They say that an admissible pair (u^*, x^*) satisfies the normal-form core Pontryagin maximum principle together with an adjoint variable ψ associated with (u^*, x^*) if the following normal-form maximum condition holds:

$$\widetilde{\mathrm{H}}(x^*(t),t,u^*(t),\psi(t)) = \overline{H}(x^*(t),t,\psi(t)) \text{ for a.a. } t \geq 0. \tag{5.7}$$

The normal-form stationary condition holds:

$$\overline{H}(x^*(t),t,\psi(t)) = \delta \int_t^\infty e^{-\delta t} g(x^*(s),u^*(s))\, ds \text{ for all } t \geq 0. \tag{5.8}$$

Theorem. *Let assumptions* (A1) – (A5) *be satisfied. There exists a* $u_0 \in U$ *such that* $f(x_0,u_0) > 0$, *and for every admissible pair* (u,x), *it holds that* $f(x(t),u(t)) \geq 0$ *for a.a.* $t \geq 0$. *Let* (u^*, x^*) *be an optimal pair. Then there exists an adjoint variable* ψ *associated with* (u^*, x^*) *such that*

(i) (u^*, x^*) *satisfies relations of the normal-form core Pontryagin maximum principle together with* ψ *;*

(ii) (u^*, x^*) *and* ψ *satisfy the normal-form stationarity condition* (5.8)*;*

(iii) $\psi(t) > 0$ *for all* $t \geq 0$.

Corollary. *Let the assumptions of Theorem be satisfied, and let*

$$x_0 \geq 0$$

$$g(x^*(t), u^*(t)) \geq 0 \text{ for a.a. } t \geq 0$$

$$\textit{and } \frac{\partial f(x^*(t), u^*(t))}{\partial x} \geq A \text{ for a.a. } t \geq 0$$

where A is a matrix of order n such that A > 0*. Then there exists an adjoint variable* ψ *associated with* (u^*, x^*) *such that statements* (i), (ii), *and* (iii) *of Theorem hold true and, moreover,* ψ *satisfies the transversality condition*

$$\lim_{t\to\infty} \langle x^*(t), \psi(t) \rangle = 0. \tag{5.9}$$

One can easily check that the dynamics of capital (4.2) and the utility function (4.1) for the logarithmic consumption index satisfy all conditions of the Theorem and Corollary, and, hence, the normal-form core Pontryagin maximum principle and the transversality condition can be applied to construction of optimal trajectories in the considered model of economic growth.

6. Application of the Pontryagin Maximum Principle

We solve the problem (4.3) in the framework of Pontryagin maximum principle. Introducing the shadow price $\tilde{\psi} = \tilde{\psi}(t)$ for the accumulated capital per worker stock $k = k(t)$ we compile the Hamiltonian of the problem

$$\tilde{H}(s, k, t, \tilde{\psi}) = [\ln f(k) + \ln(1-s)]\, e^{-\delta t} + \tilde{\psi}\,(s\, f(k) - \lambda\, k), \tag{6.1}$$

which measures the current flow of utility from all sources.

Substituting a new variable

$$\psi = \tilde{\psi}\, e^{\delta t}, \tag{6.2}$$

into the Hamiltonian (6.1), we obtain

$$\tilde{H}(s, k, t, \psi) = e^{-\delta t}\big(\ln f(k) + \ln(1-s) + \psi\,(s\, f(k) - \lambda\, k)\big). \tag{6.3}$$

For convenience, we exclude the exponential term from the Hamiltonian. Introducing notation

$$H(s, k, \psi) = e^{\delta t}\tilde{H}(s, k, t, \psi), \tag{6.4}$$

one can rewrite the Hamiltonian of the problem in the following way:

$$H(s, k, \psi) = \ln f(k) + \ln(1-s) + \psi\,(s\, f(k) - \lambda\, k). \tag{6.5}$$

Lemma 6.1. The Hamiltonian function $H(s,k,\psi)$ is strictly concave with respect to variable s.

Proof. Calculating the second derivative

$$\frac{\partial^2 H}{\partial s^2} = -\frac{1}{(1-s)^2}, \tag{6.6}$$

one can easily observe that it is negative. The negative sign means the strict concavity of the Hamiltonian with respect to variable s.

Remark 6.1. Basing on the property of strict concavity of the Hamiltonian with respect to variable s, we compose conditions of the maximum principal of Pontryagin without restrictions on control parameter s and then check that the obtained optimal solution satisfies these restrictions.

The maximum value of utility flow is achieved when the optimal condition takes place

$$\frac{\partial H}{\partial s} = -\frac{1}{1-s} + \psi\, f(k) = 0 \tag{6.7}$$

at the optimal investment level

$$s^0 = 1 - \frac{1}{\psi\, f(k)}. \tag{6.8}$$

For shadow prices one can compose the dynamics of adjoint equation

$$\dot{\psi} = \delta\,\psi - \frac{\partial H}{\partial k} = \delta\,\psi - \frac{f'(k)}{f(k)} - \psi\,(s\, f'(k) - \lambda), \tag{6.9}$$

which balances the increment in flow and the change in price.

Substituting optimal investments s^0 (6.8) into shadow price ψ (6.9) and capital per worker k (4.2) dynamics, we compile the Hamiltonian system of differential equations for the problem (4.3):

$$\begin{cases} \dot{\psi} = \psi\left(\delta + \lambda - f'(k)\right) \\ \dot{k} = f(k) - \lambda\, k - \dfrac{1}{\psi} \end{cases}. \tag{6.10}$$

7. Qualitative Analysis of the Hamiltonian System

Let us introduce the cost function for the capital per worker growth as the multiplication of capital per worker k and shadow price ψ :

$$z = \psi\, k \tag{7.1}$$

One can differentiate cost function z with respect to time t using product rule:

$$\dot{z} = \dot{\psi}\, k + \psi\, \dot{k} \tag{7.2}$$

Combining (6.9), (7.1) and (7.2), we obtain that the cost function is subject to the growth dynamics:

$$\dot{z} = z\,(\frac{f(k)}{k} + \delta - f'(k)) - 1 \tag{7.3}$$

Let us analyze the Hamiltonian system of the optimal control problem as the system of two equations for the cost function z and capital per worker stock k

$$\begin{cases} \dot{z} = z\,(\dfrac{f(k)}{k} + \delta - f'(k)) - 1 \\ \dot{k} = f(k) - \lambda\, k - \dfrac{k}{z} \end{cases} \tag{7.4}$$

Lemma 7.1. The value of the growth rate of cost function z subject to dynamics (7.3) for $\dot{z}(t) > 0$ is larger than the discount parameter δ .

Proof. Let us estimate the value of the function in brackets of equation (7.3). First, let us examine the sign of the following function of capital per worker k

$$n(k) = f(k) - k\, f'(k) \tag{7.5}$$

This function $n(k)$ is equal to zero when $k = 0$

$$n(0) = f(0) - 0\, f'(0) = 0 \tag{7.6}$$

Let us show that $n(k)$ is a monotonically growing function for $k > 0$. Really, let us find a first derivative of $n(k)$ with respect to k

$$\frac{\partial n(k)}{\partial k} = f'(k) - f'(k) - k\, f''(k) = -k\, f''(k) > 0 \tag{7.7}$$

One can see that it is positive due to negativity of the second derivative of the production function (3.15). Inequality (7.7) implies that function $n(k)$ is positive. On the other hand, this means that the following inequality takes place

$$n(k)+\delta=\frac{f(k)}{k}+\delta-f'(k)>\delta\,, \tag{7.8}$$

which proofs the Lemma.

Remark 7.1. Lemma 7.1 implies that for $\dot{z}(t)>0$ cost function z subject to dynamics (7.3) in the Hamiltonian system (7.4) does not satisfy the transversality condition $\lim_{t\to+\infty} e^{-\delta t} z(t)=0$ as the growth rate of the cost function increases the level of the discount parameter δ.

Let us find the steady state of the Hamiltonian system (7.4). We solve the following system of equations:

$$\begin{cases} \dot{z}=z\left(\dfrac{f(k)}{k}+\delta-f'(k)\right)-1=0 \\ \\ \dot{k}=f(k)-\lambda k-\dfrac{k}{z}=0 \end{cases} \tag{7.9}$$

Lemma 7.2. There exists a unique steady state for the Hamiltonian system (7.4).

Proof. Solving the system of equations (7.9) with respect to capital per worker k we obtain that the steady state is the solution of the following equation:

$$f'(k)=\lambda+\delta \tag{7.10}$$

Let us recall the properties of the per worker LINEX production function (3.15)-(3.16). The first derivative $f'(k)$ in the left side of equation (7.10) is positive and monotonically declining to zero. Geometrically it implies that there exists a point of intersection of $f'(k)$ and constant positive function $\lambda+\delta$. The k coordinate of this unique point of intersection coincides with the solution k^* of equation (7.10).

This means that there is a unique steady state (k^*, z^*) of the system (7.4) that is calculated as follows:

$$\begin{cases} f'(k^*)=\delta+\lambda \\ \\ \dfrac{1}{z^*}=\dfrac{f(k^*)}{k^*}-\lambda \end{cases} \tag{7.11}$$

Here parameter k^* is the steady state of capital per worker defined by equation (7.10). This proves the Lemma.

Remark 7.2. One can write an explicit equation for capital per worker substituting the first derivative of the per worker LINEX production function (3.14) to (7.10). Then the steady state of capital per worker k^* is a solution of the equation:

$$k^{\alpha-1}\,\tilde{u}^{(1-\alpha-\beta)}\left(\alpha + k\gamma - \xi\frac{\tilde{u}}{k}\right)\exp\left\{\gamma k + \mu\frac{1}{\tilde{u}} + \xi\frac{\tilde{u}}{k}\right\} = \delta + \lambda \tag{7.12}$$

It is hard to solve this equation analytically and further we calculate the values of the steady state (k^*, z^*) using a numerical algorithm.

Lemma 7.3. At the steady state (k^*, z^*) (7.11) the optimal investment level s^0 (6.8) is bounded by some positive value which is less than the unit value.

Proof. Let us assess the value of optimal investment plan s^0 at the steady state. Rewriting expression (6.8) in (k, z) coordinates, we obtain

$$s^0 = 1 - \frac{k}{z\, f(k)} \tag{7.13}$$

One can calculate s^0 at the steady substituting (k^*, z^*) to (7.13)

$$s^0(k^*) = \lambda\frac{k^*}{f(k^*)} \tag{7.14}$$

Let us estimate the value of the following function

$$g(k^*) = \frac{k^*}{f(k^*)} - \frac{1}{f'(k^*)} = \frac{k^* f'(k^*) - f(k^*)}{f(k^*) f'(k^*)} < 0 \tag{7.15}$$

Function $g(k^*)$ is negative due to positiveness of the production function $f(k)$ and its derivative $f'(k)$ (3.15) with respect to capital per worker k, and to positiveness of function $n(k)$ (7.5). Inequality (7.15) implies

$$\frac{k^*}{f(k^*)} < \frac{1}{f'(k^*)} \tag{7.16}$$

Using (7.16) one can estimate the value of the optimal investment level

$$s^0(k^*) = \lambda\frac{1}{f'(k^*)} < \lambda\frac{1}{f'(k^*)} \tag{7.17}$$

Substituting into this relation the value of $f'(k^*)$ (7.10) we obtain

$$s^0(k^*) < \frac{\lambda}{\lambda + \delta} < 1. \tag{7.18}$$

Remark 7.3. Let us introduce the following notation

$$a = \frac{\lambda}{\lambda + \delta} \quad \textbf{(7.19)}$$

Namely, this value of parameter a one can take as the upper bound for the control restrictions $s \in [0, a]$ (4.3).

8. Analysis of the Saddle Type Character of the Steady State

Let us linearize the nonlinear Hamiltonian system (7.4) in the neighborhood of the steady state (k^*, z^*) (7.11). We introduce functions $F_1(k,z)$ and $F_2(k,z)$ of two variables:

$$\begin{cases} F_1(k,z) = f(k) - \lambda k - \dfrac{k}{z} \\ F_2(k,z) = z\left(\dfrac{f(k)}{k} + \delta - f'(k)\right) - 1 \end{cases} \quad \textbf{(8.1)}$$

The linearized system in the neighborhood of the steady state (k^*, z^*) can be presented by the system of differential equations:

$$\begin{cases} \dot{k} = 0 + \dfrac{\partial F_1}{\partial k}(k^*, z^*)(k - k^*) + \dfrac{\partial F_1}{\partial z}(k^*, z^*)(z - z^*) \\ \dot{z} = 0 + \dfrac{\partial F_2}{\partial k}(k^*, z^*)(k - k^*) + \dfrac{\partial F_2}{\partial z}(k^*, z^*)(z - z^*) \end{cases} \quad \textbf{(8.2)}$$

One can calculate the corresponding partial derivatives of functions $F_1(k,z)$ and $F_2(k,z)$:

$$\frac{\partial F_1}{\partial k} = f'(k) - \lambda - \frac{1}{z}, \quad \textbf{(8.3)}$$

$$\frac{\partial F_1}{\partial z} = \frac{k}{z^2}, \quad \textbf{(8.4)}$$

$$\frac{\partial F_2}{\partial k} = -z\,\frac{f(k) - k\,f'(k) + k^2\,f''(k)}{k^2}, \quad \textbf{(8.5)}$$

$$\frac{\partial F_2}{\partial z} = \frac{f(k)}{k} + \delta - f'(k). \quad \textbf{(8.6)}$$

Substituting the values of partial derivatives (8.3)-(8.6) at the steady state into (8.2) we obtain the linearized Hamiltonian system in the neighborhood of the steady state (k^*, z^*):

$$\begin{cases} \dot{k} = \left(f'(k^*) - \lambda - \dfrac{1}{z^*} \right)(k - k^*) + \dfrac{k^*}{(z^*)^2}(z - z^*) \\ \dot{z} = \left(-z^* \dfrac{f(k^*) - k^* f'(k^*) + (k^*)^2 \; f''(k^*)}{(k^*)^2} \right)(k - k^*) + \left(\dfrac{f(k^*)}{k^*} + \delta - f'(k^*) \right)(z - z^*) \end{cases} \quad \textbf{(8.7)}$$

Introducing matrix A

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} f'(k^*) - \lambda - \dfrac{1}{z^*} & \dfrac{k^*}{(z^*)^2} \\ -\dfrac{z^*}{(k^*)^2}\left(f(k^*) - k^* f'(k^*) + (k^*)^2 f''(k^*) \right) & \dfrac{f(k^*)}{k^*} + \delta - f'(k^*) \end{pmatrix}, \quad \textbf{(8.8)}$$

one can rewrite the linear system of equations (8.7) in the matrix form:

$$\begin{pmatrix} \dot{k} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} (k - k^*) \\ (z - z^*) \end{pmatrix} \quad \textbf{(8.9)}$$

Here a_{ij}, $i = 1,2$, $j = 1,2$ are elements of matrix A.

Let us find eigenvalues of matrix A. We solve the following equation

$$\det(A - \chi E) = \det \begin{pmatrix} a_{11} - \chi & a_{12} \\ a_{21} & a_{22} - \chi \end{pmatrix} = 0 \quad \textbf{(8.10)}$$

Here parameter χ denotes eigenvalue and $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is an identity matrix.

Calculating the determinant (8.10) we obtain the characteristic equation for eigenvalues:

$$\chi^2 - (a_{11} + a_{22})\,\chi + a_{11}a_{22} - a_{12}a_{21} = 0 \quad \textbf{(8.11)}$$

Let us rewrite this quadratic equation in the following form

$$\chi^2 - \chi\, tr + d = 0 \quad \textbf{(8.12)}$$

with coefficients

$$d = \det A = a_{11}a_{22} - a_{12}a_{21} \text{ – the determinant of matrix } A, \quad \textbf{(8.13)}$$

$$tr = \text{trace} A = a_{11} + a_{22} \text{ – the trace of matrix } A. \tag{8.14}$$

Lemma 8.1. The value of the trace tr of matrix A is positive and the value of the determinant d of matrix A is negative at the steady state (k^*, z^*).

Proof. Let us calculate the trace of matrix A as the sum of the elements on the main diagonal:

$$tr = f'(k^*) - \lambda - \frac{1}{z^*} + \frac{f(k^*)}{k^*} + \delta - f'(k^*) = \left(\frac{f(k^*)}{k^*} - \lambda \right) - \frac{1}{z^*} + \delta \tag{8.15}$$

Substituting $\frac{1}{z^*} = \frac{f(k^*)}{k^*} - \lambda$ from (7.11) we obtain

$$tr = \frac{1}{z^*} - \frac{1}{z^*} + \delta = \delta > 0 \tag{8.16}$$

The trace tr of matrix A is equal to the positive discount parameter $\delta > 0$.

Let us calculate the determinant d (8.13) of matrix A. Calculating the product of main diagonal elements and substituting into this product expressions for the steady state (k^*, z^*) (7.11), one can obtain:

$$a_{11}\, a_{22} = \frac{1}{z^*} \left(f'(k^*) - \frac{f(k^*)}{k^*} \right) \tag{8.17}$$

For the product of elements a_{12} and a_{21} we have

$$a_{12}\, a_{21} = \frac{1}{z^* k^*} \left(f(k^*) - k^* f'(k^*) + (k^*)^2 f''(k^*) \right) \tag{8.18}$$

The determinant d of matrix A is equal

$$d = a_{11}\, a_{22} - a_{12} a_{21} = \frac{k^*}{z^*} f''(k^*) < 0 \tag{8.19}$$

The value of the determinant d of matrix A at the steady state is negative due to the positiveness of coordinates $k^* > 0$ and $z^* > 0$, and negativeness of the second derivative $f''(k^*)$ (3.15) of the LINEX per worker production function.

Lemma 8.2. There are two real eigenvalues of system (8.7). One of them is positive and another one is negative.

Proof. Let us analyze the characteristic quadratic equation (8.12). The roots of this equation are given by the quadratic formula:

$$\chi_{1,2} = \frac{tr \pm \sqrt{(tr)^2 - 4d}}{2} \tag{8.20}$$

The roots of equation (8.12) are real due to the positiveness of the discriminant:

$$(tr)^2 - 4d > 0 \tag{8.21}$$

since the determinant is negative, $d < 0$, according to Lemma 8.1.

Let us analyze the sign of the first solution of equation (8.12)

$$\chi_1 = \frac{tr - \sqrt{(tr)^2 - 4d}}{2} \tag{8.22}$$

Since the value of the determinant d is negative (8.19), then we have the following inequality

$$(tr)^2 - 4d > (tr)^2, \tag{8.23}$$

which is equivalent to the inequality

$$\sqrt{(tr)^2 - 4d} > tr \tag{8.24}$$

due to the positiveness of the trace $tr > 0$ (8.16).

Thus, the first eigenvalue of system (8.7) is negative $\chi_1 < 0$.

One can see that the value of the second eigenvalue is positive

$$\chi_2 = \frac{tr + \sqrt{(tr)^2 - 4d}}{2} > 0 \tag{8.25}$$

since it is presented by the sum of positive values. That completes the proof of Lemma.

Lemma 8.3. The value of positive eigenvalue χ_2 (8.25) is larger than the discount parameter $\delta > 0$.

Proof. Let us substitute the value of the trace tr (8.16) to (8.25). From inequality (8.24) we have

$$\sqrt{\delta^2 - 4d} > \delta \tag{8.26}$$

This leads to the inequality

$$\chi_2 = \frac{\delta + \sqrt{\delta^2 - 4d}}{2} > \frac{\delta + \delta}{2} = \delta \tag{8.27}$$

The last inequality proofs the Lemma.

Remark 8.1. Since eigenvalues χ_1 and χ_2 are real and have different signs then the steady state (k^*, z^*) is a saddle point. It means that for the linearized system only two trajectories converge to the steady state along the direction defined by the eigenvector corresponding to the negative eigenvalue.

Remark 8.2. According to the Grobman-Hartman theorem (see [7]) nonlinear system (7.4) admits a trajectory as well as linear system (8.7) which converges to equilibrium (k^*, z^*) and is tangent to the eigenvector corresponding to the negative eigenvalue.

Remark 8.3. In the case when the per worker production function $f(k)$ is strictly concave in capital k then one can prove that the trajectory which converges to equilibrium (k^*, z^*) is the unique optimal trajectory.

9. Qualitative Analysis of the Vector Field of the Hamiltonian System

The vector field of the Hamiltonian system (7.4) is defined by signs of the following equations

$$h_1(k) = \frac{k}{f(k) - k\,f'(k) + \delta\,k} \tag{9.1}$$

$$h_2(k) = \frac{k}{f(k) - \lambda\,k} \tag{9.2}$$

Let us estimate derivative of function $h_2(k)$

$$\frac{\partial h_2(k)}{\partial k} = \left(\frac{k}{f(k) - \lambda\,k}\right)' = \frac{f(k) - \lambda\,k - k(f'(k) - \lambda)}{(f(k) - \lambda\,k)^2} =$$

$$\frac{f(k) - \lambda\,k - k\,f'(k) - k\,\lambda}{(f(k) - \lambda\,k)^2} = \frac{f(k) - k\,f'(k)}{(f(k) - \lambda\,k)^2}$$

In the domain where $f(k) - k\,f'(k) > 0$, this derivative is positive and, hence, function $h_2(k)$ is a monotonically growing function. Let us note that such property is valid for functions satisfying conditions of Lemma 7.1.

One can prove that under reasonable assumptions the following relations hold for functions $h_1(k)$ and $h_2(k)$

$$\begin{cases} h_1(k) - h_2(k) > 0, \ k < k^* \\ h_1(k) - h_2(k) = 0, \ k = k^* \\ h_1(k) - h_2(k) < 0, \ k > k^* \end{cases} \tag{9.3}$$

where k^* is the first coordinate of the steady state (7.11).

Really, let us estimate the difference between these functions

$$h_1(k) - h_2(k) = \frac{k}{f(k) - k\, f'(k) + \delta k} - \frac{k}{f(k) - \lambda k} =$$

$$\frac{-\lambda k^2 + k^2 \ f'(k) - \delta k^2}{\left(f(k) - \lambda k\right)\left(f(k) - k\, f'(k) + \delta k\right)} = \frac{-\lambda k^2 + k^2 \ f'(k) - \delta k^2}{\left(f(k) - \lambda k\right)\left(f(k) - k\, f'(k) + \delta k\right)} =$$

$$\frac{k^2 \left(f'(k) - (\lambda + \delta)\right)}{\left(f(k) - \lambda k\right)\left(f(k) - k\, f'(k) + \delta k\right)}.$$

It is reasonable to assume that $f(k) - \lambda k > 0$ as it defines the positive rate of capital growth. According to Lemma 7.1, relation $f(k) - k\, f'(k) + \delta k > 0$ is also strictly positive. Hence, the sign of the difference is defined by relation $f'(k) - (\lambda + \delta)$ for the steady state (7.11). It is positive when $k < k^*$, and negative when $k > k^*$.

The steady state (k^*, z^*) is the unique (see Lemma 7.2).

Let us note that function $h_1(k)$ may not have a monotonic property since its derivative is defined by the following relation

$$\frac{\partial h_1(k)}{\partial k} = \left(\frac{k}{f(k) - k\, f'(k) + \delta k}\right)' = \frac{f(k) - k\, f'(k) + \delta k - k(f'(k) - f'(k) - k\, f''(k) + \delta)}{\left(f(k) - k\, f'(k) + \delta k\right)^2} =$$

$$= \frac{f(k) - k\, f'(k) + k^2\, f''(k)}{\left(f(k) - k\, f'(k) + \delta k\right)^2},$$

and the numerator $f(k) - k\, f'(k) + k^2\, f''(k)$ of the last ratio may change the sign.

The qualitative portrait of the vector field reflecting the indicated above properties of functions $h_1(k)$ and $h_2(k)$ is given on Fig. 9.1.

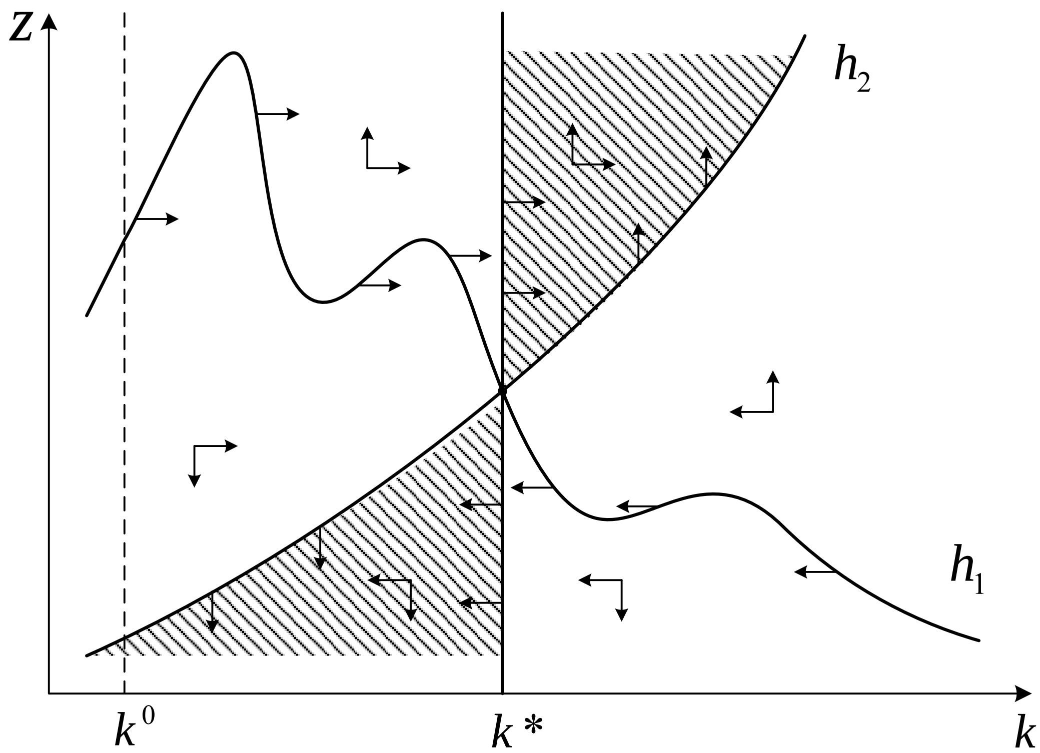

Fig. 9.1. Qualitative analysis of the vector field of the Hamiltonian system

Let us describe the domain which is strongly invariant with respect to dynamics of the Hamiltonian system (7.4). For this purpose, we indicate the vertical line $l^0 = \{(k,z): k = k^*\}$. The invariant domain V^0 lies between the vertical line l^0 and the curve h_2, and is determined by the following formula

$$V^0 = \{(k,z): z \le h_2(k) \ \mathit{for}\ k < k^*;\ z \ge h_2(k) \ \mathit{for}\ k > k^*\} \cup l^0 \setminus (k^*, z^*) \quad \textbf{(9.4)}$$

The properties of the vector field clearly demonstrate that if a trajectory of the Hamiltonian system starts or enters this domain then it remains in this domain on the infinite horizon and does not reach the steady state.

Let us consider trajectories of the Hamiltonian system which start at the initial point k^0 which is located to the left of the steady state $k^0 < k^*$ and does not belong to the invariant set V^0. For these trajectories three variants are possible due to nonzero horizontal velocity $\dot{k} > 0$:

1. at some moment of time τ the trajectory falls on the curve h_2 and $k(\tau) < k^*$; in this case the trajectory enters the invariant domain V^0 and stays in it forever;
2. at some moment of time τ the trajectory reaches the vertical line l^0 and $z(\tau) > z^*$; in this case the trajectory also enters the invariant domain V^0 and stays in it forever;
3. the trajectory tends to the steady state (k^*, z^*) on the infinite horizon.

It proves that a trajectory which satisfies the normal-form core Pontryagin maximum principle condition and transversality condition and starts at the initial point $k^0 < k^*,\ (k^0, z^0) \notin V^0$ should converge to the steady state (k^*, z^*). Analogously to proof of Lemma 6.3. from the paper [4] one can show that the trajectory converging to the steady state is unique. Then, due to the existence and uniqueness result for the Pontryagin maximum principle, this trajectory converging to the steady state is the optimal trajectory.

10. Numerical Algorithm

To construct synthetic trajectories of optimal economic growth we design a numerical algorithm for simulation of the model.

At the first step of algorithm we calculate the value of steady state (k^*, z^*). To make this, we solve equation (7.10) numerically. In other words, we find the unique point of intersection of first derivative $f'(k)$ of the LINEX production function with respect to k and the constant positive scalar function $p(k) = \lambda + \delta$.

The point of intersection of these two functions can be found numerically using method of graduation of variable k. We choose initial k_1 and final k_2 values of k and split the segment $[k_1, k_2]$ into two equal parts by point $k_s = (k_1 + k_2)/2$. Then we calculate the values of function $f'(k)$ in these three points to find which segment contains the point of intersection $[k_1, k_s]$ or $[k_s, k_2]$. This method is based on the property of declination of $f'(k)$. If the segment $[k_1, k_2]$ contains the point of intersection then the following conditions are satisfied:

$$\begin{cases} f'(k_1) > \delta + \lambda \\ \\ f'(k_2) < \delta + \lambda \end{cases} \tag{10.1}$$

Basing on this rule, we shift one of the ends of the segment to point k_s. And repeat this procedure until we find the point of intersection k^* of functions $f'(k)$ and $p(k) = \lambda + \delta$ with an arbitrarily given accuracy $\varsigma > 0$. This unique point k^* of intersection is the solution of equation (7.10) and is the first coordinate of a steady state. We calculate value z^* from the system of equations (7.11).

We consider the Hamiltonian system (7.9) of differential equations for Pontryagin maximum principle and linearize it in a neighborhood of steady state (k^*, z^*) (8.7). We find the negative eigenvalue χ_1 (8.22) and coordinates (x_1, x_2) of the corresponding eigenvector for construction of the optimal trajectory.

To construct a synthetic trajectory we choose a small precision parameter $\varepsilon > 0$. This parameter defines a neighborhood of the steady state (k^*, z^*). We integrate the linear system along the eigenvector with coordinates (x_1, x_2) in this neighborhood starting from the point (k^*, z^*). As a result, we obtain the point with coordinates:

$$\begin{pmatrix} k_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} k^* \\ z^* \end{pmatrix} + \varepsilon \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{10.2}$$

We integrate the system of nonlinear equations (7.9) starting from the point (k_0, z_0) in the reverse time. The numerical integration can be performed in a discrete Euler scheme:

$$\begin{cases} z_i = z_{i-1} - \left(z_{i-1} \left(\frac{f(k_{i-1})}{k_{i-1}} + \delta - f'(k_{i-1}) \right) - 1 \right) \Delta t \\ \\ k_i = k_{i-1} - \left(f(k) - \lambda k - \frac{z}{k} \right) \Delta t \end{cases} \tag{10.3}$$

with a time step Δt.

The stopping criterion for numerical integration in the reverse time is the value $k(0) = k^0$ of capital per worker coordinate k at the initial moment of time (see the statement of the optimal control problem (4.3)). We save the values of the coordinates

(k, z) of the optimal trajectories for each moment of time t and then represent them in real time in order to construct optimal trajectories in the direct time. The accuracy of the proposed numerical algorithm can be estimated using constructions of papers [6, 9, 15].

See illustration of the algorithm on Fig. 10.1.

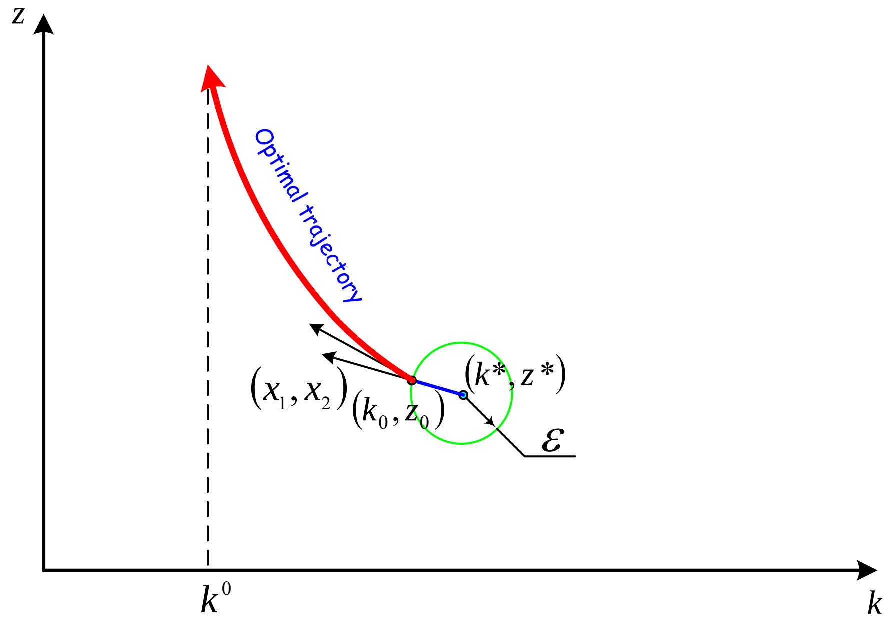

Fig. 10.1. Illustration of the algorithm

11. Results of Numerical Simulation

The numerical algorithm is realized in the software for constructing optimal trajectories and optimal levels of investments. The model is simulated using the elaborated software with the following input parameters. The average level of per worker useful work (3.13) is taken on the level $\tilde{u} = 5$. Parameters of the per worker LINEX production function $f(k)$ (3.14) are identified using econometric analysis within the frame of SPSS Sigma Stat 3.0 package. Their values are estimated by the following figures: $A = 0.44$, $\alpha = 2.216$, $\beta = -0.909$, $\gamma = -0.369$, $\mu = 8.221$, $\xi = 0.0208$.

The discount parameter δ (4.1) and the rate of capital depreciation and dilution λ (4.2) are identified at the following levels: $\delta = 0.05$, $\lambda = 0.03$. The model is simulated for the following parameters of numerical integration of the Hamiltonian system (10.2)-(10.3): precision parameter $\varepsilon = 0.001$, time step $\Delta t = 0.0001$. The stopping criterion $k(0) = k^0$ (4.3) for integration in reverse time is $k^0 = 1$.

The results of numerical solution of equation (7.10) is shown on Fig. 11.1.

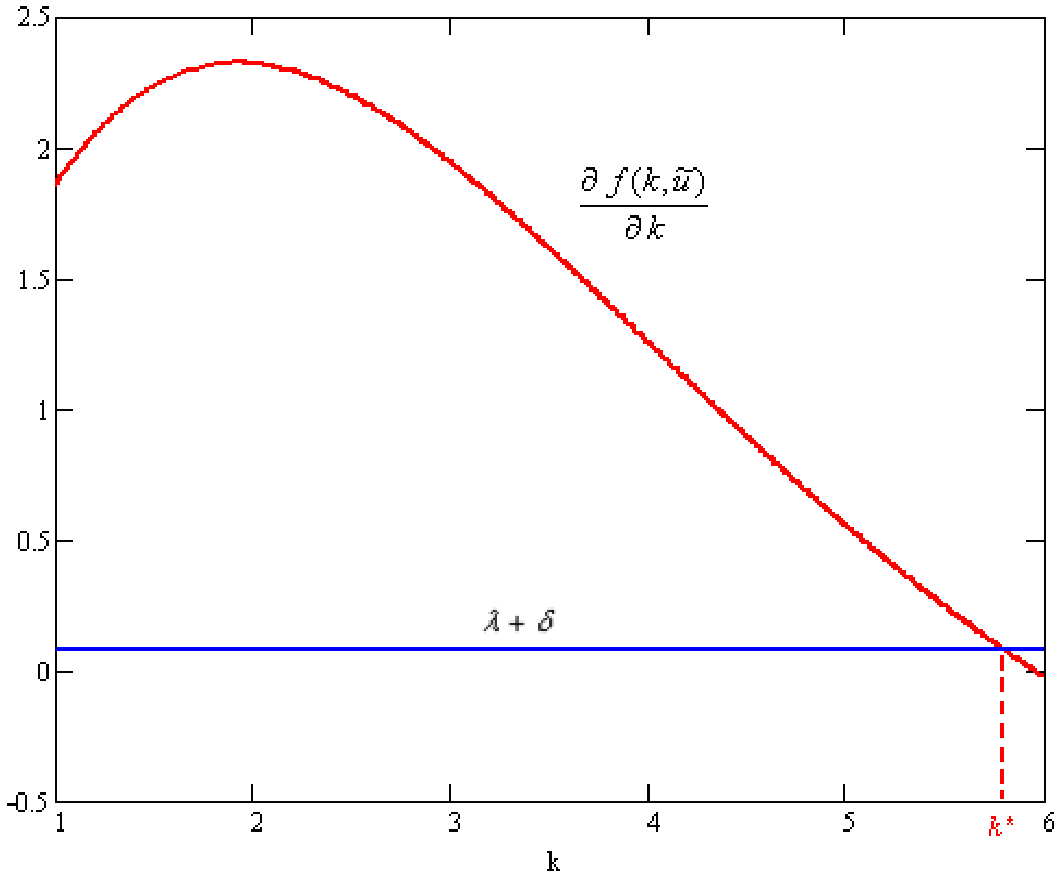

Fig. 11.1. Result of numerical calculation of the steady state

The results of numerical calculation of the steady state are presented by the following values:

$$\begin{pmatrix} k^* \\ z^* \end{pmatrix} = \begin{pmatrix} 5.80314 \\ 0.72441 \end{pmatrix}$$

The eigenvalues (8.10) for the linearized Hamiltonian system (8.7) are calculated as follows: $\chi_1 = -1.39129$, $\chi_2 = 1.44129$. According to the theoretical results only the negative eigenvalue corresponds to the optimal trajectory. The coordinates of the eigenvector corresponding to the eigenvalue χ_1 are estimated as follows:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -0.999 \\ 0.0055 \end{pmatrix}$$

The numerical calculations determine the structure of the vector field of the Hamiltonian system depicted on Fig. 11.2.

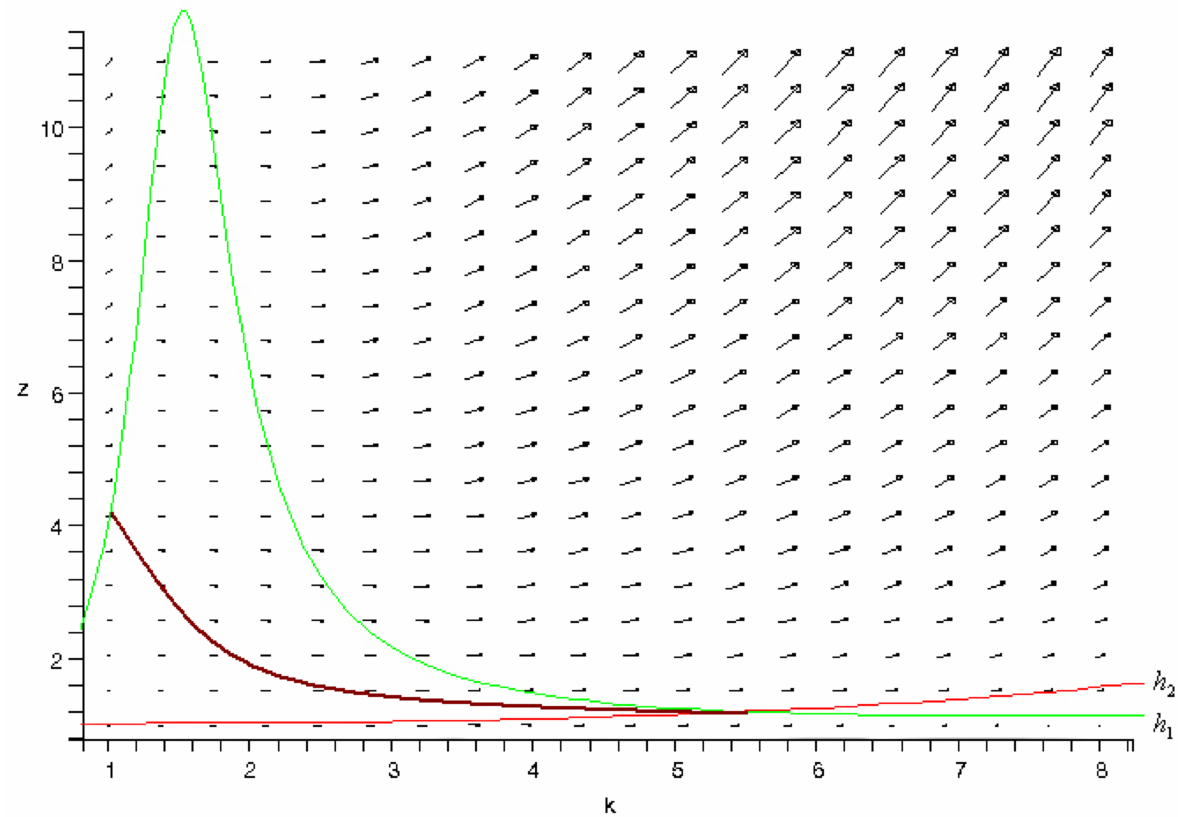

Fig. 11.2. Numerical simulations for the vector field of the Hamiltonian system

The system is linearly integrated along the eigenvector according to algorithm (10.2). The initial coordinates for numerical integration (10.3) of the nonlinear system in the reverse time are selected on the basis of the precision parameter ε:

$$\begin{pmatrix} k_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} 5.802 \\ 0.724 \end{pmatrix}$$

The results of construction of the synthetic trajectory is presented on Fig. 11.3. The nonlinear Hamiltonian system of equations is integrated in the reverse time until the stopping criterion $k^0 = 1$.

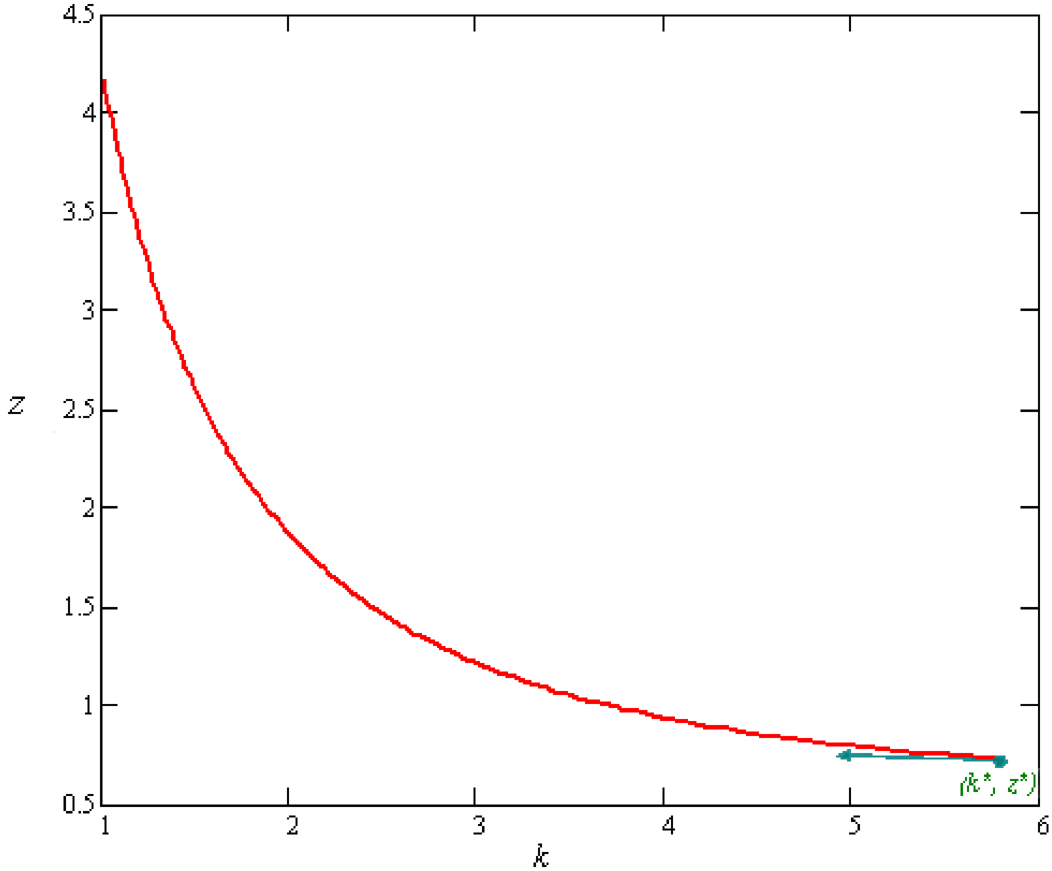

Fig. 11.3. Numerical integration of the Hamiltonian system

12. Comparison of Real and Synthetic Trajectories and Forecasting

In this part of the research we compare synthetic trajectories of optimal economic growth with actual trajectories presented by the US data. This block completes the methodology considered in the scheme depicted on Fig. 1.1. The results of comparison of synthetic trajectories of optimal growth and real data for capital per worker is presented on Fig. 12.1.

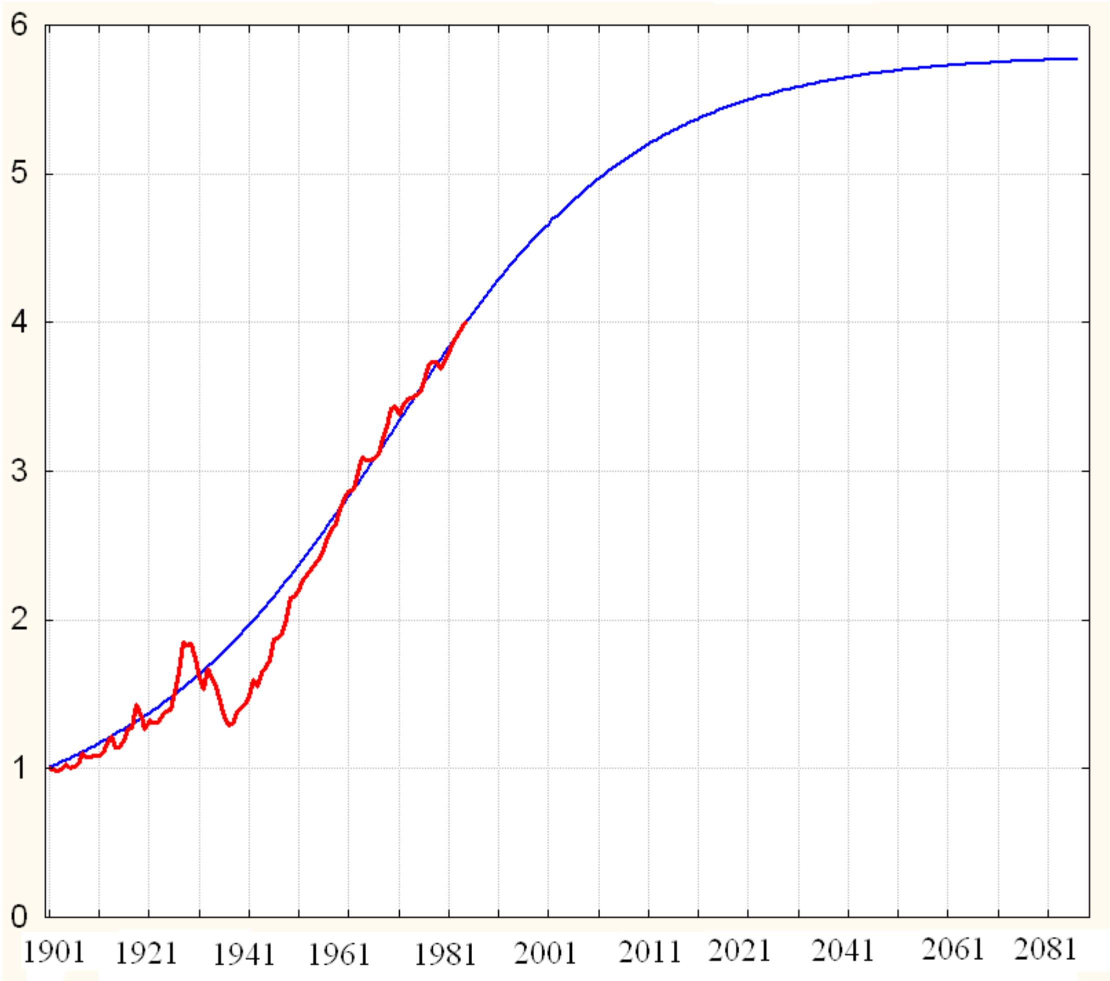

Fig. 12.1. Comparison of real and synthetic trajectories

On Fig. 12.1. the optimal trajectory is shown in blue and the real values of capital per worker from the US data are depicted in red. One can see that results obtained in the optimal control problem fit well to the actual growth trends of capital stock. It is worth to note that the optimal trajectory even follows the structural changes in the data during the postwar economic crisis.

The qualitative analysis of the synthetic optimal trajectory shows that it has S-shape: in the period 1900-1970 it is subject to the effect of increasing returns, in 1970 one can observe the inflection point, and in the period after 1970 it is subject to the effect of decreasing returns. The S-shape shows that at the beginning of the century growth of capital had increasing returns to scale in time. This fact can be interpreted from the economical point of view by high influence of electrification on economic growth. The second qualitative feature of the synthetic optimal trajectory consists in the fact that it converges to the steady state approximately in 2080 and this convergence indicates saturation of the capital stock.

The trajectories of the GDP growth are shown on Fig. 12.2.

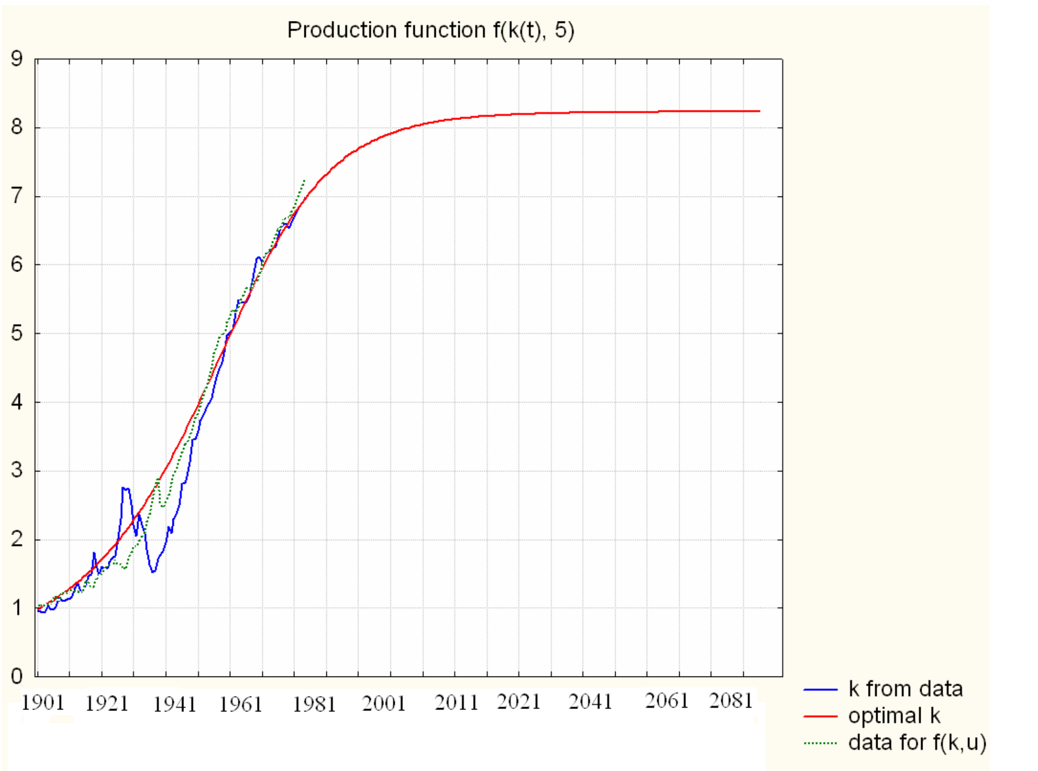

Fig. 12.2. Simulation of future scenarios of economic growth

The trajectory of optimal growth is depicted in the red color, the data is shown in green, and the graph of the per worker LINEX production function calibrated econometrically is indicated in blue. The comparison of synthetic trajectories with actual trajectories and econometric results shows good fitness of all three objects. This fact reveals an adequate choice of components of the model: the LINEX production function, economical model of capital dynamics and the utility function in the optimal control problem.

Basing on the results of the model one can simulate the future scenarios of economic growth. For the data of US such forecasts show that by the year 2041 capital per worker would saturate at the level equal to 5.8 basic levels normalized to 1900 in time series. It is important to note that saturation of GDP per worker could start around the year 2011.

13. Analysis of the Steady State for the Model with Two Economic Variables

In this section we analyze the steady state for the model whose dynamics include the variable for the useful work. Following the economic model for the capital per worker growth (3.12), one can obtain differential equation for the useful work per worker growth. We assume that one part s_1 of savings s is invested into construction of new capital

$$\dot{k} = s_1 f(k,u) - \lambda k, \tag{13.1}$$

and another part s_2 of savings s is invested into accumulation of per worker useful work

$$\dot{u} = s_2 f(k,u) - \vartheta u. \tag{13.2}$$

Here $\vartheta > 0$ is a constant rate of depreciation of useful work.

Let us treat investments s_1 and s_2 as control parameters. One can consider an optimal control problem for maximization of the integral discounted consumption index presented by the utility function

$$J = \int_0^{+\infty} \left[\ln f(k(t),u(t)) + a_1 \ln(1 - s_1(t)) + a_2 \ln(1 - s_2(t))\right] e^{-\delta t} dt \to \max \tag{13.3}$$

Here δ is a time discount parameter, a_1 and a_2 are given positive scalars, $f(k,u)$ is a per worker LINEX production function (3.6) of two variables. The problem is subject to initial conditions for capital per worker $k(0) = k^0$ and useful work per worker $u(0) = u^0$.

Using conditions of Pontryagin maximum principle one can obtain the Hamiltonian system of equations of the problem

$$
\begin{cases}
\dot{k} = f - \lambda k - \dfrac{a_1 k}{z_1} \\
\dot{u} = f - \vartheta u - \dfrac{a_2 u}{z_2} \\
\dot{z}_1 = \left(\delta + \dfrac{f}{k} - \dfrac{\partial f}{\partial k}\right) z_1 - \dfrac{k}{u}\dfrac{\partial f}{\partial k} z_2 + \dfrac{(a_1 + a_2 - 1)}{f}\dfrac{\partial f}{\partial k} k - a_1 \\
\dot{z}_2 = -\dfrac{u}{k}\dfrac{\partial f}{\partial u} z_1 + \left(\delta + \dfrac{f}{u} - \dfrac{\partial f}{\partial u}\right) z_2 + \dfrac{(a_1 + a_2 - 1)}{f}\dfrac{\partial f}{\partial u} u - a_2
\end{cases}
\quad \textbf{(13.4)}
$$

Here z_1 and z_2 are cost functions of capital per worker and useful work per worker respectively.

The research is focused on the qualitative analysis of the Hamiltonian system (13.4) and development of numerical algorithm for calculation of the steady state and construction of synthetic optimal trajectories of economic growth.

The steady state of the Hamiltonian system is described by the following four-dimensional system of algebraic equations

$$
\begin{cases}
z_1 = \dfrac{a_1 k}{f - \lambda k} \\
z_2 = \dfrac{a_2 u}{f - \vartheta u} \\
\left(\delta + \dfrac{f}{k} - \dfrac{\partial f}{\partial k}\right)\left(\dfrac{a_1 k}{f - \lambda k}\right) - \dfrac{k}{u}\dfrac{\partial f}{\partial k}\left(\dfrac{a_2 u}{f - \vartheta u}\right) + \dfrac{(a_1 + a_2 - 1)}{f}\dfrac{\partial f}{\partial k} k - a_1 = 0 \\
-\dfrac{u}{k}\dfrac{\partial f}{\partial u}\left(\dfrac{a_1 k}{f - \lambda k}\right) + \left(\delta + \dfrac{f}{u} - \dfrac{\partial f}{\partial u}\right)\left(\dfrac{a_2 u}{f - \vartheta u}\right) + \dfrac{(a_1 + a_2 - 1)}{f}\dfrac{\partial f}{\partial u} u - a_2 = 0
\end{cases}
\quad \textbf{(13.5)}
$$

Substituting expression for z_1 and z_2 from the first two equations to the third and the fourth equation, we obtain:

$$
\begin{cases}
\left(\delta k + f - k\dfrac{\partial f}{\partial k}\right)\dfrac{a_1}{(f - \lambda k)} - k\dfrac{\partial f}{\partial k}\dfrac{a_2}{(f - \vartheta u)} + \dfrac{(a_1 + a_2 - 1)}{f}\dfrac{\partial f}{\partial k} k - a_1 = 0 \\
-u\dfrac{\partial f}{\partial u}\dfrac{a_1}{(f - \lambda k)} + \left(\delta u + f - u\dfrac{\partial f}{\partial u}\right)\dfrac{a_2}{(f - \vartheta u)} + \dfrac{(a_1 + a_2 - 1)}{f}\dfrac{\partial f}{\partial u} u - a_2 = 0
\end{cases}
$$

To calculate the values of steady state, we introduce functions $F_1(k,u)$ and $F_2(k,u)$ for the transformed left hand sides of equations for the steady state:

$$\begin{cases} F_1 = a_1\left(\dfrac{\partial f}{\partial k} - \delta - \lambda\right) + \dfrac{\partial f}{\partial k} a_2 \dfrac{(f-\lambda k)}{(f-\vartheta u)} + \dfrac{(a_1 + a_2 - 1)}{f} \dfrac{\partial f}{\partial k}(f - \lambda k) \\ \\ F_2 = a_2\left(\dfrac{\partial f}{\partial u} - \delta - \vartheta\right) a_2 + \dfrac{\partial f}{\partial u} a_1 \dfrac{(f-\vartheta u)}{(f-\lambda k)} - \dfrac{(a_1 + a_2 - 1)}{f} \dfrac{\partial f}{\partial u}(f - \vartheta u) \end{cases}$$

Then, one can find the steady state by numerical integration of the following system of equations:

$$\begin{cases} k_i = k_{i-1} + F_1(k_{i-1}, u_{i-1})\,\Delta t \\ \\ u_i = u_{i-1} + F_2(k_{i-1}, u_{i-1})\,\Delta t \end{cases} \qquad \textbf{(13.6)}$$

The experiments on numerical calculation give the following figures for the steady state: $\left(k^*, u^*, z_1^*, z_2^*\right) = \left(6.101, 28.109, 0.613, 2.854\right)$.

The estimation of eigenvalues and eigenvectors of the linearized system at the steady state demonstrates the saddle character of this equilibrium. Really, calculations in Maple give the following figures for four eigenvalues written in the row

$$\left(\chi_1, \chi_2, \chi_3, \chi_4\right) = \left(-2.186, 2.236, 0.246, -0.196\right)$$

One can see that the first and the fourth eigenvalues are negative, and the second and the third eigenvalues are positive.

Calculations in Maple of eigenvectors provide the following matrix of four vectors written by columns and corresponding to the calculated eigenvalues

$$\left(y_1, y_2, y_3, y_4\right) = \begin{pmatrix} -9.370 & -3.878 & 0.249 & 0.462 \\ 0.355 & 0.144 & 6.503 & 12.103 \\ 0.347 & -0.914 & -0.003 & -0.019 \\ 0.016 & 0.192 & 0.988 & 0.292 \end{pmatrix}$$

On the basis of this analysis one can propose an algorithm for constructing the optimal trajectory of economic growth as the trajectory which converges to the steady state along an eigenvector belonging to the plane generated by basic eigenvectors corresponding to negative eigenvalues.

From the economical point of view the existence of the unique steady state specifies the saturation levels for growth of capital and useful work per worker. The proportion of saturation levels $r^* = u^* / k^* = 28.109 / 6.101 = 4.607$ shows that growth of capital per worker is saturated much quicker than growth of useful work per worker. It is expected that the following scenario would be quite plausible: in the first period, growth of GDP is determined mainly by capital growth; in the second period, both factors capital and useful work are significant for GDP growth; further, in the third period, one can observe saturation of capital while useful work is far from saturation – in this period, growth of GDP would be determined mainly by growth of useful work; finally, after saturation of useful work all proportions of economic growth are stabilized at the steady state.

14. Conclusion

This paper is devoted to development of an optimization model for describing economic growth of a country. Methods and approaches from various disciplines are combined in this research. Namely, elements of the theory of economic growth, optimal control theory, statistics, econometric analysis, and numerical methods are used in modeling. The complete cycle of research, including data analysis, parameters identification, dynamic optimization of trajectories of economic growth, and matching optimal trajectories with the real data dynamics, is fulfilled. Using methods of econometric analysis, we identify coefficients of the LINEX production function, one of the main components of the model, from the real data on the US economy. Then we consider dynamics for the capital growth in the model of economic growth of GDP of a country. In this part the sequential derivation of differential equations of the model is presented including economic explanations and assumptions. The next part of the paper is devoted to the statement of the optimal control problem, to its solution in the framework of the Pontryagin maximum principle and analysis of the Hamiltonian system of differential equations. The obtained theoretical results are used for construction of a numerical algorithm for calculation of the steady state and designing optimal synthetic trajectories of economic growth. The numerical algorithm is realized in the elaborated software for calibration of model parameters and simulation of optimal trajectories. The graphical output of real and synthetic trajectories of economic growth allows to compare results of modeling and trends of real data. This comparison shows good coincidence of actual and synthetic trajectories and, thus, demonstrates the adequateness of all components of the model: the LINEX production function, economical model of capital dynamics and the utility function in the optimal control problem. The qualitative analysis of the synthetic optimal trajectory shows that it has S-shape: in the period 1900-1970 it is subject to the effect of increasing returns, in 1970 one can observe the inflection point, and in the period after 1970 it is subject to the effect of decreasing returns. The S -shape shows that at the beginning of the century growth of capital had increasing returns to scale in time. This fact can be interpreted from the economical point of view by high influence of electrification on economic growth. The second qualitative feature of the synthetic optimal trajectory consist in the fact that it converges to the steady state approximately in 2080 and this convergence indicates saturation of the capital stock. It is important to note that saturation of GDP per worker could start around the year 2011. For the model with two economic factors: capital and useful work per worker, the qualitative analysis of the steady state is given in order to describe potential scenarios of the balanced economic growth.

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