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A Model of Technological Growth Under Emission Constraints

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Abstract

We suggest and analyze a model of global technological growth under a prescribed constraint on the annual emission of greenhouse gases (GHG). The model assumes that industrial GHG emission is positively related to the world's production output driven by the development of the "production" technology stock. "Cleaning" technology is developed in parallel to keep the annual GHG emission within a "safety" zone. The ratio between annual investment in "cleaning" technology and annual investment in "production" technology acts as a time-variable control parameter in the model. Under a set of natural assumptions we find an optimal control which maximizes an integral utility characterizing the rate of economic growth over a given time period. In substantial terms, the optimal control strategy suggests that "production" technology is developed at a maximum rate until a critical point is reached, at which the annual emission hits the prescribed upper bound. In the subsequent period investment in "production" and "cleaning" technology is planned so that the annual emission "tracks" the prescribed upper bound. One should note that the proposed control strategy optimal with respect to the chosen utility, is the most risky one since it assumes a minimum distance to the boundary of the "safety" zone.

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1 Introduction

The issue of the reduction of emission of greenhouse gases (GHG) is to a considerable extent associated with climate change. Some works (see, e.g., [6], [1]) relate the increase in GHG emission driven by global economic growth to the raise of the average temperature in the world. Even a small temperature increase is believed to produce a strong negative impact on the environment. Accordingly, it is believed that a reduction of GHG emission should reduce this negative impact. The reduction of industrial GHG emission implies investment in development of "cleaning" mechanisms or/and new technology producing less GHG emissions. However, an excessive investment in development of "cleaning" mechanisms may lead to a recession in technological growth, which may in turn restrict possibilities of investment in "cleaning" in the future. As a result, the atmospheric GHG concentration may, in a time perspective, become high in spite of today's efforts to decreasing it; in the same time humanity may face a shortage of resources for developing both "cleaning" technology and "production" technology.

A principal goal of environmental economics is therefore to find an optimal balance in development of "production" technology and "cleaning" technology so that the effectiveness of economic growth is maximized under the constraint that a "safe" level of negative environmental effects is not exceeded. While planning an optimal policy, it should be taken into account that the average life time of CO_2 in the atmosphere is estimated as 10 years, from which it follows that the greenhouse effect would not be reduced immediately even if the production process was stopped. Preventive measures are being worked out, based on various data assessment, measurement and modeling methods.

A common way to identify key parameters of these measures is simulation of economic growth scenarios and their impact on climate change ([8], [7]). Usually such approach requires extremely complex modeling, involving a variety of uncertain parameters and a number of components and links, some of which are yet not well understood. Our study follows the aggregated modeling approach suggested by endogenous economic growth theory [6]. Based on the simplified model of economic growth under an emission constraint, constructed in [3], we extend it in three aspects: we introduce a time-varying upper constraint for annual GHG emission; we represent the utility flow as an arbitrary convex function of the production output; and we take into account the depreciation of the technology stock. The model assumes that industrial GHG emission is positively related to the world's production output driven by the development of the "production" technology stock. "Cleaning" technology is developed in parallel to keep the annual GHG emission within a "safety" zone. The ratio between annual investment in "cleaning" technology and annual investment in "production" technology acts as a time-variable control parameter in the model.

The substantial part of this research, including a model calibration, an uncertainty analysis and economic interpretations is presented in [4]. In this paper we provide a rigorous mathematical justification for the methodology used in [4]. Under a set of natural assumptions we find an optimal control which maximizes an integral utility characterizing the rate of economic growth over a given time period. In substantial terms, the optimal control strategy suggests that "production" technology is developed at a maximum rate until a critical point is reached, at which the annual emission hits the prescribed upper bound. In the subsequent period investment in "production" and "cleaning" technology is planned so that the annual emission "tracks" the prescribed upper bound. A rigorous proof of this statement whose mathematical formulation is Theorem 5.1 is the principal technical goal of this paper. One should note that the proposed control strategy optimal with respect to the chosen utility, is the most risky one since it assumes a minimum distance to the boundary of the "safety" zone.

2 Model

Let $Y(t)$ stand for the total annual production output, $T(t)$ stand for the "production" technology stock used for production, and $C(t)$ stand for the stock of "cleaning" technology used to reduce GHG emissions. Let $\vartheta > 0$ be a fixed time horizon. Here and in what follows, t is time varying from 0 to ϑ. Using a simplest form of the standard Cobb-Douglas production function, we assume

$$Y(t) = aT(t) \tag{2.1}$$

where $a > 0$. Let $u^* \in (0,1)$ be the total fraction of the production output, annually allocated for developing both "production" and "cleaning" technology stocks, and

$$u(t) \in [0, u^*] \tag{2.2}$$

be the fraction of the production output $Y(t)$, allocated for developing the "production" technology stock $T(t)$. The complementary fraction, $u^* - u(t)$, of the production output $Y(t)$ is allocated for developing the "cleaning" technology stock $C(t)$. Based on this, using (2.1) and introducing the depreciation of technology, we set

$$\begin{array}{rcl} \dot{T}(t) & = & u(t)aT(t) - \gamma T(t), \\ \dot{C}(t) & = & (u^* - u(t))aT(t) - \gamma C(t), \end{array} \tag{2.3}$$

where $\gamma > 0$ is a discount factor. We also denote

$$T(0) = T_0, \quad C(0) = C_0. \tag{2.4}$$

Besides, for system (2.3) we introduce the utility index

$$J = \int_0^\vartheta e^{-\rho t} f(Y(t))dt, \tag{2.5}$$

commonly used to assess the effectiveness of an economy [2], [8]. Here $f(\cdot) : [0,\infty) \mapsto [0,\infty)$ is a continuous increasing function, and $\rho > 0$ is a discount rate.

We model the annual GHG emission as

$$E(t) = \alpha_0 \frac{Y(t)}{C(t)} = \alpha_0 a \frac{T(t)}{C(t)} = \alpha \frac{T(t)}{C(t)},$$

where $\alpha_0 > 0$, $\alpha = \alpha_0 a$, and impose the constraint

$$E(t) \le E^*(t), \tag{2.6}$$

where $E^*(t) > 0$ is a fixed function describing a time-varying admissible level of emission. We assume that at the initial time $t = 0$ constraint (2.6) is satisfied:

$$E_0 < E_0^*, \quad \text{where} \quad E_0 = E(0) = \alpha \frac{T_0}{C_0}, \; E_0^* = E^*(0). \tag{2.7}$$

In our analysis $u(t)$ acts as a control variable. The set of all admissible controls, denoted by U, is the collection of all measurable functions $u(\cdot)$ on $[0, \vartheta]$ which satisfy (2.2).

Thus we consider an optimal control problem which can be represented in the following standard form [5]:

$$\begin{aligned} \text{maximize} \quad J_0 &= \int_0^\vartheta e^{-\rho t} f(Y(t))dt, \\ Y(t) &= aT(t), \\ \dot{T}(t) &= u(t)aT(t) - \gamma T(t), \\ \dot{C}(t) &= (u^* - u(t))aT(t) - \gamma T(t), \\ T(0) &= T_0, \quad C(0) \; = \; C_0, \\ u(\cdot) &\in U, \\ \alpha \frac{T(t)}{C(t)} &\leq E^*(t), \\ t &\in [0, \vartheta]. \end{aligned} \tag{2.8}$$

An admissible control process in (2.8) is a triple $(T(\cdot), C(\cdot), u(\cdot))$ satisfying the differential equations (2.3), initial condition (2.4) and the state constraint (2.6). We assume that the set of all admissible control processes is nonempty.

We assume that
(A1) $E^*(\cdot)$ is monotonically decreasing on $[0, \vartheta]$;
(A2) the rate of the decrease of $E^*(\cdot)$ is small enough, more accurately, for each $t \in [0, \vartheta]$

$$0 < |\dot{E}^*(t)| < \frac{au^*}{\alpha} E^{*2}(t).$$

3 Equivalent problem formulation

In our analysis we use an equivalent formulation of problem (2.8). From (2.3) and (2.4) we get that for each $t \in [0, \vartheta]$

$$T(t) = T_0 e^{ap(t) - \gamma t} \quad \text{where} \quad p(t) = \int_0^t u(s)ds, \tag{3.1}$$

and

$$C(t) = e^{-\gamma t} \left(C_0 + au^* T_0 \int_0^t e^{ap(s)} ds - T_0(e^{ap(t)} - 1) \right).$$

Now constraint (2.6) takes the form

$$v(t) \geq 0, \tag{3.2}$$

where

$$v(t) = 1 + \frac{C_0}{T_0} + au^* \int_0^t e^{ap(s)} ds - \left(1 + \frac{\alpha}{E^*(t)} \right) e^{ap(t)}. \tag{3.3}$$

Note that

$$\dot{v}(t) = au^* e^{ap(t)} - au(t)e^{ap(t)}\left(1 + \frac{\alpha}{E^*(t)}\right) + \frac{\alpha \dot{E}^*(t)}{E^{*2}(t)} e^{ap(t)} \tag{3.4}$$

and

$$v(0) = v_0 = \frac{C_0}{T_0} - \frac{\alpha}{E_0^*}. \tag{3.5}$$

Constraint (3.2) is satisfied for $t = 0$ by (2.7), therefore

$$v_0 > 0.$$

Consider the utility index (2.5). Taking into account (2.1) and (3.1), we get

$$J = \int_0^{\vartheta} e^{-\rho t} f(aT_0 e^{ap(t)})dt = \int_0^{\vartheta} e^{-\rho t}\phi(p(t))dt \tag{3.6}$$

where $\phi(\cdot) : [0,\infty) \mapsto [0,\infty) : p \mapsto \phi(p) = f(aT_0 e^{ap})$ is a continuous increasing function.

Summarizing, we represent the original optimization problem (2.8) in the following equivalent form:

$$\begin{array}{rrcl} \text{maximize} & J(u) & = & \int_0^{\vartheta} e^{-\rho t}\phi(p(t))dt, \\ & p(t) & = & \int_0^t u(s)ds, \\ & \dot{v}(t) & = & au^* e^{ap(t)} - au(t)e^{ap(t)}\left(1 + \frac{\alpha}{E^*(t)}\right) + \frac{\alpha \dot{E}^*(t)}{E^{*2}(t)} e^{ap(t)}, \\ & v(0) & = & v_0 \geq 0, \\ & v(t) & \geq & 0, \\ & u(t) & \in & [0, u^*], \\ & t & \in & [0, \vartheta]. \end{array} \tag{3.7}$$

As usual, a solution to problem (3.7) is said to be an optimal control in this problem.

Remark 3.1 Further on, we use a more informative notation for variables (3.1) and (3.3): $p(t) = p(t,u)$, $v(t) = v(t,u)$, thus stressing their dependence on a control $u = u(\cdot)$.

We denote the optimal value in problem (3.7) as J^0, and the set of all optimal controls in problem (3.7) as U^0.

4 Discrete approximation

In this section we introduce a discrete approximation to problem (3.7). Take a uniform time grid

$$t_i^\delta = i\delta \quad (i = 0, 1, \ldots, m), \quad t_m^\delta = \vartheta$$

and introduce approximate piece-wise constant controls

$$u^\delta(t) = u_i^\delta \in \{0, u^*\} \quad (t \in [t_i^\delta, t_{i+1}^\delta),\ i = 0, 1, \ldots, m-1)$$

identified with the vectors

$$u^\delta = (u_0^\delta, u_1^\delta, \ldots, u_{m-1}^\delta) \in \{0, u^*\}^m.$$

For each approximate control $u^\delta(\cdot)$ we define the following piece-wise approximation $p^\delta(\cdot, u^\delta)$ to $p(\cdot) = p(\cdot, u)$ (3.1):

$$p^\delta(t, u^\delta) = p_0^\delta(u^\delta) = 0 \quad (t \in [t_0^\delta, t_1^\delta)),$$

$$p^\delta(t, u^\delta) = p_i^\delta(u^\delta) = \delta \sum_{j=0}^{i-1} u_j^\delta \quad (t \in [t_i^\delta, t_{i+1}^\delta),\ i = 1, \ldots, m-1),$$

identified with the vector

$$p^\delta(t, u^\delta) = (p_0^\delta(u^\delta), p_1^\delta(u^\delta), \ldots, p_{m-1}^\delta(u^\delta)) \quad (t \in [0, \vartheta]).$$

Remark 4.1 Note that the proposed approximation allows us to choose such an approximate control $u^\delta(\cdot)$ that for each $t \in [0, \vartheta]$

$$|p(t, u) - p(t, u^\delta)| \le u^* \delta$$

and

$$|p(t, u^\delta) - p^\delta(t, u^\delta)| \le u^* \delta.$$

This leads to

$$|p(t, u) - p^\delta(t, u^\delta)| \le 2u^* \delta. \tag{4.1}$$

We define an approximation $v^\delta(\cdot, u^\delta)$ to $v(\cdot) = v(\cdot, u)$ (3.3) by

$$\dot{v}^\delta(t, u^\delta) = au^* e^{ap^\delta(t,u^\delta)} - au^\delta(t) e^{ap^\delta(t,u^\delta)} \left(1 + \frac{\alpha}{E^*(t)}\right) + \frac{\alpha \dot{E}^*(t)}{E^{*2}(t)} e^{ap^\delta(t,u^\delta)} \quad (t \in [0, \vartheta])$$

or

$$\dot{v}^\delta(t, u^\delta) = au^* e^{ap_i^\delta(u^\delta)} - au_i^\delta e^{ap_i^\delta(u^\delta)} \left(1 + \frac{\alpha}{E^*(t)}\right) + \frac{\alpha \dot{E}^*(t)}{E^{*2}(t)} e^{ap_i^\delta(u^\delta)} \tag{4.2}$$

$$(t \in [t_i^\delta, t_{i+1}^\delta),\ i = 0, 1, \ldots, m-1).$$

Following (3.5), we set

$$v^\delta(0, u^\delta) = v_0.$$

Finally, for each approximate control $u^\delta(\cdot)$ we define an approximate utility value by

$$I(u^\delta) = \int_0^\vartheta e^{-\rho t} \phi(p^\delta(t, u^\delta)) dt = \sum_{i=0}^{m-1} h_i^\delta \phi(p_i^\delta(u^\delta))$$

where

$$h_i^\delta = \int_{t_i^\delta}^{t_{i+1}^\delta} e^{-\rho t} dt \quad (i = 0, 1, \ldots, m-1).$$

Introduce a parameter $\beta > 0$ and an approximate constraint

$$v^\delta(t_i^\delta, u^\delta) \ge -\beta \quad (i = 0, 1, \ldots, m-1).$$

The approximate optimization problem (further called the (δ, β) -problem) can be represented as follows

$$\begin{aligned} \text{maximize} \quad I(u^\delta) &= \sum_{i=0}^{m-1} h_i^\delta \phi(p_i^\delta(u^\delta)), \\ u^\delta &= (u_0^\delta, \ldots, u_{m-1}^\delta) \in \{0, u^*\}^m \\ v^\delta(t_i^\delta, u^\delta) &\ge -\beta, \quad (i = 0, 1, \ldots, m-1). \end{aligned} \tag{4.3}$$

We assume that the set of all admissible controls in (4.3) is nonempty. Denote the optimal value by $I^{\delta 0}$, and the set of all optimal controls by $U^{\delta 0}$.

Lemma 4.1 Let $\delta_k > 0$ $(k = 0, 1, \ldots)$, $\delta_k \to 0$ $(k \to \infty)$. Then there exist $\beta_k > 0$ $(k = 0, 1, \ldots)$, $\beta_k \to 0$ $(k \to \infty)$, such that

(i) $I^{k0} \to I^0$,

(ii) $u^{k0} \to U^0$ weakly in $L^2[0, \vartheta]$

where $u^{k0} = u^{\delta_k 0}(\cdot) \in U^{\delta_k 0} \subset U$ and $I^{k0} = I^{\delta_k 0}$ are, respectively, an optimal control and the optimal value in the (δ_k, β_k)-problem.

Proof.

1. We use simplified notations:

$$\begin{aligned} u^{\delta_k}(\cdot) &= u^k(\cdot) = u^k, \\ p^{\delta_k}(\cdot, u^{\delta_k}) &= p^k(\cdot, u^k), \\ v^{\delta_k}(\cdot, u^{\delta_k}) &= v^k(\cdot, u^k). \end{aligned}$$

Put

$$\beta_k = \sup_{u \in U} \left| v^k(\cdot, u^k) - v(\cdot, u) \right|_C \quad (k = 0, 1, \ldots). \tag{4.4}$$

We will show that sequence (β_k) satisfies the conditions of the lemma. Suppose the contrary: there exist a subsequence (k_j) and an $\varepsilon > 0$ such that $\beta_{k_j} > \varepsilon$ for each $j = 0, 1, \ldots$. Without loss of generality we set $\beta_k > \varepsilon$ for each $k = 0, 1, \ldots$. Consider elements $\bar{u}_k \in U$ that provide the supremum in (4.4) at least with accuracy $\varepsilon/2$, i.e.

$$\sup_{u \in U} \left| v^k(\cdot, u^k) - v(\cdot, u) \right|_C - \left| v^k(\cdot, \bar{u}_k^k) - v(\cdot, \bar{u}_k) \right|_C \leq \frac{\varepsilon}{2} \quad (k = 0, 1, \ldots),$$

where $\bar{u}_k^k$ is the approximation to $\bar{u}_k$ on the δ_k-grid.

Hence,

$$\left| v^k(\cdot, \bar{u}_k^k) - v(\cdot, \bar{u}_k) \right|_C \geq \varepsilon \quad (k = 0, 1, \ldots). \tag{4.5}$$

We will complete the first part of the proof by arriving at a contradiction to the latter inequality.

2. As $\bar{u}_k(\cdot) \in U$ and U is a weak compact in $L^2[0, \vartheta]$, there exist subsequence of $(\bar{u}_k(\cdot))$ weakly converging to some control $\bar{u}(\cdot) \in U$. Without loss of generality we set

$$\bar{u}_k(\cdot) \to \bar{u}(\cdot) \quad \text{weakly in} \quad L^2[0, \vartheta].$$

From (4.1) we have

$$|p(t, \bar{u}_k) - p^k(t, \bar{u}_k^k)| \leq 2u^* \delta_k,$$

and while $\delta_k \to 0$

$$\left| \int_0^t \bar{u}_k(s) ds - \int_0^t \bar{u}_k^k(s) ds \right| \to 0.$$

The latter fact together with the weak convergence of $(\bar{u}_k(\cdot))$ to $\bar{u}(\cdot)$ yields that

$$\bar{u}_k^k(\cdot) \to \bar{u}(\cdot) \quad \text{weakly in} \quad L^2[0, \vartheta].$$

3. Using (3.4), we obtain

$$v(t, \bar{u}_k) = v_0 + \int_0^t a e^{ap(s, \bar{u}_k)} \left[u^* - \bar{u}_k(s) \left(1 + \frac{\alpha}{E^*(s)} \right) + \frac{\alpha \dot{E}^*(s)}{a E^{*2}(s)} \right] ds.$$

From the approximation condition (4.1) we have

$$e^{ap(\cdot, \bar{u}_k)} \to e^{ap(\cdot, \bar{u})} \quad \text{in} \quad L^2[0, \vartheta].$$

From the weak convergence of $(\bar{u}_k(\cdot))$ to $\bar{u}(\cdot)$ we have

$$u^* - \bar{u}_k(\cdot)\left(1 + \frac{\alpha}{E^*(\cdot)}\right) + \frac{\alpha \dot{E}^*(\cdot)}{aE^{*2}(\cdot)} \to$$

$$u^* - \bar{u}(\cdot)\left(1 + \frac{\alpha}{E^*(\cdot)}\right) + \frac{\alpha \dot{E}^*(\cdot)}{aE^{*2}(\cdot)} \quad \text{weakly in} \quad L^2[0, \vartheta].$$

Consequently

$$v(t, \bar{u}_k) \to v_0 + \int_0^t a e^{ap(s,\bar{u})}\left[u^* - \bar{u}(s)\left(1 + \frac{\alpha}{E^*(s)}\right) + \frac{\alpha \dot{E}^*(s)}{aE^{*2}(s)}\right] ds. \tag{4.6}$$

Similarly we find that

$$v^k(t, \bar{u}_k^k) \to v_0 + \int_0^t a e^{ap(s,\bar{u})}\left[u^* - \bar{u}(s)\left(1 + \frac{\alpha}{E^*(s)}\right) + \frac{\alpha \dot{E}^*(s)}{aE^{*2}(s)}\right] ds. \tag{4.7}$$

Thus, (4.6) and (4.7) lead to

$$\left| v^k(\cdot, \bar{u}_k^k) - v(\cdot, \bar{u}_k) \right|_C \to 0,$$

which contradicts (4.5). We proved that $\beta_k \to 0$.

4. Consider a control $u^0(\cdot) \in U^0 \subset U$ optimal in problem (3.7). For each δ_k-grid we consider its approximation on each $u^{0k}(\cdot)$. According to 1-3 control $u^{0k}(\cdot)$ is admissible in problem (4.3) with $\beta = \beta_k$, $\delta = \delta_k$. Let

$$\nu_k = |J^0 - I(u^{0k})|.$$

Obviously,

$$\begin{aligned}
\nu_k &= |J^0 - I(u^{0k})| \\
&= \left| \int_0^\vartheta e^{-\rho t} \phi(p(t, u^0)) dt - \sum_{i=0}^{m-1} h_i \phi(p_i^k(u^{0k})) \right| \\
&= \left| \sum_{i=0}^{m-1} \int_{t_i^k}^{t_{i+1}^k} e^{-\rho t} \phi(p(t, u^0)) dt - \sum_{i=0}^{m-1} \int_{t_i^k}^{t_{i+1}^k} e^{-\rho t} dt \ \phi(p_i^k(u^{0k})) \right| \\
&\le \sum_{i=0}^{m-1} \int_{t_i^k}^{t_{i+1}^k} e^{-\rho t} \left| \phi(p(t, u^0)) - \phi(p_i^k(u^{0k})) \right| dt.
\end{aligned}$$

Since $\phi(\cdot)$ is continuous and (4.1) holds we find that $\nu_k \to 0$. Then for each $k = 0, 1, \ldots$

$$I^{k0} \ge J^0 - \varepsilon_k.$$

Letting $k \to \infty$, we get

$$\lim_{k \to \infty} I^{k0} \ge J^0. \tag{4.8}$$

Now we consider controls $u^{k0}(\cdot) \in U$ optimal in problems (4.3) with $\beta = \beta_k$, $\delta = \delta_k$. Since U is a weak compact in $L^2[0, \vartheta]$ there exist a subsequence of $(u^{k0}(\cdot))$, which converges to some $\tilde{u}(\cdot) \in U$ weakly in $L^2[0, \vartheta]$. Without loss of generality we set

$$u^{k0}(\cdot) \to \tilde{u}(\cdot) \quad \text{weakly in} \quad L^2[0, \vartheta].$$

Reasoning as in 1-3 we find that control $\tilde{u}(\cdot)$ is admissible in problem (3.7). Hence,

$$J(\tilde{u}) \leq J^0.$$

Similarly to a previous reasoning we get

$$I^{k0} \to J(\tilde{u}).$$

Therefore

$$\lim_{k\to\infty} I^{k0} \leq J^0. \tag{4.9}$$

We see that (4.8) and (4.9) prove (i). Statement (ii) follows from the weak compactness of U in $L^2[0, \vartheta]$.

5 Solution of (δ, β) -problem

In this section we find a solution to the (δ, β) -problem (4.3). Let $u^{\delta 0}(\cdot)$ be an arbitrary control optimal in (4.3). By (4.2) we have

$$\dot{v}^\delta(t, u^{\delta 0}) = \begin{cases} \lambda^+(t)e^{ap_i^\delta(u^{\delta 0})} & \text{if} \quad u_i^{\delta 0} = 0 \\ \lambda^-(t)e^{ap_i^\delta(u^{\delta 0})} & \text{if} \quad u_i^{\delta 0} = u^*, \end{cases} \quad (t \in [t_i^\delta, t_{i+1}^\delta),\ i = 0, 1, \ldots, m-1); \tag{5.1}$$

where due to assumptions (A1) and (A2)

$$\lambda^+(t) \;=\; au^* + \frac{\alpha \dot{E}^*(t)}{E^{*2}(t)} > 0, \tag{5.2}$$

$$\lambda^-(t) \;=\; -\frac{\alpha a u^*}{E^*(t)} + \frac{\alpha \dot{E}^*(t)}{E^{*2}(t)} < 0. \tag{5.3}$$

Note that from (5.1) and (5.2), (5.3) we have

$$v^\delta(t_{i+1}^\delta, u^{\delta 0}) = v^\delta(t_i^\delta, u^{\delta 0}) + \begin{cases} \lambda_i^{\delta +} e^{ap_i^\delta(u^{\delta 0})} & \text{if} \quad u_i^{\delta 0} = 0 \\ \lambda_i^{\delta -} e^{ap_i^\delta(u^{\delta 0})} & \text{if} \quad u_i^{\delta 0} = u^*, \end{cases} \quad (i = 0, 1, \ldots, m-1) \tag{5.4}$$

where

$$\lambda_i^{\delta +} \;=\; \int_{t_i^\delta}^{t_{i+1}^\delta} \lambda^+(t)dt, \tag{5.5}$$

$$\lambda_i^{\delta -} \;=\; \int_{t_i^\delta}^{t_{i+1}^\delta} \lambda^-(t)dt. \tag{5.6}$$

As $u^{\delta 0}(\cdot)$ is admissible in (4.3), we have

$$v^\delta(t_i^\delta, u^{\delta 0}) \geq -\beta \quad (i = 0, 1, \ldots, m). \tag{5.7}$$

We call a point $t_j^\delta < t_m^\delta$ *critical* if the replacement of $u_j^{\delta 0}$ with u^* leads to the violation of constraint (5.7) at point t_{j+1}^δ, i.e.

$$v^\delta(t_{j+1}^\delta, u^{\delta 0}) < -\beta \quad (j = 0, 1, \ldots, m-1)$$

or

$$v^\delta(t_j^\delta, u^{\delta 0}) + e^{ap_j^\delta(u^{\delta 0})} \int_{t_j^\delta}^{t_{j+1}^\delta} \left(-\frac{\alpha a u^*}{E^*(t)} + \frac{\alpha \dot{E}^*(t)}{E^{*2}(t)} \right) dt < -\beta \quad (j = 0, 1, \ldots, m-1).$$

Each point $t_j^\delta < t_m^\delta$ that is not critical is called *regular*.

Remark 5.1 Let $t_j^\delta < t_m^\delta$ be a critical point. Then $u_j^{\delta 0} = 0$.

Lemma 5.1 Let $t_j^\delta < t_{m-2}^\delta$ be a regular point. Then $u_j^{\delta 0} = u^*$.

Proof. Suppose the contrary:
$$u_j^{\delta 0} = 0.$$
Consider an admissible control $\bar{u}^\delta = (\bar{u}_0^\delta, \dots, \bar{u}_{m-1}^\delta) \in \{0, u^*\}^m$ and complete the proof by arriving at a contradiction with the optimality of $u^{\delta 0}$ in problem (3.7), namely we will prove that
$$I(\bar{u}^\delta) > I(u^{\delta 0}).$$
1. First we suppose
$$u_i^{\delta 0} = 0 \quad (i = j+1, \dots, m-2). \tag{5.8}$$
We define control $\bar{u}^\delta$ by
$$\begin{aligned} \bar{u}_i^\delta &= u_i^{\delta 0} \quad (i = 0, \dots, j-1), \\ \bar{u}_j^\delta &= u^*, \\ \bar{u}_i^\delta &= u_i^{\delta 0} \quad (i = j+1, \dots, m-2). \end{aligned} \tag{5.9}$$
Then
$$p_0^\delta(\bar{u}^\delta) = 0 = p_0^\delta(u^{\delta 0});$$
$$p_i^\delta(\bar{u}^\delta) = \delta \sum_{l=0}^{j-1} \bar{u}_l^\delta = \delta \sum_{l=0}^{j-1} u_l^{\delta 0} = p_i^\delta(u^{\delta 0}) \quad (i = 1, \dots, j);$$
$$p_{j+1}^\delta(\bar{u}^\delta) = \delta \sum_{l=0}^{j-1} \bar{u}_l^\delta + \delta u^* > \delta \sum_{l=0}^{j-1} u_l^{\delta 0} = p_{j+1}^\delta(u^{\delta 0});$$
$$p_i^\delta(\bar{u}^\delta) = \delta \sum_{l=0}^{j-1} \bar{u}_l^\delta + \delta u^* + \delta \sum_{l=j+1}^{m-1} \bar{u}_l^\delta > \delta \sum_{l=0}^{j-1} u_l^{\delta 0} + \delta \sum_{l=j+1}^{m-1} u_l^{\delta 0} = p_i^\delta(u^{\delta 0}) \quad (i = j+1, \dots, m-1).$$
Thus finally we have
$$p_i^\delta(\bar{u}^\delta) = p_i^\delta(u^{\delta 0}) \quad (i = 0, \dots, j-1), \tag{5.10}$$
$$p_i^\delta(\bar{u}^\delta) > p_i^\delta(u^{\delta 0}) \quad (i = j+1, \dots, m-1). \tag{5.11}$$
Now consider $v^\delta(\cdot, \bar{u}^\delta)$. From (5.9), (5.4) we have
$$v^\delta(t_i^\delta, \bar{u}^\delta) = v^\delta(t_i^\delta, u^{\delta 0}) \geq -\beta \quad (i = 0, \dots, j).$$
Furthermore
$$v^\delta(t_{j+1}^\delta, \bar{u}^\delta) = v^\delta(t_j^\delta, \bar{u}^\delta) + \lambda_j^{\delta -} e^{a p_j^\delta(\bar{u}^\delta)} = v^\delta(t_j^\delta, u^{\delta 0}) + \lambda_j^{\delta -} e^{a p_j^\delta(u^{\delta 0})}.$$
By assumption t_j^δ is regular. Hence,
$$v^\delta(t_j^\delta, u^{\delta 0}) + \lambda_j^{\delta -} e^{a p_j^\delta(u^{\delta 0})} \geq -\beta,$$
from which it follows that
$$v^\delta(t_{j+1}^\delta, \bar{u}^\delta) \geq -\beta.$$

Since
$$\dot{v}^\delta(t,\bar{u}^\delta) = \lambda^+(t)e^{ap_i^\delta(\bar{u}^\delta)} > 0 \quad (t \in [t_i^\delta, t_{i+1}^\delta),\ i = j+1,\dots,m-2),$$
function $t \mapsto v^\delta(t,\bar{u}^\delta)$ is monotonically increasing on $[t_{j+2}^\delta, t_{m-2}^\delta]$. Therefore
$$v^\delta(t_i^\delta,\bar{u}^\delta) > v^\delta(t_{j+1}^\delta,\bar{u}^\delta) \geq -\beta. \quad (i = j+2,\dots,m-1).$$
Summarizing, we have
$$v^\delta(t_i^\delta,\bar{u}^\delta) \geq -\beta \quad (i = 0,\dots,m-1).$$
In other words, $\bar{u}^\delta$ is an admissible control in (4.3).

For the corresponding value of the utility index we get
$$I(\bar{u}^\delta) = \sum_{i=0}^{m-1} h_i^\delta p_i^\delta(\bar{u}^\delta) > \sum_{i=0}^{m-1} h_i^\delta p_i^\delta(u^{\delta 0}) = I(u^{\delta 0})$$
(see (5.10), (5.11)). Thus we obtained a contradiction proving the statement of the lemma in case (5.8).

2. Now let (5.8) do not hold, i.e. $u_i^{\delta 0} = u^*$ $(i = j+1,\dots,m-2)$. Set
$$u_i^{\delta 0} = 0 \quad (i = j+1,\dots,k-1), \quad u_k^{\delta 0} = u^*.$$
We put
$$\begin{aligned} \bar{u}_i^\delta &= u_i^{\delta 0} \quad (i = 0,\dots,m-2,\ i \neq j,k), \\ \bar{u}_j^\delta &= u^*, \\ \bar{u}_k^\delta &= 0. \end{aligned} \tag{5.12}$$
Then
$$p_0^\delta(\bar{u}^\delta) = 0 = p_0^\delta(u^{\delta 0}) \quad (i = 1,\dots,j);$$
$$p_i^\delta(\bar{u}^\delta) = \delta\sum_{l=0}^{j-1} \bar{u}_l^\delta = \delta\sum_{l=0}^{j-1} u_l^{\delta 0} = p_i^\delta(u^{\delta 0});$$
$$p_{j+1}^\delta(\bar{u}^\delta) = \delta\sum_{l=0}^{j-1} \bar{u}_l^\delta + \delta u^* > \delta\sum_{l=0}^{j-1} u_l^{\delta 0} = p_{j+1}^\delta(u^{\delta 0}) \quad (i = j+2,\dots,k);$$
$$p_i^\delta(\bar{u}^\delta) = \delta\sum_{l=0}^{j-1} \bar{u}_l^\delta + \delta u^* + \delta\sum_{l=j+1}^{k} \bar{u}_l^\delta > \delta\sum_{l=0}^{j-1} u_l^{\delta 0} + \delta\sum_{l=j+1}^{k} u_l^{\delta 0} = p_i^\delta(u^{\delta 0})$$
$$(i = k+1,\dots,m-1);$$

$$p_i^\delta(\bar{u}^\delta) = \delta\sum_{l=0}^{j-1} \bar{u}_l^\delta + \delta u^* + \delta\sum_{l=j+1}^{k-1} \bar{u}_l^\delta + \delta\sum_{l=k+1}^{m-1} \bar{u}_l^\delta = \delta\sum_{l=0}^{j-1} u_l^{\delta 0} + \delta\sum_{l=j+1}^{k-1} u_l^{\delta 0} + \delta u^* + \delta\sum_{l=k+1}^{m-1} u_l^{\delta 0} = p_i^\delta(u^{\delta 0}).$$
Thus finally we have
$$p_i^\delta(\bar{u}^\delta) = p_i^\delta(u^{\delta 0}) \quad (i = 0,\dots,j \text{ and } k+1,\dots,m-2), \tag{5.13}$$
$$p_i^\delta(\bar{u}^\delta) > p_i^\delta(u^{\delta 0}) \quad (i = j+1,\dots,k). \tag{5.14}$$

Now consider $v^\delta(\cdot, \bar{u}^\delta)$. From (5.9), (5.4) we have

$$v^\delta(t_i^\delta, \bar{u}^\delta) = v^\delta(t_i^\delta, u^{\delta 0}) \geq -\beta \quad (i = 0, \ldots, j).$$

Furthermore,

$$v^\delta(t_{j+1}^\delta, \bar{u}^\delta) = v^\delta(t_j^\delta, \bar{u}^\delta) + \lambda_j^{\delta -} e^{ap_j^\delta(\bar{u}^\delta)} = v^\delta(t_j^\delta, u^{\delta 0}) + \lambda_j^{\delta -} e^{ap_j^\delta(u^{\delta 0})}.$$

Due to the regularity of t_j^δ we get

$$v^\delta(t_j^\delta, u^{\delta 0}) + \lambda_j^{\delta -} e^{ap_j^\delta(u^{\delta 0})} \geq -\beta,$$

from which it follows that

$$v^\delta(t_{j+1}^\delta, \bar{u}^\delta) \geq -\beta.$$

For $i = j+1, \ldots, k$

$$\dot{v}^\delta(t_i^\delta, \bar{u}^\delta) - \dot{v}^\delta(t_i^\delta, u^{\delta 0}) = \lambda^+(t_i)(e^{ap_i^\delta(\bar{u}^\delta)} - e^{ap_i^\delta(u^{\delta 0})}).$$

Using (5.14) and the admissibility of $u^{\delta 0}$ in problem (4.3) we get

$$\dot{v}^\delta(t_i^\delta, \bar{u}^\delta) > \dot{v}^\delta(t_i^\delta, u^{\delta 0}) \geq -\beta \quad (i = j+1, \ldots, k).$$

Consider a $t \in [t_k^\delta, t_{k+1}^\delta)$:

$$\begin{aligned} \dot{v}^\delta(t, \bar{u}^\delta) - \dot{v}^\delta(t, u^{\delta 0}) &= \lambda^+(t) e^{ap_k^\delta(\bar{u}^\delta)} - \lambda^-(t) e^{ap_k^\delta(u^{\delta 0})} \\ &> (\lambda^+(t) - \lambda^-(t)) e^{ap_k^\delta(u^{\delta 0})}; \end{aligned}$$

here we used (5.14). Notice that

$$\begin{aligned} v^\delta(t_j^\delta, \bar{u}^\delta) &= v^\delta(t_j^\delta, u^{\delta 0}), \\ \dot{v}^\delta(t_i^\delta, \bar{u}^\delta) &> \dot{v}^\delta(t_i^\delta, u^{\delta 0}) \quad (i = j+1, \ldots, k), \end{aligned}$$

which together with (5.13) leads to

$$\begin{aligned} v^\delta(t_k^\delta, \bar{u}^\delta) - v^\delta(t_k^\delta, u^{\delta 0}) &> v^\delta(t_{j+1}^\delta, \bar{u}^\delta) - v^\delta(t_{j+1}^\delta, u^{\delta 0}) \\ &= \lambda_j^{-\delta} e^{ap_j^\delta(\bar{u}^\delta)} - \lambda_j^{+\delta} e^{ap_j^\delta(u^{\delta 0})} \\ &= (\lambda_j^{-\delta} - \lambda_j^{+\delta}) e^{ap_j^\delta(u^{\delta 0})}. \end{aligned}$$

Taking into account (5.4) we have

$$\begin{aligned} v(t_{k+1}^\delta, \bar{u}^\delta) - v(t_{k+1}^\delta, u^{\delta 0}) &> v^\delta(t_k^\delta, \bar{u}^\delta) - v^\delta(t_k^\delta, u^{\delta 0}) + (\lambda_k^{\delta +} - \lambda_k^{\delta -}) e^{ap_k^\delta(u^{\delta 0})} \\ &> (\lambda_k^{\delta +} - \lambda_k^{\delta -}) e^{ap_k^\delta(u^{\delta 0})}) - (\lambda_j^{\delta +} - \lambda_j^{\delta -}) e^{ap_j^\delta(u^{\delta 0})} \\ &> (\lambda_k^{\delta +} - \lambda_k^{\delta -} - \lambda_j^{\delta +} + \lambda_j^{\delta -}) e^{ap_j(u^{\delta 0})}. \end{aligned}$$

Consider the round brackets in the latter inequality. Obviously we have:

$$\begin{aligned} \lambda_k^{\delta +} - \lambda_j^{\delta +} &= \int_{t_k^\delta}^{t_{k+1}^\delta} \left(au^* + \alpha \frac{\dot{E}^*(t)}{E^{*2}(t)} \right) dt - \int_{t_j^\delta}^{t_{j+1}^\delta} \left(au^* + \alpha \frac{\dot{E}^*(t)}{E^{*2}(t)} \right) dt \\ &= au^*\delta + \alpha \int_{t_k}^{t_{k+1}} \frac{dE^*(t)}{E^{*2}(t)} - au^*\delta - \alpha \int_{t_j}^{t_{j+1}} \frac{dE^*(t)}{E^{*2}(t)} \\ &= \alpha \left(\frac{1}{E^*(t)} \bigg|_{t_k^\delta}^{t_{k+1}^\delta} - \frac{1}{E^*(t)} \bigg|_{t_j^\delta}^{t_{j+1}^\delta} \right); \end{aligned}$$

$$\begin{aligned}
\lambda_k^{\delta-} - \lambda_j^{\delta-} &= \int_{t_k^\delta}^{t_{k+1}^\delta} \left(-\frac{\alpha a u^*}{E^*(t)} + \alpha \frac{\dot{E}^*(t)}{E^{*2}(t)} \right) dt - \int_{t_j^\delta}^{t_{j+1}^\delta} \left(-\frac{\alpha a u^*}{E^*(t)} + \alpha \frac{\dot{E}^*(t)}{E^{*2}(t)} \right) dt \\
&= -\alpha a u^* \left(\int_{t_k^\delta}^{t_{k+1}^\delta} \frac{dt}{E^*(t)} - \int_{t_j^\delta}^{t_{j+1}^\delta} \frac{dt}{E^*(t)} \right) + \alpha \left(\frac{1}{E^*(t)} \Big|_{t_k^\delta}^{t_{k+1}^\delta} - \frac{1}{E^*(t)} \Big|_{t_j^\delta}^{t_{j+1}^\delta} \right) \\
&= -\alpha a u^* \left(\int_{t_k^\delta}^{t_{k+1}^\delta} \frac{dt}{E^*(t)} - \int_{t_j^\delta}^{t_{j+1}^\delta} \frac{dt}{E^*(t)} \right) + \alpha \left(\frac{1}{E^*(t)} \Big|_{t_k^\delta}^{t_{k+1}^\delta} - \frac{1}{E^*(t)} \Big|_{t_j^\delta}^{t_{j+1}^\delta} \right).
\end{aligned}$$

Combining these estimates, we finally get

$$\lambda_k^{\delta+} - \lambda_k^{\delta-} - \lambda_j^{\delta+} + \lambda_j^{\delta-} = \alpha a u^* \left(\int_{t_k^\delta}^{t_{k+1}^\delta} \frac{dt}{E^*(t)} - \int_{t_j^\delta}^{t_{j+1}^\delta} \frac{dt}{E^*(t)} \right). \tag{5.15}$$

Assumption (A1) guarantees that

$$\int_{t_k^\delta}^{t_{k+1}^\delta} \frac{dt}{E^*(t)} > \int_{t_j^\delta}^{t_{j+1}^\delta} \frac{dt}{E^*(t)}.$$

Thus we have

$$v^\delta(t_{k+1}^\delta, \bar{u}^\delta) > v^\delta(t_{k+1}^\delta, u^{\delta 0}) \geq -\beta.$$

Furthermore, we have

$$\dot{v}^\delta(t_i^\delta, \bar{u}^\delta) = v^\delta(t_i^\delta, u^{\delta 0}) \quad (i = k+1, \ldots, m-1),$$

implying

$$v^\delta(t_i^\delta, \bar{u}^\delta) > v^\delta(t_i^\delta, u^{\delta 0}) \quad (i = k+1, \ldots, m-1).$$

Summarizing, we have

$$v^\delta(t_i^\delta, \bar{u}^\delta) \geq -\beta \quad (i = 0, \ldots, m-1). \tag{5.16}$$

In other words, $\bar{u}(\cdot)^\delta$ is admissible in problem (4.3).

For the utility index we have

$$I(\bar{u}^\delta) = \sum_{i=0}^{m-1} h_i^\delta p_i^\delta(\bar{u}^\delta) > \sum_{i=0}^{m-1} h_i^\delta p_i^\delta(u^{\delta 0}) = I(u^{\delta 0}) \tag{5.17}$$

(see (5.12), (5.13)). We arrived at a contradiction which proves the statement of the lemma in case (5.12). The proof is completed.

Introduce the extreme control

$$U = (u^*, \ldots, u^*)$$

and set

$$L = \{i = 1, \ldots, m-2 : \ v^\delta(t_i^\delta, U) < -\beta\}.$$

If $L \neq \emptyset$, put

$$i^* = \min_L i.$$

Fix a constant $K > 0$ such that

$$|\dot{v}^\delta(t, u)| < K \quad (t \in [0, \vartheta], \ u \in \{0, u^*\}^m). \tag{5.18}$$

Remark 5.2 Note that from (5.5), (5.6) for each $u \in \{0, u^*\}^m$ we have

$$\begin{array}{rcl} 0 & < & \lambda_i^{\delta+} \;<\; Ke^{-ap_i^\delta(u)}\delta, \\ -Ke^{-ap_i^\delta(u)}\delta & < & \lambda_i^{\delta-} \;<\; 0. \end{array}$$

Lemma 5.2 Let $L = \emptyset$. Then $u_i^{\delta 0} = u^*$ $(i = 0, \dots, m-3)$.

Proof. Suppose the contrary. Let $u_i^{\delta 0} = 0$ for some $i \in \{0, \dots, m-3\}$. Without loss of generality we assume that $u_0^{\delta 0} = \dots = u_{i-1}^{\delta 0} = u^*$. Since $u^{\delta 0}$ is admissible in problem (4.3), we have

$$v^\delta(t_i^\delta, u^{\delta 0}) = v^\delta(t_i^\delta, U) \geq -\beta. \tag{5.19}$$

By assumption $i+1 \notin L$, combining with (5.19), we see that t_i^δ is a regular point. By Lemma 4.1 $u_i^{\delta 0} = u^*$. A contradiction with the initial assumption proves the lemma.

Lemma 5.3 Let $L \neq \emptyset$. Then
(i) $u_i^{\delta 0} = u^*$ $(i = 0, \dots, i^* - 2)$,
(ii) $-\beta \leq v^\delta(t, u^{\delta 0}) \leq -\beta + 2K\delta \quad t \in [t_{i^*}^\delta, t_{m-2}^\delta]$.

Proof. Prove (i). Suppose the contrary: $u_i^{\delta 0} = 0$ for some $i \in \{0, \dots, i^* - 2\}$. Without loss of generality we assume $u_0^{\delta 0} = \dots = u_{i-1}^{\delta 0} = u^*$. Since $u^{\delta 0}$ is an admissible control in (4.3) we have

$$v^\delta(t_i^\delta, u^{\delta 0}) = v^\delta(t_i^\delta, U) \geq -\beta. \tag{5.20}$$

Since $i + 1 < i^* - 1 < i^*$, we get that $i + 1 \notin L$. Combining with (5.20), we find that t_i^δ is a regular point. By Lemma 4.1 $u_i^{\delta 0} = u^*$. A contradiction with the initial assumption proves (i).

Let us prove (ii). From statement (i) we have $v^\delta(t_{i^*-1}^\delta, u^{\delta 0}) = v^\delta(t_{i^*-1}^\delta, U)$, which together with $i^* \in L$ leads to a conclusion that $t_{i^*}^\delta$ is a critical point. Thus,

$$v^\delta(t_{i^*}^\delta, u^{\delta 0}) < -\beta$$

or

$$v^\delta(t_{i^*-1}^\delta, u^{\delta 0}) + \lambda_{i^*-1}^{\delta-} e^{ap_{i^*-1}^\delta(u^{\delta 0})} < -\beta.$$

Using Remark 5.2, we have

$$v^\delta(t_{i^*-1}^\delta, u^\delta 0) < -\beta + K\delta. \tag{5.21}$$

Since $u^{\delta 0}$ is an admissible control, we have

$$v^\delta(t_{i^*-1}^\delta, u^{\delta 0}) \geq -\beta. \tag{5.22}$$

Besides, from the definition of K (5.18),

$$|v^\delta(t_{i^*}^\delta, u^{\delta 0}) - v^\delta(t_{i^*-1}^\delta, u^{\delta 0})| < K\delta. \tag{5.23}$$

Summing (5.21) and (5.22), we have

$$-\beta \leq v^\delta(t_{i^*-1}^\delta, u^{\delta 0}) \leq -\beta + K\delta. \tag{5.24}$$

Modifying (5.23), we obtain

$$v^\delta(t_{i^*}^\delta, u^{\delta 0}) - K\delta \leq v^\delta(t_{i^*-1}^\delta, u^{\delta 0}) \leq v^\delta(t_{i^*}^\delta, u^{\delta 0}) + K\delta. \tag{5.25}$$

Finally summing (5.24) and (5.25), we get

$$-\beta - K\delta \le v^\delta(t^\delta_{i^*-1}, u^{\delta 0}) \le -\beta + K\delta$$

or

$$|v^\delta(t^\delta_{i^*}, u^{\delta 0}) + \beta| \le 2K\delta.$$

Since $u^{\delta 0}$ is admissible, $v^\delta(t^\delta_{i^*}, u^{\delta 0}) \ge -\beta$. Thus,

$$-\beta \le v^\delta(t^\delta_{i^*-1}, u^{\delta 0}) \le -\beta + 2K\delta.$$

Let us now prove statement (ii) for each $t \in [t^\delta_{i^*}, t^\delta_{m-2}]$. Put

$$t^\delta_k = \max\ t^\delta_i :\ -\beta \le v^\delta(t, u^{\delta 0}) \le -\beta + 2K\delta \quad (t \in [t^\delta_{i^*}, t^\delta_i]).$$

Note that

$$-\beta \le v^\delta(t^\delta_k, u^{\delta 0}) \le -\beta + 2K\delta. \tag{5.26}$$

If $t^\delta_k = t^\delta_{m-2}$ statement (ii) is proved. Let us consider the case where $t^\delta_k < t^\delta_{m-2}$. Then

$$v^\delta(t^\delta_{i^*-1}, u^{\delta 0}) \notin [-\beta, -\beta + 2K\delta]$$

for some $t \in [t^\delta_k, t^\delta_{k+1}]$. Since u^0 is admissible, the right inequality in (ii) is violated, i.e.

$$v^\delta(t, u^{\delta 0}) > -\beta + 2K\delta.$$

By (5.18)

$$v^\delta(t, u^{\delta 0}) - v^\delta(t^\delta_k, u^{\delta 0}) < K\delta$$

or

$$v^\delta(t^\delta_k, u^{\delta 0}) > v^\delta(t, u^{\delta 0}) - K\delta > -\beta + K\delta.$$

Hence, t^δ_k is a regular point. Therefore by Lemma 4.1 $u^{\delta 0}_k = u^*$.

Consider a $t \in [t^\delta_k, t^\delta_{k+1})$. We have

$$\dot v^\delta(t, u^{\delta 0}) = \lambda^-(t) e^{a p^\delta_k(u^{\delta 0})} < 0.$$

Therefore

$$v^\delta(t^\delta_k, u^{\delta 0}) \ge v^\delta(t, u^{\delta 0}) \le -\beta + 2K\delta$$

(here we used (5.26)). Thus we obtained a contradiction. The proof is completed.

Theorem 5.1 The control

$$u^0(t) = \begin{cases} u^*, & \text{if} \quad t \le \xi, \\ \frac{u^*}{1+\frac{\alpha}{E^*(t)}} + \frac{\alpha}{a}\frac{\dot E^*(t)}{E^{*2}(t)\left(1+\frac{\alpha}{E^*(t)}\right)}, & \text{if} \quad t > \xi, \end{cases} \tag{5.27}$$

where ξ is the single root of equation

$$E^*(t) = E_0 e^{a u^* t} \tag{5.28}$$

is optimal in problem (3.7).

Proof.

1. In accordance with Lemma 5.2 for each $k = 0, 1, \ldots$ the optimal control in the corresponding approximate problem (4.3) satisfies the equality

$$u^{k0}(t) = u^* \quad (t \le t_k),$$

where t_k is defined by statement (i) of Lemma 5.2. Let us prove that $t_k \to \xi$ (where ξ is defined by (5.28)). Consider some $k \in \{0, 1, \ldots\}$ and suppose that $t_k \ge \xi$. From the construction of t_k we obviously have that

$$v^k(t_k, u^{k0}) \ge -\beta_k. \tag{5.29}$$

The structure of optimal control $u^0(\cdot)$ (5.27), (5.28) leads to

$$v(\xi, u^0) = 0,$$

which together with the definition of β_k (4.4) leads to

$$v^k(\xi, u^{0k}) \le \beta_k.$$

Let $\Delta_k > 0$ be such a quantity that the point $t_k - \Delta_k$ is the nearest point of the δ_k-grid from the right to the point ξ, i.e. $\xi \le t_k - \Delta_k < \xi + \delta_k$. Then

$$v^k(t_k - \Delta_k, u^{0k}) \le \beta_k, \tag{5.30}$$

$$|v^k(t_k - \Delta_k, u^{0k}) - v^k(t_k - \Delta_k, u^{k0})| \le \beta_k. \tag{5.31}$$

Thus, taking into account (5.29), (5.30), (5.31) we get

$$\begin{aligned} |v^k(t_k - \Delta_k, u^{k0}) - v^k(t_k, u^{k0})| &\le |v^k(t_k - \Delta_k, u^{k0}) - v^k(t_k - \Delta_k, u^{0k})| \\ &+ |v^k(t_k - \Delta_k, u^{0k}) - v^k(t_k, u^{k0})| \\ &\le \beta_k + 2\beta_k = 3\beta_k. \end{aligned}$$

The latter inequality yields

$$\Delta_k \le \frac{3\beta_k}{Q},$$

where

$$Q = \inf_{t \in [0,\vartheta]} |\lambda^-(t)| > 0.$$

Hence,

$$\xi \le t_k \le \xi + \Delta_k.$$

A similar estimation holds for $t_k < \xi$. Therefore, $t_k \to \xi$.

2. Due to Lemma 4.1 $u^{k0}(\cdot) \to u^0(\cdot)$ weakly in $L^2[0, \xi]$; hence,

$$\int_0^t u^{k0}(s)ds \to \int_0^t u^0(s)ds \quad (t \in [0, \xi]). \tag{5.32}$$

On the other hand, due to Lemma 5.2 $u^{k0}(t) = u^*$ $(t \le t_k)$; hence,

$$\int_0^t u^{k0}(s)ds = \begin{cases} u^*\xi, & \text{if} \quad \xi \le t_k, \\ u^* t_k + \int_{t_k}^{\xi} u^{k0}(s)ds, & \text{if} \quad \xi > t_k. \end{cases}$$

Therefore,

$$\int_0^t u^{k0}(s)ds \to u^* t \quad (t \in [0, \xi]). \tag{5.33}$$

Because of the uniqueness of the limit, (5.32) and (5.33) lead to $u^0(t) = u^*$ $(t \le \xi)$.

3. From Lemma 4.1 and condition (4.1) it follows that $v^k(\cdot, u^{k0}) \to v(\cdot, u^0)$ uniformly. Then in accordance with Lemma 4.1 and statement (ii) of Lemma 5.2 for $t \in (\xi, \vartheta]$ we have $v(t, u^0) = 0$ and $\dot{v}(t, u^0) = 0$, which defines $u^0(t)$ on $(\xi, \vartheta]$ in accordance with (5.27). Thus, we completed the proof of the theorem.

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