MIGRATION AND SETTLEMENT: MEASUREMENT AND ANALYSIS

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RR-78-13
August 1978

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PREFACE

Interest in human settlement systems and policies has been a critical part of urban-related work at IIASA since its inception. During the past three years this interest has given rise to a concentrated research activity focusing on migration dynamics and settlement patterns. Four sub-tasks have formed the core of this research effort:

- the study of spatial population *dynamics*;
- the definition and elaboration of a new research area called *demometrics* and its application to migration analysis and spatial population forecasting;
- the analysis and design of migration and settlement *policy*; and
- a *comparative study* of national migration and settlement patterns and policies.

This report brings together four articles that describe major results of IIASA's research on the measurement and analysis of migration and population redistribution patterns. It complements a collection of seven papers published recently as a special issue of the journal *Environment and Planning, A* (May, 1978) and with that issue stands as the final report on methodological contributions of the Migration and Settlement Task at IIASA. The proceedings of the 1978 September Conference on the Comparative Study and the forthcoming report on computer programs will conclude the Task's series of final reports.

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Human Settlements
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July 1978

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SPATIAL POPULATION DYNAMICS

by Andrei Rogers and Frans Willekens*

1. INTRODUCTION

The evolution of every regional human population is governed by the interaction of births, deaths, and migration. Individuals are born into a population, age with the passage of time, reproduce, and ultimately leave the population because of death or outmigration. These events and flows enter into an accounting relationship in which the growth of a regional population is determined by the combined effects of natural increase (births minus deaths) and net migration (inmigrants minus outmigrants). This paper focuses on such relationships and seeks to identify and clarify some of the more fundamental population dynamics that are involved.

In considering how fertility, mortality, and migration combine to determine the growth, age composition, and spatial distribution of a multiregional population, we address several theoretical and empirical issues already studied by Ansley Coale [1]. But Coale restricts his attention to the evolution of populations that are *closed* to migration (i.e., populations that are undisturbed by in- or outmigration). Since his focus is primarily on national populations, such an assumption does not seriously weaken the significance of his principal conclusions. Regional scientists, however, are generally confronted by problems involving regional populations that are very *open* to migration. Hence they cannot successfully apply the received body of theory of classical single-region mathematical demography. This paper seeks to help remedy that situation by generalizing some of Coale's results to multiregional population systems.

We proceed in three stages. First, we consider several well-defined regularities that are exhibited by the fertility, mortality, and migration schedules of human populations. Next, we study some of the principal population dynamics that connect such schedules with the growth, age composition, and spatial distribution of multiregional populations that are subjected to them. Finally, we examine some of the spatial implications of zero population growth.

2. THE COMPONENTS OF MULTIREGIONAL GROWTH

The proportional allocation of a multiregional population among its constituent regions and the age compositions of its regional populations are deter-

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mined by the recent history of fertility, mortality, and internal migration to which it has been subject. At any given moment its crude regional rates of birth, death, migration, and growth are all governed by the interaction of its regional age compositions and regional shares with the prevailing regime of growth that is defined by the current regional age-specific schedules of fertility, mortality, and migration. Knowledge of such schedules for a sufficiently long past period of time enables one to obtain current regional age compositions, regional shares, and regional component rates, inasmuch as the influence of a past population distribution on the current one declines over time and ultimately disappears entirely; see, for example, Coale [1], Lopez [16]. If the regime of growth is held fixed for a long enough period of time, then as we shall show in Section 3, the population evolves into a *stable* population with fixed regional age compositions and regional shares and a constant annual rate of growth.

Consider a regional female population for which the annual regional rates of fertility, mortality, and migration at age x and time t are denoted by $m_j(x,t)$, $\mu_j(x,t)$, and $\nu_{jk}(x,t)$, respectively. If $c_j(x,t)$ is the population's age composition and $SHA_j(t)$ is its regional share of the total multiregional population, then, denoting the last age of life by ω, we may define

$$\begin{aligned} b_j(t) &= \int_0^\omega c_j(x,t)m_j(x,t)dx \\ d_j(t) &= \int_0^\omega c_j(x,t)\mu_j(x,t)dx \\ o_j(t) = \sum_{\substack{k=1\\k\neq j}}^{m} o_{jk}(t) &= \sum_{\substack{k=1\\k\neq j}}^{m} \int_0^\omega c_j(x,t)\nu_{jk}(x,t)dx \qquad (2.1) \\ i_j(t) = \sum_{\substack{k=1\\k\neq j}}^{m} i_{kj}(t) &= \sum_{\substack{k=1\\k\neq j}}^{m} \frac{SHA_k(t)}{SHA_j(t)} o_{kj}(t) \\ r_j(t) &= b_j(t) - d_j(t) - o_j(t) + i_j(t) \end{aligned}$$

to be its annual crude rates of birth, death, outmigration, inmigration, and growth, respectively.

We begin this section of our paper by identifying several regularities in the age schedules of the components of multiregional population growth. The variations with age that are exhibited by such schedules are summarized and subsequently used to develop an improved understanding of how changing levels and patterns of fertility, mortality, and migration influence the evolution of particular regional age compositions and regional shares in a multiregional population.

Fertility

Age-specific rates of childbearing in human populations are shaped by both biological and social factors. The capacity to bear children generally begins at an age α of about 15 and ends by age β which is normally close to 50. In between these ages the fertility curve is unimodal, attaining its peak somewhere between ages 20 and 35. The precise form of this curve depends on a number of social variables, among which age at marriage and the degree of contraception practiced are of paramount importance.

Figure 1A illustrates several fertility schedules which exhibit a general pattern that persists across a wide variety of regional populations. In all, childbearing begins early in the teenage years, rises to a peak in the twenties or thirties, and then declines regularly to zero by age 50. A useful summary measure of this *pattern* is the mean age of the schedule

$$\bar{m} = \frac{\int_{\alpha}^{\beta} x m(x) dx}{\int_{\alpha}^{\beta} m(x) dx}$$

The *level* of fertility is given by the area under the curve, which is called the *total fertility rate* (*TFR*) if the schedule refers to live births of both sexes and the *gross reproduction rate* (*GRR*) if to female births alone. This level may be interpreted as the number of children an average woman would have if the particular fertility schedule prevailed during her lifetime and mortality were ignored.

After a study of the relative age patterns of age-specific fertility rates in 52 countries with different levels of fertility, Rele [20] concludes that they follow, on average, the ratio 1:7:7:6:4:1 for the six quinquennial reproductive age groups between ages 15 to 45. Coale and Demeny [3] go a step further and distinguish between four such patterns to summarize a similar collection of published national age-specific birth rates by means of four basic fertility schedules, each of which is scaled to a *GRR* of unity and associated with a particular mean age $\bar{m}$. Figure 1B shows the curves of their fertility schedule with a mean age of 29 as its level is increased from a *GRR* of unity to a *GRR* of 3.

Mortality

Observed age-specific death rates of both high and low mortality populations exhibit a remarkably regular pattern. They normally show a moderately high mortality immediately after birth, after which they drop to a minimum between ages 10 to 15, then increase slowly until about age 50, and thereafter rise at an increasing rate until the last years of life. Moreover, in each mortality schedule the death rates experienced at different ages are highly intercorrelated, because if health conditions, for example, are good or poor for one age group in a population they also will tend to be good or poor for all other age groups in that population. Hence if mortality at a particular age in one schedule exceeds that of the same age in another, the first is likely to also have higher death rates at every other age as well. Because of this property, demographers normally do not find it necessary to use an index such as the mean age of the mortality schedule in order to differentiate patterns of mortality (although they may group schedules

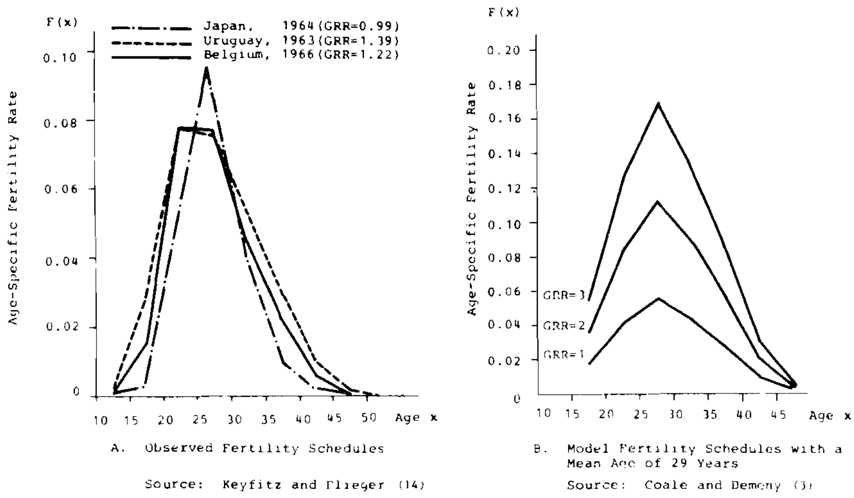


FIGURE 1. Observed and Model Female Fertility Schedules

into separate "families"). Generally only the level of a mortality schedule is defined by specifying its implicit expectation of life at birth e (0), and it is assumed that the age pattern of the schedule follows that found in most observed curves of mortality.

Figure 2A presents several observed schedules of mortality which illustrate the normal age pattern. Mortality is high during infancy, ranging anywhere from 18 to 60 per thousand live births; it is low between ages 10 through 15, falling to a value in the range of 0.28 to 0.42 per thousand; it then rises, gradually at first and more sharply after the late fifties, to values that in the late sixties lie between 20 to 30 per thousand.

After an extensive study of national populations, Coale and Demeny [3] conclude that four families of mortality schedules adequately embrace the principal variations in the age patterns that they discovered; "one of these age patterns characterizes the mortality experienced in Norway, Sweden, and Iceland; another the mortality schedules of central and parts of eastern Europe; a third the schedules of Spain, Portugal, and southern Italy; and a fourth encompasses mortality in western Europe, northern America, Oceania, Japan, and Taiwan" [1; p. 9]. They designate these four families by the labels NORTH EAST, SOUTH, and WEST, respectively, and go on to calculate 24 "model" life tables for each of these age patterns of mortality at levels of mortality ranging from a life expectancy of 20 years to one of 77.5. Figure 2B illustrates several typical mortality schedules drawn from their WEST family.

Migration

As in the case of mortality, migration rates among the different age and sex groups of a population are highly intercorrelated, with high (or low) migration

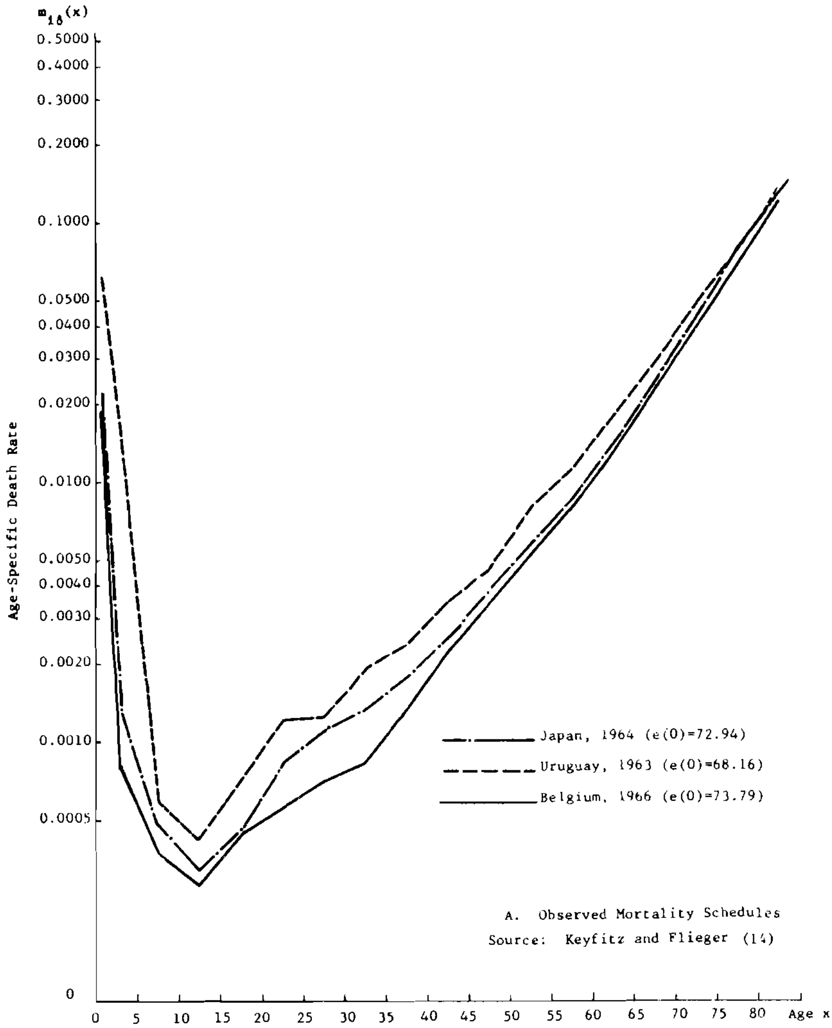


FIGURE 2. Observed and Model Female Mortality Schedules

rates among one segment of the population implying high (or low) migration rates for other segments of the same population. This association occurs because migration often is a response to changing economic conditions, and if these are good or poor for one segment of a population, they also are likely to be good or poor for other segments as well.

Demographers have long recognized the strong regularities that persist among age-specific schedules of migration, the most prominent being the high concentration of migration among young adults; see, for example, Long [15], Lowry [17]. Rates of migration are also high among children, starting with a

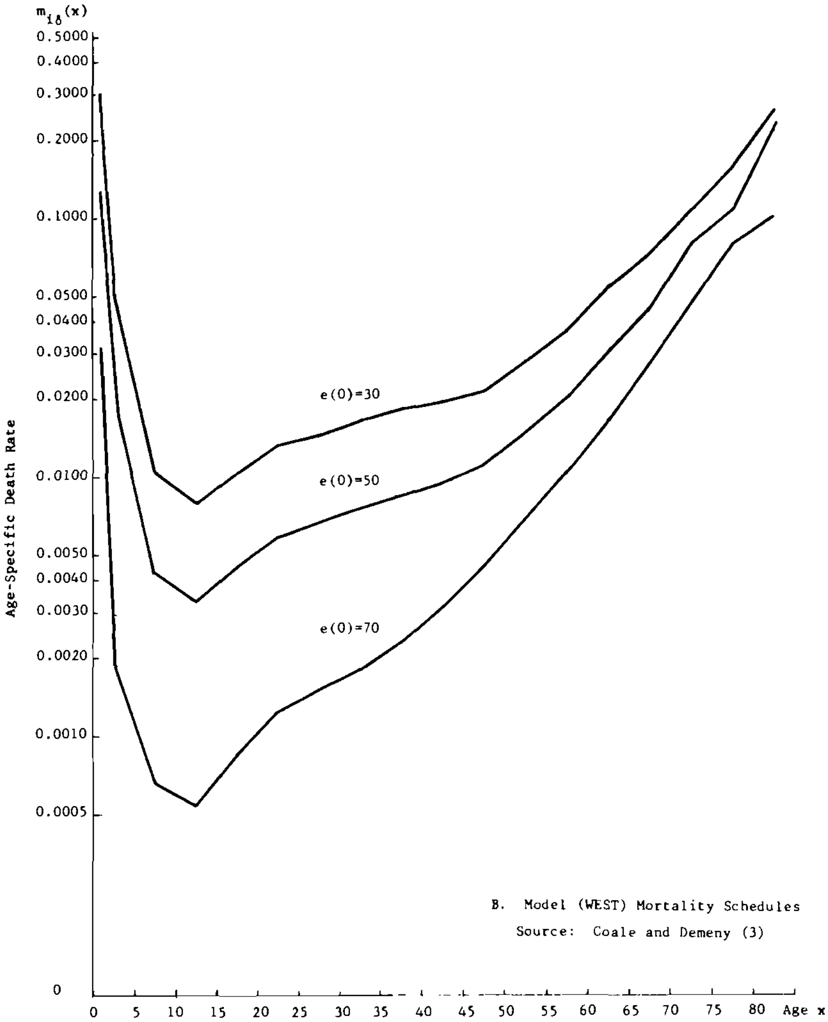


FIGURE 2. Observed and Model Female Mortality Schedules (Continued)

peak during the first year of life and dropping to a low point at about age 16. Beyond that age the curve turns sharply upward to another peak near age 22, declining regularly thereafter except for a slight hump around 62 through 65, the principal ages of retirement.

The empirical regularities are not surprising. Young adults exhibit the highest migration rates because they are much less constrained by ties to their community. They are more likely to be renters than home owners, their children generally are not yet in school, and job seniority is not an important consideration. Since children normally move only as members of a family, their migration pattern mirrors that of their parents. Inasmuch as younger children generally have younger parents, the migration rates of infants are higher than those of adolescents. Finally, the small hump in the age profile between ages 62 to 65 describes migration after retirement and usually reflects moves made to sunnier and milder climates.

Figure 3A repeats the fundamental age pattern of migration described above but expresses it in terms of 5-year age intervals. In consequence, the low rate of migration at age 16 is aggregated with the substantially higher rates that follow it, thereby shifting the low point among teenagers to a younger age. An analogous shift occurs with respect to the principal peak. The overall profile, however, remains essentially unchanged, with peaks occurring at infancy, during the young adult ages, and at retirement. Variations in the location of the principal peak and in the levels of migration to major retirement areas indicate that as in the case of mortality, age patterns of migration may usefully be disaggregated into families which are distinguished by the location and relative height of their peaks. Alternatively, such a disaggregation may be carried out, in the manner of fertility schedules, by means of the mean age of migration

$$\bar{n}_{ij} = \frac{\int_0^\omega x\nu_{ij}(x)dx}{\int_0^\omega \nu_{ij}(x)dx}$$

which readily may be used to classify migration schedules into "young" and "old" categories, perhaps with suitable gradations in between.

Two alternative ways of formally specifying the *level* of migration from one region to another are immediately suggested by our discussion of fertility and mortality schedules. The first adopts the fertility point of view and defines the migration level from region i to region j in terms of the area under the relevant migration schedule, designating it the *gross migra-production rate*, GMR_{ij} say. The second adopts a mortality perspective and defines the same migration level in terms of the fraction of an average person's lifetime that is spent in the region of destination. Specifically,

$${}_i\theta_j = \frac{{}_ie_j(0)}{{}_ie(0)}$$

is said to be the migration level with respect to region j of individuals born in region i. The numerator in the fraction represents the number of years expected to be lived in region j, on the average, by individuals born in region i and having a total life expectancy of ${}_ie(0)$ years. We adopt the latter perspective in this paper and in Figure 3B demonstrate its application by illustrating several typical model migration schedules. These are developed in a paper by Rogers and Castro [23] which also deals with the important problem of disaggregating such schedules into families containing "young" and "old" age profiles.

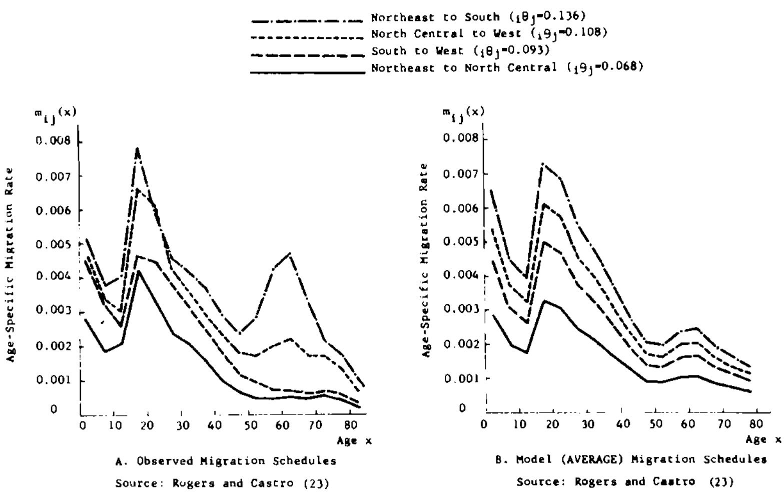


FIGURE 3. Observed and Model Female Migration Schedules (USA)

Regional Age Compositions and Regional Shares

The equations in (2.1) show how regional age compositions and regional shares together with age schedules of fertility, mortality, and migration determine the principal regional component rates of multiregional population growth and change. A single set of such age schedules can produce quite different crude regional rates of birth, death, and migration if combined with differing sets of regional age compositions and regional shares. Consequently such rates may be unsatisfactory summary measures of the components of multiregional population growth.

By way of illustration, consider the empirical age compositions set out in Figure 4A. Belgium had lower female mortality rates at every age in 1966 than did Uruguay in 1963, but it had a higher crude death rate ($11.15 > 9.67$). Japan, on the other hand, had lower fertility rates in 1964 than Belgium at every age save one, but it exhibited a higher crude birth rate ($16.91 > 15.17$). In each case, the cause of the apparent anomaly was the difference in the age compositions of the populations compared. Belgium had a much larger proportion of its population over 65 than did Uruguay. Japan had a substantially larger proportion of its population in the childbearing ages than did Belgium. Because these differences in age composition occurred at ages where the respective rates in the relevant schedule were high or low, changes in the age composition biased the values of the consolidated rates in the expected directions.

Changes in regional shares have an analogous but somewhat different way of helping to shape regional component rates. Regional shares serve as weights in the consolidation process. Hence the same outmigration rate originating from a region that is twice the size of another will develop twice the impact on the size

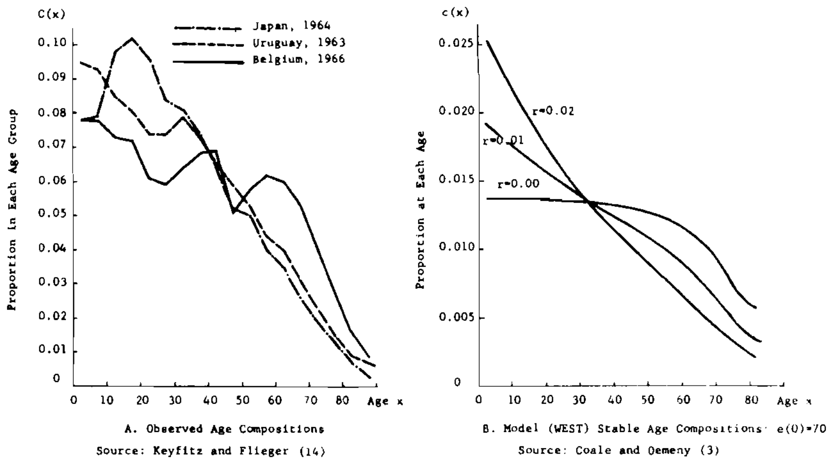


FIGURE 4. Observed and Model Female Age Compositions

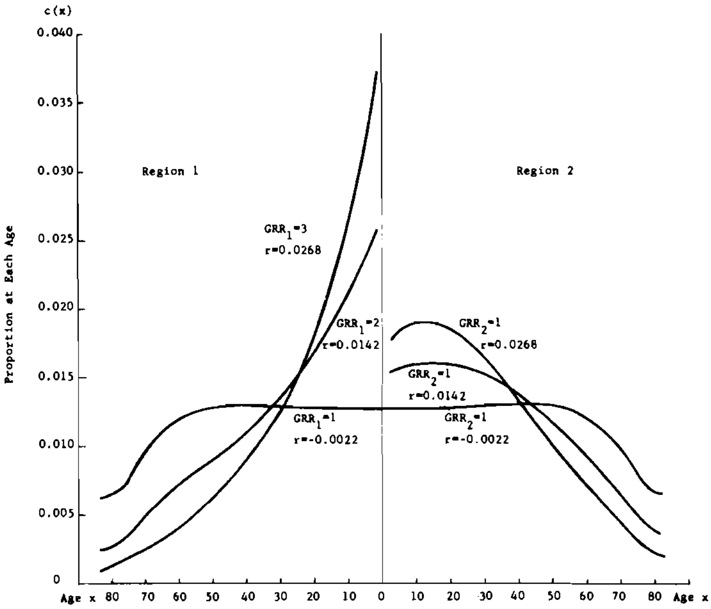
of the population in the destination region. Moreover, since any idiosyncracies in the age profile of a sending region's migration schedule are transmitted to the receiving region's population, large sources of heavy outmigration can have substantial impact on the values assumed by all of the component rates in a destination region.

Finally, while it is important to underscore the powerful influence that regional age compositions and regional shares have in shaping regional component rates, one must also recognize that past records of fertility, mortality, and migration play an equally important role in the determination of present regional age compositions and shares, inasmuch as the latter arise out of a history of regional births, deaths, and internal migration. For example, a region experiencing high levels of fertility will have a relatively younger population, but if it also is the origin of high levels of outmigration a large proportion of its young adults will move to other regions, producing a higher growth rate in the destination regions while lowering the average age of its own population.[1] This suggests that inferences made about fertility, say, on the basis of a model that ignores migration may be seriously in error. For example, Figure 4C illustrates the significant impact on the ultimate stable age composition and regional share of region 2 that is occasioned by a doubling and tripling of fertility levels in region 1 while

[1]The mean age of a regional population, like the mean ages of the fertility and migration schedules, is a summary measure of *pattern* and is defined as

$$a_j(t) = \frac{\int_0^\omega x c_j(x,t)\,dx}{\int_0^\omega c_j(x,t)\,dx} = \int_0^\omega x c_j(x,t)\,dx.$$

The regional share $SHA_j(t)$, on the other hand, is a summary measure of *level*.



C. Model (WEST) Stable Regional Age Compositions: $_1e(0) = {_2e(0)} = 70$; $_1\theta_2 = {_2\theta_1} = 0.3$

Source: Rogers and Castro (23)

FIGURE 4. Observed and Model Female Age Compositions (Continued)

holding everything else constant. The mean age of the population in region 2 declines by 5.1 and 8.9 years, respectively, while its regional share decreases by 24 percent in the first instance and by 36 percent in the second. It is to spatial population dynamics of this kind that we now turn in the remainder of this paper.

3. THE SPATIAL DYNAMICS OF STABLE POPULATIONS

The regional age compositions and regional shares of a closed multiregional population are completely determined by that population's recent history of fertility, mortality, and internal migration. A particularly useful way of understanding the evolution of such age compositions and shares is to imagine them as describing a population which has been subjected to fertility, mortality, and migration schedules that have remained unchanged for a long period of time, say a century. The population that develops under such circumstances is said to have been subjected to a fixed regime of growth and is called a *stable multiregional population.* Its principal characteristics are: unchanging regional age compositions and regional shares; constant regional annual rates of birth, death, and migration; and a fixed multiregional annual rate of growth that also is the annual rate of population increase in each and every region.

A frequently raised objection to the use of stable population theory is the implausibility of the assumption of an unchanging regime of growth. Such an objection confuses projection with prediction. No one truly believes that fertility, mortality, and migration schedules will remain unchanged for a prolonged period of time; yet our understanding of current demographic rates can be substantially enhanced by a projection of their long-run consequences. Keyfitz [13; p. 347] has likened such projections to "microscopic examinations" because they magnify the effects of differences in current rates in order to more easily identify their true meaning. Others have called them "speedometer readings" to emphasize their monitoring function and hypothetical nature; see, for example, Coale [1; p. 52] and Rogers [21; p. 426]. But perhaps the most vivid interpretation of the role of infinite horizon projections was offered by Gale [6; p. 2] in the context of economic planning: "To describe the situation figuratively, one is guiding a ship on a long journey by keeping it lined up with a point on the horizon even though one knows that long before that point is reached the weather will change (but in an unpredictable way) and it will be necessary to pick up a new course with a new reference point, again on the horizon rather than just a short distance ahead." In this section of our paper we examine the stable multiregional populations that evolve out of particular histories of fertility, mortality, and internal migration. By tracing through the ultimate consequences of alternative fixed regimes of growth, we strive for a further understanding of the spatial dynamics of the hypothetical populations that they describe.

Characteristics of Stable Multiregional Populations

Imagine a multiregional female population that has been exposed to a fixed regime of growth for a very long period of time.[2] The number of women aged x to $x + dx$ in this population at time t are survivors of those born x years ago, $x \leqslant t$, and therefore may be denoted by $\boldsymbol{P}(x)\{\boldsymbol{B}(t-x)\}dx$, where $\boldsymbol{P}(x)$ is a matrix of regional probabilities of surviving to age x and $\{\boldsymbol{B}(t)\}$ is a vector of regional births at time t. If subjected to a regime of fertility described by $\boldsymbol{M}(x)$, a diagonal matrix of age-specific annual rates of childbearing at age x, these women at time t give birth to $\boldsymbol{M}(x)\boldsymbol{P}(x)\{\boldsymbol{B}(t-x)\}dx$ baby girls per year. Integrating over all ages x, we obtain the multiregional Lotka renewal equation

$$\begin{aligned} \{\boldsymbol{B}(t)\} &= \int_0^t \boldsymbol{M}(x)\boldsymbol{P}(x)\ \{\boldsymbol{B}(t-x)\}dx \\ &= \int_0^t \boldsymbol{\Phi}(x)\{\boldsymbol{B}(t-x)\}dx, \text{ for } t \geqslant \beta, \end{aligned} \tag{3.1}$$

where β denotes the last age of childbearing and $\boldsymbol{\Phi}(x)$ is the multiregional net maternity function.[3]

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[2]We adopt the normal convention of mathematical demography and focus on the female population. It should be clear, however, that our arguments apply to any single-sex population: male, female, or total.

[3]Contrary to conventional matrix notation, we use a transposed ordering of subscripts so as to preserve a left-to-right ordering of successive regions of residence in the usual "matrix-

Following the procedure used in the single-region model by Keyfitz [8; Ch. 5] we observe that the trial solution $\{\boldsymbol{B}(t)\} = \{\boldsymbol{Q}\}\, e^{rt}$ satisfies (3.1) provided r takes on a particular value, which we shall derive presently. Substituting the trial solution into (3.1) transforms that equation into the *multiregional characteristic equation*

$$\{\boldsymbol{Q}\} = \int_\alpha^\beta e^{-rx}\boldsymbol{M}(x)\boldsymbol{P}(x)\{\boldsymbol{Q}\}dx = \left[\int_\alpha^\beta e^{-rx}\boldsymbol{\Phi}(x)dx\right]\{\boldsymbol{Q}\} = \boldsymbol{\Psi}(r)\{\boldsymbol{Q}\}, \tag{3.2}$$

where $\Psi(r)$ is the *multiregional characteristic matrix*. Note that the range of integration has been narrowed to embrace only the ages of childbearing α through β.

By moving from (3.1) to (3.2) we have reduced our problem to one of finding the value of the constant r that satisfies the characteristic equation:

$$\{\boldsymbol{Q}\} = \boldsymbol{\Psi}(r)\{\boldsymbol{Q}\}.$$

Rewriting that equation as:

$$[\boldsymbol{\Psi}(r) - \boldsymbol{I}]\{\boldsymbol{Q}\} = \{\boldsymbol{0}\} \tag{3.3}$$

we observe that $\{\boldsymbol{Q}\}$ is the characteristic vector that corresponds to the unit dominant characteristic root of $\boldsymbol{\Psi}(r)$, and r is the number which gives that matrix such a value for its dominant characteristic root.[4]

The system of equations in (3.3) can have only one maximal real root r and any complex roots that satisfy (3.3) must occur in complex conjugate pairs whose real components are smaller than the maximal real root. Consequently, the birth sequence $\{\boldsymbol{B}(t)\} = \sum_{h=1}^{\infty}\{\boldsymbol{Q}_h\}e^{r_h t}$ is increasingly dominated by its first term as t becomes large. Thus, ultimately,

$$\{\boldsymbol{B}(t)\} \doteq \{\boldsymbol{Q}_1\}e^{r_1 t} = \{\boldsymbol{Q}\}e^{rt},$$

where we omit the unit subscripts for convenience.

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times-a-vector" multiplication projection process of single-region mathematical demography. For example, the probability that a baby girl born in region j will be alive in region i at age x is denoted by ${}_jp_i(x)$ and appears as the element in the i^{th} row and j^{th} column of $\boldsymbol{P}(x)$. The multiplication of the vector of births $\{\boldsymbol{B}(t-x)\}$ by $\boldsymbol{P}(x)$ then yields a vector of sums such as $\sum_{j=1}^{m} B_j(t-x)\,{}_jp_i(x)$, in which the subscript referring to region of birth appears before the one defining the subsequent region of residence at age x. Extensions to denote several successive regions of residence, for example, ${}_jp_{ik}(x)$, are straightforward.

[4]Such a root is in fact a function which associates each value of r with the dominant characteristic root of $\boldsymbol{\Psi}(r)$ evaluated at this particular value of r. This function is continuous concave upward throughout, and its values decrease monotonically as its argument increases. Thus a dominant characteristic root of unity can occur only once, and it will always take on that value when r assumes its maximal value.

Exponentially growing births produce an exponentially growing population, $\{\boldsymbol{K}(t)\}$ say, which maintains stable regional age compositions and a constant regional allocation of the total multiregional population:

$$\begin{aligned}\{\boldsymbol{K}(t)\} &= \int_0^\omega \{\boldsymbol{k}(x,t)\}\,dx \\ &= \int_0^\omega \boldsymbol{P}(x)\{\boldsymbol{B}(t-x)\}\,dx \\ &= \int_0^\omega e^{r(t-x)}\boldsymbol{P}(x)\{\boldsymbol{Q}\}\,dx \\ &= e^{rt}\int_0^\omega e^{-rx}\boldsymbol{P}(x)\{\boldsymbol{Q}\}\,dx = e^{rt}\boldsymbol{b}^{-1}\{\boldsymbol{Q}\} = \{\boldsymbol{Y}\}\,e^{rt},\end{aligned}$$

where $\{\boldsymbol{Y}\}$ is a vector of *stable equivalent regional populations* defined by Keyfitz [9], and $\boldsymbol{b}$ is a diagonal matrix of regional intrinsic birth rates

$$b_i = \frac{1}{\int_0^\omega e^{-rx}\sum_{j=1}^m \frac{Q_j}{Q_i}\,{}_jp_i(x)dx}. \tag{3.5}$$

A multiregional population that is projected to stability under a constant regime of growth will ultimately increase by the ratio e^{5r} every five years. If this population were stable to begin with and contained Y_i individuals in each region, $i = 1, 2, \ldots, m$, then by time $5t$ it would have grown to $\{\boldsymbol{Y}\}\,e^{5rt}$. Thus, as Keyfitz suggests, the stable equivalent population of an observed population may be found by projecting the latter *forward* t periods to stable growth and then *backward* an equal length of time by dividing by e^{5rt}. The resulting hypothetical population, if increased by the ratio e^{5r} after every unit time interval of 5 years, would approach the same asymptotic levels as the projected observed population. By analogous reasoning,

$$\{\boldsymbol{Q}\} = \{\boldsymbol{B}(0)\} = \boldsymbol{b}\{\boldsymbol{Y}\} \tag{3.6}$$

may be referred to as the vector of *stable equivalent regional births.*

The number of j-born persons at age x in region i in a stable multiregional population is equal to the number born x years ago in region j times the proportion of those babies alive x years later in region i. Summing this quantity over all regions in the multiregional system and dividing it by the same total integrated over all ages of life gives the regional age composition

$$c_i(x) = \frac{\sum_{j=1}^m Q_j\,e^{-rx}\,{}_jp_i(x)}{\int_0^\omega \sum_{j=1}^m Q_j\,e^{-rx}\,{}_jp_i(x)dx} = b_i e^{-rx}\sum_{j=1}^m \frac{Q_j}{Q_i}\,{}_jp_i(x)$$

or, in matrix form,

$$\{\boldsymbol{c}(x)\} = \boldsymbol{b}\,e^{-rx}\,\boldsymbol{Q}^{-1}\,\boldsymbol{P}(x)\,\{\boldsymbol{Q}\} = \boldsymbol{Q}^{-1}\boldsymbol{c}(x)\,\{\boldsymbol{Q}\}, \tag{3.7}$$

where

$$\boldsymbol{c}(x) = \boldsymbol{b}\, e^{-rx}\, \boldsymbol{P}(x) \tag{3.8}$$

and $\boldsymbol{Q}$ is a diagonal matrix with the elements of $\{\boldsymbol{Q}\}$ along its diagonal.

Having found the stable age composition of each regional population we may proceed to develop a number of demographic measures that describe other important characteristics of such stable multiregional populations. First, the mean age of the population in region j is given by

$$a_j = \int_0^\omega x\, c_j(x)dx,$$

and its intrinsic rates of birth, death, outmigration, inmigration and net migration are, respectively,

$$b_j = \int_0^\omega c_j(x)m_j(x)dx$$

$$d_j = \int_0^\omega c_j(x)\mu_j(x)dx$$

$$o_j = \int_0^\omega c_j(x) \sum_{\substack{i=1 \\ i \neq j}}^{m} \nu_{ji}(x)dx$$

$$i_j = r - b_j + d_j + o_j$$

$$n_j = i_j - o_j$$

where $\mu_j(x)$ is the instantaneous (that is, compounded momently) annual rate of mortality at age x in region j and $\nu_{ji}(x)$ is the corresponding rate of migration from region j to region i. Another useful measure is the net absence rate:

$$\Delta_j = b_j - r = d_j - n_j.$$

Finally, the share of the total multiregional population that is allocated to region j at stability may be defined in terms of stable equivalent populations as

$$SHA_j = \frac{Y_j}{\sum_{i=1}^{m} Y_i}.$$

Table 1 presents several fundamental characteristics of the stable female United States population that evolves from a projection using the 1968 growth regime. The national territory is divided into two regions: the West region defined by the U.S. Census Bureau and the rest of the United States.[5] The expectation of life of women born in the West was found to be ${}_1e(0) = 75.49$ years with ${}_1e_2(0) = 23.10$ years of that total (31 percent) expected to live

[5]The West region is comprised of the following 13 states: Alaska, Arizona, California, Colorado, Hawaii, Idaho, Montana, Nevada, New Mexico, Oregon, Utah, Washington, and Wyoming.

TABLE 1. Relations Under Stability: Female Population of the United States, 1968

	1. The West Region			2. The Rest of the United States		
	1968 Population			Proportion		
Age, x	1 + 2	1	2	1 + 2	1	2
0	8,245,762	1,411,337	6,834,425	0.0806	0.0817	0.0804
5	9,597,933	1,652,125	7,945,808	0.0938	0.0957	0.0935
10	10,000,941	1,711,965	8,288,976	0.0978	0.0992	0.0975
15	9,253,495	1,576,909	7,676,586	0.0905	0.0913	0.0903
20	8,289,804	1,474,897	6,814,907	0.0811	0.0854	0.0802
25	6,722,467	1,217,733	5,504,734	0.0657	0.0705	0.0648
30	5,721,493	1,020,086	4,701,407	0.0559	0.0591	0.0553
35	5,583,993	966,359	4,617,634	0.0546	0.0560	0.0543
40	6,042,636	1,022,598	5,020,038	0.0591	0.0592	0.0590
45	6,143,112	1,050,292	5,092,820	0.0601	0.0608	0.0599
50	5,644,471	926,417	4,718,054	0.0552	0.0537	0.0555
55	5,106,221	809,787	4,296,434	0.0499	0.0469	0.0505
60	4,500,799	684,070	3,816,729	0.0440	0.0396	0.0449
65	3,794,498	566,234	3,228,264	0.0371	0.0328	0.0380
70	3,068,152	461,793	2,606,359	0.0300	0.0267	0.0307
75	2,230,070	341,626	1,888,444	0.0218	0.0199	0.0222
80	1,381,406	217,761	1,163,645	0.0135	0.0126	0.0137
85+	949,739	152,125	797,614	0.0093	0.0088	0.0094
Total	102,276,992	17,264,114	85,012,878	1.0000	0.1688	0.8312

STABLE POPULATION:

	1 + 2	1	2
r = Rate of growth	0.00432		
Y = Stable equivalent population	121,292,482	26,989,870	94,302,612
$SHA = Y/\Sigma Y$ = Stable regional share	1.0000	0.2225	0.7775
Q = Stable equivalent births	1,920,961	410,412	1,510,549
$SBR_{21} = Q_2/Q_1$ = Stable birth ratio	3.68		
b = Birth rate	0.0158	0.0152	0.0160
d = Death rate	0.0115	0.0114	0.0115
o = Outmigration rate	—	0.0083	0.0025
i = Inmigration rate	—	0.0088	0.0024
n = Net migration rate	—	0.0005	−0.0002
a = Mean age	37.18	38.12	36.91

in the rest of the United States. Women born in the rest of the United States, on the other hand, were expected on the average to live a total of ${}_2e(0) = 74.29$ years with ${}_2e_1(0) = 6.95$ years of that total (9 percent) expected to live in the West. Fertility in the West was slightly lower than in the rest of the United States. The former had a gross reproduction rate of 1.13, whereas the latter experienced a GRR of 1.17. Symbolically,

$$\boldsymbol{e}(0) = \begin{bmatrix} 52.39 & 6.95 \\ 23.10 & 67.34 \end{bmatrix} \qquad \boldsymbol{GRR} = \begin{bmatrix} 1.13 & 0 \\ 0 & 1.17 \end{bmatrix}$$

where $\boldsymbol{e}(0) = \boldsymbol{EXP} \cdot \boldsymbol{\theta}$, and

$$\boldsymbol{EXP} = \begin{bmatrix} 75.49 & 0 \\ 0 & 74.29 \end{bmatrix} \qquad \boldsymbol{\theta} = \begin{bmatrix} 0.69 & 0.09 \\ 0.31 & 0.91 \end{bmatrix}$$

The stable projection allocates 22.25 percent of the ultimate national population to the West and accords it an annual growth rate of 4.3 per 1000, an annual birth rate of 15.2 per 1000, and a positive annual net migration rate of 0.5 per

1000. The stable population of the rest of the United States increases at the same intrinsic annual rate of growth, but its other demographic characteristics are quite different. It has a somewhat younger population, a higher annual birth rate, and exhibits a very slight net outmigration to the West. Both regional stable populations are a few years older in mean age than the corresponding observed 1968 populations.

Two Families of Model Stable Multiregional Populations

The numerical evaluation of the multiregional population growth process described above usually involves a population disaggregated into 18 five-year age groups (0–4 through 85+) of which 8 are assumed to be capable of childbearing ($\alpha = 10$ through $\beta = 50$). Thus the mathematical representation requires 8 age-specific birth rates, 18 age-specific death rates, and 18 $(m - 1)$ age-specific migration rates for each of the m regions comprising the multiregional system. We have seen, however, that among human populations such rates exhibit persistent regularities and therefore are not truly independent observations. In consequence, a remarkably accurate description of spatial population dynamics can be realized with the aid of model stable populations that have been generated using a much smaller number of indices of variation in fertility, mortality, and migration which summarize the kinds of regularities that were identified in Section 2.

In their monumental study of single-region model life tables and model stable populations, Coale and Demeny [3] present two overlapping sets of stable populations which to a large extent provide similar information. Each population is identified by two nonredundant indices of variation relating to fertility and mortality, respectively, and evolves out of a particular combination of a model life table and intrinsic rate of growth or gross reproduction rate. The former are referred to as the "growth rate" stable populations; the latter are called the *GRR* stable populations and rely on a model fertility schedule with a given mean age of childbearing m, which is assumed to be 29 years. Symbolically, the two sets of model stable populations may be expressed as:

1) Growth Rate Stable Populations: $f(e(0),r)$
2) *GRR* Stable Populations: $g(e(0), GRR)$

Model stable multiregional populations may be developed by means of a straightforward extension of the Coale and Demeny method. Underlying every such model population are: (1) a set of regional mortality levels specified by regional expectations of life at birth; (2) a set of regional fertility levels defined either by an intrinsic rate of growth and an associated proportional regional allocation of total stable equivalent births, or by a set of regional gross reproduction rates; and, finally, (3) a set of interregional migration levels between every pair of regions in the multiregional system. Symbolically, we may once again express two sets of model stable populations:

1) Growth Rate Stable Multiregional Populations: $f(\boldsymbol{EXP}, r, \boldsymbol{SBR}, \boldsymbol{\theta})$
2) *GRR* Stable Multiregional Populations: $g(\boldsymbol{EXP}, \boldsymbol{GRR}, \boldsymbol{\theta})$

where **EXP** is a diagonal matrix of regional expectations of life at birth ${}_ie(0)$, **SBR** is a matrix of stable equivalent birth ratios: $SBR_{ji} = Q_j/Q_i$; $\boldsymbol{\theta}$ is a matrix of migration levels ${}_j\theta_i$; and **GRR** is a diagonal matrix of regional gross reproduction rates GRR_i.[6]

Coale and Demeny observe that growth rate stable populations are more convenient for exploring the implications of various recorded intercensal rates of growth, whereas *GRR* stable populations are more useful in analyses of the effects of different levels of fertility and mortality. An analogous observation may be made with respect to multiregional populations. Growth rate stable multiregional populations are more convenient for examining the implications of various recorded intercensal rates of growth and regional allocations of total births, whereas *GRR* stable multiregional populations are more suitable for assessing the impacts of different combinations of regional levels of fertility, mortality, and migration.

Growth rate stable multiregional populations also may be used in connection with analyses of regional allocations of the total multiregional population. Expressing the stable regional shares in the form of a diagonal matrix **SHA**; we easily may establish that

$$\{\boldsymbol{b}\} = \boldsymbol{SHA}^{-1}\left[\sum_{x=0}^{\omega} e^{-r(x+2.5)}\boldsymbol{L}(x)\right]^{-1}\boldsymbol{SHA}\,\{\boldsymbol{1}\} \tag{3.9}$$

and with it obtain

$$\boldsymbol{SBR} = [\boldsymbol{SHA}\cdot\boldsymbol{b}]^{-1}\,\boldsymbol{1}\,[\boldsymbol{SHA}\cdot\boldsymbol{b}], \tag{3.10}$$

where $\{\boldsymbol{b}\} = \boldsymbol{b}\,\{\boldsymbol{1}\}$ and $\boldsymbol{L}(x)$ is a matrix comprised of elements ${}_jL_i(x)$ that denote the stationary population aged x to $x+4$ years in region i who were born in region j.[7] Thus we may work with r and either **SBR** or **SHA**. Hence our earlier symbolic expression for growth rate stable multiregional populations has the alternative form:

1b). Growth Rate Stable Multiregional Populations: $h(\boldsymbol{EXP}, r, \boldsymbol{SHA}, \boldsymbol{\theta})$.

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[6]Note that $\boldsymbol{e}(0) = \boldsymbol{EXP}\cdot\boldsymbol{\theta}$ and that $\boldsymbol{SBR} = \boldsymbol{Q}^{-1}\,\boldsymbol{1}\,\boldsymbol{Q}$ where $\boldsymbol{1}$ is a matrix of ones.

[7]The reciprocal of the expectation of life at birth in a single-region life table is equal to the birth rate of the stationary life-table population. Equation (3.9) with $r = 0$ may be used to establish the corresponding result for the multiregional life-table population:

$$\{\boldsymbol{b}\} = \boldsymbol{SHA}^{-1}\,\boldsymbol{e}(0)^{-1}\,\boldsymbol{SHA}\,\{\boldsymbol{1}\}$$

which, for example, in a two-region model gives

$$b_i = \frac{{}_je_j(0) - \dfrac{SHA_j}{SHA_i}\,{}_je_i(0)}{{}_ie_i(0)\,{}_je_j(0) - {}_ie_j(0)\,{}_je_i(0)}, \quad i, j = 1, 2.$$

The regional shares in this case refer of course to the regional distribution of the stationary population.

Table 2 sets out several specimen model stable multiregional populations which were generated by combining various model schedules of fertility, mortality, and migration.[8] Each of the 12 populations may be expressed symbolically by any one of the three forms listed earlier. For example, the first stable multiregional population may be expressed as a function of

$$\boldsymbol{EXP} = \begin{bmatrix} 70 & 0 \\ 0 & 70 \end{bmatrix} \quad r = -0.0022 \quad \boldsymbol{SBR} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \boldsymbol{\theta} = \begin{bmatrix} 7/10 & 3/10 \\ 3/10 & 7/10 \end{bmatrix}$$

in which $\boldsymbol{SBR}$ could be replaced by

$$\boldsymbol{SHA} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}.$$

Alternatively, the same population may be described as a function of the same $\boldsymbol{EXP}$ and $\boldsymbol{\theta}$ matrices but with r and $\boldsymbol{SBR}$ (or $\boldsymbol{SHA}$) replaced by

$$\boldsymbol{GRR} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Spatial Impacts of Changes in the Components of Multiregional Population Growth

Perhaps the simplest way to examine the spatial impacts of particular changes in schedules of fertility, mortality, and migration on an observed population is by means of population projection. Such arithmetical calculations, carried out first with the original and then with revised schedules, readily identify the effects of the differences between the two growth regimes. However, this approach suffers from a lack of generality and fails to reveal functional relationships that may exist between the changes occasioned in the population studied and its fundamental parameters. Thus mathematical demographers concerned with population dynamics such as Coale [1] and Preston [19] have focused their attention on the behavior of model populations that evolve from different growth regimes, while Goodman [7] and Keyfitz [11] have developed mathematical formulas that trace through the impacts of particular changes in age-specific rates on the population subjected to those rates. Both approaches have adopted the stable population as their basic model, and both can be extended to multiregional populations.

Model stable multiregional populations readily reveal the impacts of changes in fertility, mortality, and migration levels. By varying these levels either singly or in various combinations, we may establish the long-run consequences of

[8] To develop the fertility matrices $\boldsymbol{F}(x)$ and the life table matrices $\boldsymbol{L}(x)$ which are needed as inputs to the calculations, we used Coale and Demeny's basic fertility schedule for $\bar{m} = 29$, their "WEST" model life tables, and our own "AVERAGE" model migration schedules which are set out in Rogers and Castro [23]. Given $\boldsymbol{F}(x)$ and $\boldsymbol{L}(x)$ for all x, we evaluate $\boldsymbol{\Psi}(r)$, determine the r that gives it a dominant characteristic root of unity, and solve for the associated characteristic vector $\{\boldsymbol{Q}\}$.

TABLE 2. Model GRR Stable Multiregional (Two-Region) Female Populations with Equal Mortality Levels: ${}_1e(0) = {}_2e(0) = 70$

A. Low Fertility in Region 1

Fertility Levels: $GRR_1 = 1$

	$GRR_2 = 1$			$GRR_2 = 2$			$GRR_2 = 3$		
Equal Migration Levels: ${}_1\theta_2 = {}_2\theta_1 = 0.3$									
Parameters	1 + 2	1	2	1 + 2	1	2	1 + 2	1	2
r	−0.0022			0.0142			0.0268		
SBR_{21}	1.00			2.99			5.28		
SHA	1.0000	0.5000	0.5000	1.000	0.3832	0.6168	1.0000	0.3199	0.6801
b	0.0131	0.0131	0.0131	0.0232	0.0152	0.0282	0.0331	0.0165	0.0409
a	39.08	39.08	39.08	30.80	33.96	28.84	25.34	30.17	23.06
Unequal Migration Levels: ${}_1\theta_2 = 0.2$; ${}_2\theta_1 = 0.4$									
Parameters	1 + 2	1	2	1 + 2	1	2	1 + 2	1	2
r	−0.0022			0.0106			0.0222		
SBR_{21}	0.50			1.60			3.01		
SHA	1.0000	0.6667	0.3333	1.0000	0.5391	0.4609	1.0000	0.4550	0.5450
b	0.0131	0.0131	0.0131	0.0208	0.0148	0.0277	0.0293	0.0161	0.0404
a	39.08	39.08	39.08	32.52	35.08	29.52	27.22	31.52	23.63

B. High Fertility in Region 1

Fertility Levels: $GRR_1 = 3$

	$GRR_2 = 1$			$GRR_2 = 2$			$GRR_2 = 3$		
Equal Migration Levels: ${}_1\theta_2 = {}_2\theta_1 = 0.3$									
Parameters	1 + 2	1	2	1 + 2	1	2	1 + 2	1	2
r	0.0268			0.0311			0.0369		
SBR_{21}	0.19			0.51			1.00		
SHA	1.0000	0.6801	0.3199	1.0000	0.5884	0.4116	1.0000	0.5000	0.5000
b	0.0331	0.0409	0.0165	0.0368	0.0414	0.0303	0.0419	0.0419	0.0419
a	25.34	23.06	30.17	23.68	22.49	25.40	21.70	21.70	21.70
Unequal Migration Levels: ${}_1\theta_2 = 0.2$; ${}_2\theta_1 = 0.4$									
Parameters	1 + 2	1	2	1 + 2	1	2	1 + 2	1	2
r	0.0306			0.0332			0.0369		
SBR_{21}	0.10			0.26			0.50		
SHA	1.0000	0.7976	0.2024	1.0000	0.7367	0.2633	1.0000	0.6667	0.3333
b	0.0363	0.0413	0.0167	0.0386	0.0416	0.0305	0.0419	0.0419	0.0419
a	23.88	22.56	29.09	22.93	22.20	24.97	21.70	21.70	21.70

particular changes in the components of population growth and, in the process, obtain an improved understanding of the population dynamics that are involved. For example, consider some of the more interesting aspects of population dynamics that are revealed by the stable populations presented in Table 2 (and illustrated in Figures 4B and 4C). First, an unchanging multiregional growth regime in which regional fertility and mortality levels are identical produces identical stable regional age compositions, even though their stable regional shares vary inversely with the ratio of their migration levels; that is to say, $SHA_j/SHA_i = {}_i\theta_j/{}_j\theta_i$. Second, as in the single-region model, higher values of the intrinsic rate of growth create stable (regional) populations that taper more rapidly with age and, in consequence, include a higher proportion of the population below every age. Further, fertility affects not only the rate of growth of a stable population but, in the multiregional case, it also affects the stable regional allocations of such populations. Mortality and migration schedules also affect the form of the stable regional age compositions and the stable regional shares in an obvious way, and any idiosyncracies in the age patterns of such schedules will be reflected in the stable regional populations.

A rather surprising finding is the relative insensitivity of the regional age compositions and birth rates to migration levels. Consider, for example, the case of unequal migration levels with $GRR_1 = 1$, $GRR_2 = 3$ and $GRR_1 = 3$, $GRR_2 = 1$, respectively. In the first instance the region with the larger (by a factor of 2) outmigration has the higher fertility level; in the second case the situation is reversed. Yet in both instances the population in the region with the higher fertility level has an average age of approximately 23 years and exhibits a birth rate of approximately 41 per 1000. This insensitivity to migration behavior does not extend to systemwide measures, however. For example, the intrinsic growth rate and systemwide birth rate are considerably lower in the first case than in the second, and the higher fertility region assumes a stable regional share of only 54 percent in the first case but receives 80 percent in the second.

The compounding of regional differentials in mortality with those of fertility and migration produces complex interactions that generate even more complex patterns of growth and change. For example, in a two-region population system with fixed, identical regional schedules of fertility and migration, the regional population with the higher expectation of life at birth (i.e., with the lower mortality level) ultimately assumes the higher stable regional share of the total multiregional population and becomes the older population with the lower birth rate. As fertility in the region with the higher life expectancy is increased relative to that in the other region, the high fertility population assumes an even higher stable regional share and develops into the younger population with the higher birth rate. However, if the increase in relative fertility occurs instead in the region with the lower life expectancy, this pattern may be reversed and the regional population with the higher mortality level can become the population with the higher stable regional share, the lower average age, and the higher birth rate.

We have considered some of the spatial impacts of changes in the components of population growth by examining model stable multiregional populations. We could instead have directed our efforts toward a mathematical analysis of the impacts on the stable population of changes in rates at a particular age in the manner of Keyfitz [11]. The details of such an approach will be developed in a forthcoming paper and we, therefore, only sketch out the principal arguments here. Our approach follows Keyfitz's chain of derivations and centers on the multiregional generalization of his principal formulas.

Keyfitz begins his derivations by tracing through the effect on $p(a)$, the probability of surviving from birth to age a, of a change $\Delta\mu(x)$ in the age-specific death rate $\mu(x)$ applied to ages x to $x + \Delta x$, where $a > x + \Delta x$. He shows that the change $\Delta p(a)$ occasioned by the change $\Delta\mu(x)$ may be found by

$$\Delta p(a) \doteq -p(a)\Delta\mu(x)\Delta x, \quad a > x + \Delta x, \tag{3.11}$$

and concludes that the effect of a change in the age-specific death rate $\mu(x)$ on the expectation of life at age a is approximated by

$$\Delta e(a) \doteq -\frac{p(x)}{p(a)} e(x)\Delta\mu(x). \tag{3.12}$$

Keyfitz then goes on to identify the effects of changes in age-specific birth and death rates on stable population parameters such as the intrinsic rate of growth, the intrinsic birth and death rates, and the age composition and mean age of the stable population.

The multiregional generalizations of (3.11) and (3.12) may be shown to be, respectively,

$$\Delta \boldsymbol{P}(a) \doteq -\Delta \boldsymbol{V}(x)\Delta x \boldsymbol{P}(a) \qquad a > x + \Delta x \tag{3.13}$$

and

$$\Delta \boldsymbol{e}(a) \doteq -\Delta \boldsymbol{V}(x)\boldsymbol{e}(x)\boldsymbol{P}(x)\boldsymbol{P}(a)^{-1}, \tag{3.14}$$

where, for example, in a two-region population system

$$\boldsymbol{V}(x) = \begin{bmatrix} \mu_i(x) + \nu_{ij}(x) & -\nu_{ji}(x) \\ -\nu_{ij}(x) & \mu_j(x) + \nu_{ji}(x) \end{bmatrix},$$

in which $\mu_i(x)$ and $\nu_{ij}(x)$ denote instantaneous rates of mortality and migration, respectively.

4. THE SPATIAL DYNAMICS OF STATIONARY POPULATIONS

Increasing public concern about the sizes and growth rates of national populations has generated a vast literature on the social, economic, and environmental impacts of a reduction of fertility to replacement levels and the consequent evolution of national populations to a zero growth condition; see, for example, Coale [2] and Frejka [5]. But where people choose to live in the future presents issues and problems that are potentially as serious as those posed by the number of children they choose to have. Yet the spatial implications of reduced fertility have received very little attention and we are, in consequence, ill-equipped to develop an adequate response to questions such as the one recently posed by the Commission on Population Growth and the American Future [4; p. 13]:[9] "How would stabilization of the national population affect migration and local growth"?

The Commission observes that zero growth for the nation will mean an *average* of zero growth for local areas. This, of course, still allows for the possibility of nonzero growth in particular localities. Thus *spatial* zero growth, like *temporal* zero growth, may be viewed either as a condition that ultimately prevails uniformly or one that exists only because of a fortuitous balancing of rates of positive growth, of zero growth, and of decline. Since no obvious advantages arise from the latter case, demographers quite naturally have viewed the attainment of temporal zero growth in the long-run in terms of an indefinite continuation of temporal zero growth in the short-run. We follow this tradition in this paper and view the attainment of spatial zero growth in the long-run in terms of temporal zero growth within each region of a closed multiregional population system. In consequence, we confine our attention here to the evolution of *stationary* multiregional populations; that is, stable multiregional populations that have a zero growth rate. Thus we augment the usual twin assumptions of a fixed mortality schedule and a fixed fertility schedule set at replacement level with the assumption of a fixed migration schedule. Multiregional populations subjected to such regional growth regimes ultimately assume a persisting zero rate of growth in every region and exhibit zero growth both over time and over space.

Characteristics of Stationary Multiregional Populations

If age-specific death rates remain unchanged and the annual number of births is fixed, a population that is closed to migration will ultimately evolve into a stationary population. The characteristics of such a population are well known. The number of individuals at any age would remain fixed, and the total number of deaths would exactly equal the total number of births. Because mortality risks would be relatively low from just after birth through middle age, the age composition of such a population would be nearly rectangular until ages 50 or 60, tapering much more rapidly thereafter as death rates increase among the older population.

The maintenance of a stationary population requires that parents have only as many children as are needed to maintain a fixed number of births every

[9]A notable exception is the work of Morrison [18; p. 547], who concludes: ". . . demographic processes interact in subtle and often complex ways, and the mechanisms by which declining fertility would influence population redistribution are only partially understood. Their elucidation can furnish a clearer picture of how and under what circumstances population redistribution can be influenced by public policy."

year. This means, for example, that 1000 women must on the average produce 1000 baby girls during their lifetime. Moreover, since some women will not survive to become mothers, those who do must have slightly more than 1000 daughters in order to compensate for those who don't. Hence the gross reproduction rate must be greater than unity by an amount just sufficient to maintain a unit level of *net* reproduction. For example, about 97 to 98 percent of women in the United States today survive to the principal ages of childbearing. Consequently, those who do must have approximately 1.027 daughters, on the average, as they pass through the childbearing ages. In other words, the *GRR* must be 1.027 in order for the *NRR* to be unity.[10]

The net reproduction rate, like the total fertility rate and the gross reproduction rate, summarizes the fertility experience of a population of all ages during a single year as if it were the experience of a single cohort that passed through all ages. It is a hypothetical value that treats cross-sectional data as if it were longitudinal data in order to estimate the number of daughters that would be born per woman subjected to specified age-specific risks of fertility and mortality. A commonly used procedure for obtaining *NRR* is to multiply each female age-specific fertility rate by the corresponding probability of surviving from birth to that age and then to integrate the product over all ages of childbearing:

$$NRR = \int_{\alpha}^{\beta} p(x)m(x)dx \equiv R(0).$$

Since *NRR* may be viewed as the zero$^{\text{th}}$ moment of the net maternity function, it usually is denoted by $R(0)$, a notation which we shall adopt henceforth.

Total births in a stationary multiregional population must, of course, also equal total deaths. However, because of the redistributive effects of migration, total births in any particular region need not equal total deaths in that region. This can be readily demonstrated by means of the accounting identity connecting regional stable intrinsic rates:

$$r = b_j - d_j - o_j + i_j = b_j - d_j + n_j,$$

which with $r = 0$ defines the fundamental relationship that must hold in every region of a stationary multiregional population:

$$\hat{b}_j + \hat{n}_j = \hat{d}_j, \tag{4.1}$$

where the caret is introduced to designate a stationary population. Thus only if the net migration rate is zero will regional births equal regional deaths.

The maintenance of a stationary multiregional population requires that the

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10Because there are usually about 105 baby boys born for every 100 baby girls, mothers in a stationary population of males and females would need to have a total rate of reproduction about 3 percent more than twice 1.027. In this way we obtain the total fertility rate of 2.11 used, for example, in the United States Bureau of the Census [24] projections.

total number of births in every region remain constant over time. Thus we may substitute the trial solution vector $\{\boldsymbol{B}(t)\} = \{\hat{\boldsymbol{Q}}\}$ into (3.1) to find

$$\{\hat{\boldsymbol{Q}}\} = \left\{\int_{\alpha}^{\beta} \hat{\boldsymbol{M}}(x)\boldsymbol{P}(x)dx\right\} \{\hat{\boldsymbol{Q}}\} = \hat{\boldsymbol{R}}(0)\{\hat{\boldsymbol{Q}}\}, \tag{4.2}$$

where carets are once again used to distinguish stationary population measures, and the element in the i^{th} row and the j^{th} column of $\hat{\boldsymbol{R}}(0)$ is the stationary regional net reproduction rate in region i of women born in region j:

$$_{j}\hat{R}_{i}(0) = \int_{\alpha}^{\beta} {}_{j}p_{i}(x)\hat{m}_{i}(x)dx.$$

Equation (4.2) shows that for a stationary multiregional population to be maintained, the dominant characteristic roots of the matrix $\hat{\boldsymbol{R}}(0)$ must be unity. Consequently a reduction of fertility to replacement level may be interpreted as a reduction of the elements of $\boldsymbol{M}(x)$ to a level that reduces the dominant characteristic root of a given net reproduction matrix $\boldsymbol{R}(0)$ to unity. Such an operation transforms $\boldsymbol{M}(x)$ to $\hat{\boldsymbol{M}}(x)$ and $\boldsymbol{R}(0)$ to $\hat{\boldsymbol{R}}(0)$.

Stabilization of the regional populations in a multiregional population will alter the relative contributions of natural increase and migration to regional growth. Regional age compositions will also be affected, and in ways that are strongly influenced by the age patterns of migration. Retirement havens such as San Diego and Miami, for example, will receive proportionately higher flows of inmigrants as the national population increases in average age, whereas destinations that previously attracted mostly younger migrants will receive proportionately fewer inmigrants. Finally, as we demonstrate in the next section, the redistributional effects of stabilization depend in a very direct way on the redistribution of total births that is occasioned by the reduction in fertility.

Alternative Spatial Paths to a Stationary Multiregional Population

In his paper for the Commission on Population Growth and the American Future, Ansley Coale [2] considers three alternative paths to a stationary population: (1) maintaining births constant at the levels recorded in 1970; (2) moving to a replacement level of fertility either immediately or in the very near future; and (3) reducing childbearing levels such that total population size is held fixed beginning immediately. He finds only slight differences between the first two alternatives and rejects the third as infeasible since it would require an immediate decline in the birth rate of almost 50 percent. We shall therefore confine our attention to Coale's second alternative path and will explore a few of its spatial ramifications.

Imagine a multiregional population system growing at some positive rate of growth. This will exhibit a net reproduction matrix $\boldsymbol{R}(0)$ with a dominant characteristic root $\lambda_1(\boldsymbol{R}(0))$ greater than unity. If the rate of childbearing in each region of this multiregional population system were immediately reduced such that every woman born in that region would have a net reproduction rate of unity, then

$$ {}_i\hat{R}(0) = \sum_{j=1}^{m} {}_i\hat{R}_j(0) = 1 $$

or, in matrix form,

$$ \hat{\boldsymbol{R}}(0)'\{\boldsymbol{1}\} = \{\boldsymbol{1}\}, \qquad (4.3) $$

where the prime denotes transposition.

As in the normal practice in single-region exercises of this kind, assume that the reduction of the fertility of each regional cohort of women is achieved by reducing that cohort's age-specific fertility rates by the same fixed proportion, γ_i say. Then

$$ {}_i\hat{R}(0) = \sum_{j=1}^{m} {}_i\hat{R}_j(0) = \sum_{j=1}^{m} \int_{\alpha}^{\beta} {}_ip_j(x)\gamma_j m_j(x)dx = \sum_{j=1}^{m} \gamma_j\, {}_iR_j(0) = 1 $$

and

$$ \hat{\boldsymbol{R}}(0) = \boldsymbol{\gamma}\boldsymbol{R}(0), \qquad (4.4) $$

where $\boldsymbol{\gamma}$ is a diagonal matrix of fertility adjustment factors. Substituting (4.4) into (4.3) gives

$$ \boldsymbol{R}(0)'\boldsymbol{\gamma}\{\boldsymbol{1}\} = \{\boldsymbol{1}\} $$

whence

$$ \{\boldsymbol{\gamma}\} = [\boldsymbol{R}(0)']^{-1}\{\boldsymbol{1}\}. \qquad (4.5) $$

The adjustment factor γ_i may be re-expressed in a way that offers additional insights into its properties. According to (4.2)

$$ \hat{Q}_i = \sum_{j=1}^{m} {}_j\hat{R}_i(0)\hat{Q}_j. $$

Dividing both sides of the equation by $\hat{Q}_i$ gives

$$ 1 = \sum_{j=1}^{m} \frac{\hat{Q}_j}{\hat{Q}_i}\, {}_j\hat{R}_i(0) = \hat{R}_i(0), $$

where $R_i(0)$ may be defined to be the net reproduction of women *living* in region i (as distinguished from the net reproduction of those *born* in region i). But

$$ \hat{R}_i(0) = \sum_{j=1}^{m} \frac{\hat{Q}_j}{\hat{Q}_i} \gamma_i\, {}_jR_i(0) = \gamma_i R_i(0) = 1, $$

where we define

$$ R_i(0) = \sum_{j=1}^{m} \frac{\hat{Q}_j}{\hat{Q}_i}\, {}_jR_i(0). $$

Hence

$$\gamma_i = \frac{1}{R_i(0)},$$

and

$$\{\boldsymbol{R}(0)\} = \boldsymbol{\gamma}^{-1}\{\mathbf{1}\} = \hat{\boldsymbol{Q}}^{-1}\boldsymbol{R}(0)\{\hat{\boldsymbol{Q}}\}, \tag{4.6}$$

where

$$\{\hat{\boldsymbol{Q}}\} = \hat{\boldsymbol{Q}}\{\mathbf{1}\}.$$

The vector $\{\hat{\boldsymbol{Q}}\}$ in (4.6) is the characteristic vector associated with the unit dominant characteristic root of $\hat{\boldsymbol{R}}(0)$ and denotes the total number of births in each region of a stationary multiregional population. The proportional allocation of total births that it defines is directly dependent on the transformation that is applied to change $\boldsymbol{R}(0)$ to $\hat{\boldsymbol{R}}(0)$, a particular example of which is given by (4.4). Since in a stationary multiregional population the regional *stationary* equivalent population $\hat{Y}_i$ is equal to the quotient $\hat{Q}_i/\hat{b}_i$, we see that the different ways in which $\boldsymbol{R}(0)$ is transformed into $\hat{\boldsymbol{R}}(0)$ become, in fact, alternative "spatial paths" to a stationary multiregional population.

A numerical example may be instructive at this point. The net reproduction behavior of the urban and rural female populations of the United States in 1968 is crudely approximated by the net reproduction matrix

$$\boldsymbol{R}(0) = \begin{bmatrix} {}_uR_u(0) & {}_rR_u(0) \\ {}_uR_r(0) & {}_rR_r(0) \end{bmatrix} = \begin{bmatrix} 0.85 & 0.45 \\ 0.25 & 0.90 \end{bmatrix}$$

where, for example, ${}_rR_u(0) = 0.45$ denotes the net reproduction rate in urban areas of rural-born women. In other words, under the regime of growth observed in 1968, each woman born in a rural area will, on the average, replace herself in the succeeding generation by 1.35 daughters, one third of whom will be born in urban areas. Urban-born women, on the other hand, have a lower net reproduction rate (i.e., ${}_uR(0) = 1.10 < {}_rR(0)$), which when combined with the net reproduction rate of rural-born women gives the United States female population an overall net reproduction rate of $\lambda_1(\boldsymbol{R}(0)) = 1.21$, where $\lambda_1(\boldsymbol{R}(0))$ is the dominant characteristic root of the net reproduction matrix $\boldsymbol{R}(0)$.

About 73 percent of the 1968 United States female population lived in urban areas. A projection to stable growth under the 1968 growth regime reduces that allocation to approximately two thirds of the stable population and yields an intrinsic growth rate of slightly under one percent per annum. What would be the spatial allocation under a similar projection, but one in which fertility was immediately reduced to a level of one daughter per urban- or rural-born woman? To obtain an estimate of the regional shares in the stationary population that would evolve out of such a projection we need to first derive the fertility adjustment factors γ_u and γ_r, respectively. Calculations carried out using (4.5) give $\gamma_u = 1$ and $\gamma_r = 0.6$, whence

$$\boldsymbol{R}(0) = \begin{bmatrix} 0.85 & 0.45 \\ 0.15 & 0.55 \end{bmatrix} \tag{4.7}$$

Note that both groups of women now exhibit unit rates of net reproduction, and observe that the dominant characteristic root of $\hat{\boldsymbol{R}}(0)$ is unity.

The characteristic vector associated with the unit dominant characteristic root of $\hat{\boldsymbol{R}}(0)$ indicates that three quarters of the total births in the stationary multiregional population will occur in urban areas. Since $\hat{Q}_i = \hat{b}_i\hat{Y}_i$,

$$\frac{\hat{Y}_i}{\hat{Y}_j} = \frac{\hat{Q}_i}{\hat{Q}_j} \cdot \frac{\hat{b}_j}{\hat{b}_i}, \tag{4.8}$$

a result that equates the ratio of stationary regional shares to the corresponding stationary birth ratio times the reciprocal of the corresponding ratio of intrinsic birth rates. Since the stationary birth ratio of urban to rural births is given by $\hat{\boldsymbol{R}}(0)$ to be 3 (i.e., 0.75 to 0.25) and because the ratio of rural to urban intrinsic birth rates is likely to be close to unity (it comes out to be 1.07 in the projection) we conclude that in the stationary multiregional population about three quarters of the population would reside in urban areas.[11]

We have observed earlier that the proportional allocation of total births in a stationary multiregional population depends directly on the transformation by which $\boldsymbol{R}(0)$ is changed to $\hat{\boldsymbol{R}}(0)$. Alternative transformations are in effect alternative spatial paths to such a population inasmuch as they lead to alternative spatial allocations of the total multiregional population. This can be easily illustrated by considering an alternative fertility reduction program which reduces the aggregate net reproduction rate to unity by reducing each regional fertility schedule by the same proportion, γ say. That is,

$$\hat{\hat{\boldsymbol{R}}}(0) = \gamma \boldsymbol{R}(0), \tag{4.9}$$

where

$$\gamma = \frac{1}{\lambda_1(\boldsymbol{R}(0))}.$$

In the context of our numerical illustration this means that the fertility of urban-born women would now be reduced to below replacement levels whereas that of rural-born women would be permitted to exceed replacement fertility levels. That is,

$$\hat{\hat{\boldsymbol{R}}}(0) = \frac{1}{1.211}\boldsymbol{R}(0) = \begin{bmatrix} 0.70 & 0.37 \\ 0.21 & 0.74 \end{bmatrix}, \tag{4.10}$$

and ${}_u\hat{\hat{R}}(0) = 0.91$, ${}_r\hat{\hat{R}}(0) = 1.11$.

The spatial implications of this alternative path to a stationary multiregional population are quite different, as can be seen by calculating the characteristic vector associated with the unit dominant characteristic root of $\hat{\hat{\boldsymbol{R}}}(0)$. The characteristic vector in this case allocates approximately 55 percent of total multiregional births to urban areas. Since the ratio of rural to urban intrinsic birth

[11]This result of course refers to regional designations that existed in 1968. In light of the continuing urbanization of rural regions it is probably a conservative estimate.

rates would now be somewhat higher than unity, however, we should expect a correspondingly higher concentration in urban areas than is indicated by this allocation of total births.

On the Momentum of Multiregional Population Growth

Differences between most observed population age compositions and those of stationary populations make it virtually impossible to attain zero growth in the near future. A population's birth rate and growth rate depend on its fertility schedule and its age composition. Consequently whether and how long a population continues to grow after achieving a net reproduction rate of unity depends on that population's age composition and its degree of divergence from that of a stationary population. The ratio by which the ultimate stationary population exceeds a current population is the *momentum* of that population, a quantity that recently has been given analytical content by Keyfitz [10] who shows that the momentum of a population numbering K individuals and having an age composition close to stable may be approximated by the expression

$$\frac{\hat{Y}}{K} = \frac{b\,e(0)}{r\,\mu}\left(\frac{R(0)-1}{R(0)}\right), \qquad (4.11)$$

where b is the birth rate, r the rate of growth, $e(0)$ the expectation of life, and $R(0)$ the net reproduction rate, all measured before the drop in fertility, and μ is the mean age of childbearing afterward. The derivation assumes that the population is approximately stable before the decline in fertility so that b and r are intrinsic stable rates of the initial (nonstationary) regime of growth.

Straightforward population projection calculations may be used to obtain the future population that evolves from any particular observed or hypothetical regime of growth. Therefore (4.11) is not needed to obtain a numerical estimate of an ultimate stationary population. However Keyfitz's simple momentum formula gives us an understanding of the population dynamics that are hidden in the arithmetical computations of a population projection. It identifies in an unambiguous way the five parameters of a current population that determine the size of the ultimate stationary population.

In order to evaluate the accuracy of Keyfitz's momentum formula we have carried out a two-region projection of the 1968 United States female population on the assumption that age-specific fertility rates in each region drop immediately to replacement levels. Table 3 shows that the ultimate total stationary multiregional population exceeds its 1968 level by about a third. Equation (4.11) estimates the momentum to be about the same:[12]

$$\frac{\hat{Y}}{K(1968)} = \frac{0.01878}{0.00432} \cdot \frac{74.3}{26.3}\,\frac{(1.12–1)}{1.12} = 1.31.$$

[12]Unlike Keyfitz we do not use the observed birth rate but divide total stable births Q by the current population, i.e., $b = Q/K(1968) = 1{,}920{,}961/102{,}276{,}992 = 0.01878$. That is why our approximation is more accurate than similar ones reported by Keyfitz.

TABLE 3. Relations Under Stationarity: Female Population of the United States, 1968

	1. The West Region			2. The Rest of the United States		
	Stationary Equivalent Population			Proportion		
Age, x	1 + 2	1	2	1 + 2	1	2
0	8,919,063	1,999,286	6,919,777	0.0662	0.0645	0.0668
5	8,801,161	1,960,681	6,840,480	0.0654	0.0632	0.0660
10	8,786,342	1,954,891	6,831,451	0.0653	0.0630	0.0659
15	8,767,142	1,973,196	6,793,946	0.0651	0.0636	0.0656
20	8,739,604	1,993,989	6,745,615	0.0649	0.0643	0.0651
25	8,706,948	1,987,121	6,719,827	0.0647	0.0641	0.0648
30	8,663,633	1,968,097	6,695,537	0.0643	0.0635	0.0646
35	8,596,752	1,948,595	6,648,156	0.0638	0.0628	0.0642
40	8,493,131	1,924,987	6,568,144	0.0631	0.0621	0.0634
45	8,341,530	1,894,516	6,447,014	0.0620	0.0611	0.0622
50	8,126,033	1,853,200	6,272,832	0.0604	0.0598	0.0605
55	7,822,168	1,793,484	6,028,684	0.0581	0.0578	0.0582
60	7,401,017	1,711,320	5,689,697	0.0550	0.0552	0.0549
65	6,805,483	1,594,855	5,210,627	0.0505	0.0514	0.0503
70	5,963,701	1,425,125	4,538,576	0.0443	0.0460	0.0438
75	4,814,365	1,182,097	3,632,267	0.0358	0.0381	0.0351
80	3,410,328	866,383	2,543,945	0.0253	0.0279	0.0245
85+	3,480,965	982,171	2,498,794	0.0259	0.0317	0.0241
Total	134,639,366	31,013,997	103,625,370	1.0000	0.2303	0.7697

STATIONARY POPULATION

$\hat{r}$ = Rate of growth	0.00000		
$\hat{Y}$ = Stationary equivalent population	134,639,366	31,013,997	103,625,370
$\hat{SHA} = \hat{Y}/\Sigma\hat{Y}$ = Stationary regional share	1.0000	0.2303	0.7697
$\hat{Q}$ = Stationary equivalent births	1,805,735	406,374	1,399,361
$\hat{SBR}_{21} = \hat{Q}_2/Q_1$ = Stationary birth ratio	3.44		
$\hat{b}$ = Birth rate	0.0134	0.0131	0.0135
$\hat{d}$ = Death rate	0.0134	0.0133	0.0135
$\hat{o}$ = Outmigration rate	—	0.0079	0.0024
$\hat{i}$ = Inmigration rate	—	0.0080	0.0024
$\hat{n}$ = Net migration rate	—	0.0002	–0.0000
$\hat{a}$ = Mean age	39.65	40.45	39.41

A multiregional generalization of Keyfitz's momentum formula may be shown to be

$$\left\{\frac{\hat{Y}}{K}\right\} = \frac{1}{r} e(0)\, \boldsymbol{R}(1)^{-1}[\boldsymbol{R}(0) - \omega(r)]\{\boldsymbol{b}\}, \tag{4.12}$$

where $\boldsymbol{R}(1)$ is a matrix with elements ${}_jR_i(1) = {}_j\mu_i \cdot {}_jR_i(0)$, and where total multiregional stable births Q are allocated to regions according to the stationary proportions defined by the characteristic vector associated with the unit dominant characteristic root of $\hat{\boldsymbol{R}}(0)$. Evaluating (4.12) with the same two-region data that produced the results in Tables 1 and 3 gives:

$$\begin{bmatrix} \dfrac{\hat{Y}_i}{K_i(1968)} \\ \dfrac{\hat{Y}_j}{K_j(1968)} \end{bmatrix} = \frac{1}{0.00432} \begin{bmatrix} 52.39 & 6.95 \\ 23.10 & 67.34 \end{bmatrix} \begin{bmatrix} 0.0509 & -0.0045 \\ -0.0161 & 0.0382 \end{bmatrix}$$

$$\times \begin{bmatrix} 0.0833 & 0.0098 \\ 0.0350 & 0.1108 \end{bmatrix} \begin{bmatrix} 0.02377 \\ 0.01777 \end{bmatrix} = \begin{bmatrix} 1.74 \\ 1.23 \end{bmatrix}.$$

A comparison of these regional momenta with those found by population projection and set out in Table 3.1 reveals that the quality of approximation afforded by (4.12) is adequate (1.74 and 1.23 as approximations of 1.80 and 1.22, respectively).

Equation (4.12) is not as practically useful as its single-region counterpart because it is much more difficult to come up with accurate guesses or estimates of the values taken on by the many parameters. Thus a more effective procedure may be to first estimate the ultimate size of the total stationary multiregional population using Keyfitz's formula and then rely on (4.8) to allocate that total to the various regions of the multiregional system. Such a procedure requires estimates of the stationary birth rate ratios.

5. CONCLUSION

It is an unassailable fact of life that current rates of population growth cannot prevail for a very long period of time in the future. Coale [2], for example, points out that were the United States population to continue to increase at its present rate of about one percent a year, there would be more than one American per square foot of land in less than 1300 years. In an analogous vein, Keyfitz [12] observes that Mexico cannot continue with its current rate of growth for as long as the lifetimes of children now born, for if it did its population would increase sixteen-fold to about 800 million people in that span of time. Of what use then are population projections developed on the basis of constant rates?

Knowledgeable users of population projections know full well that the assumptions that generated them are certain to be violated. This is especially true of multiregional projections that assume fixed schedules of internal migration. Unlike fertility or mortality, migration is functionally related to two populations instead of one. Thus it is patently unrealistic to assume that age-specific rates of migration between two regional populations will remain unaffected by changes in the relative sizes of these populations over time. Nonlinear models of growth therefore deserve an important place in any agenda of demographic research. Nevertheless, the knowledgeable user of demographic models will still find that linear multiregional models of population growth can indeed provide a better understanding of the spatial dynamics of such growth. As Keyfitz [12; p. 573] notes, they "permit experiments out of which we obtain causal knowledge; they explain data; they focus research by identifying theoretical and practical issues; they systematize comparative study across space and time; they reveal formal analogies between problems that on the surface are quite different; they even help assemble data."

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Environment and Planning A, 1977, volume 9, pages 653-674

Sensitivity analysis in multiregional demographic models

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Received 7 January 1977, in revised form 16 March 1977

Abstract. The theory of multiregional mathematical demography investigates how fertility, mortality, and migration combine to shape the growth of multiregional population systems. Population dynamics have been studied for cases where the structural parameters, namely the age-specific rates of fertility, mortality, and migration, are fixed. This paper addresses the question of how the system behaves under changing structural parameters. By applying the technique of matrix differentiation, sensitivity functions are derived which link changes in multiregional life-table statistics and in population projections to changes in the age-specific rates. A review of the technique, which may be used for the sensitivity analysis of any matrix model, is given in the appendix.

The field of mathematical demography is concerned with the mathematical description of how fertility and mortality combine to determine the characteristics of populations and to shape their growth. Traditionally, demographers have restricted their attention to fertility and mortality (Coale, 1972; Keyfitz, 1968), and have in fact assumed that populations are 'closed' to migration, that is, populations are undisturbed by in- and out-migration. This is an unrealistic assumption, especially in population analysis at the subnational level. The introduction of migration into mathematical demography has been pioneered by Rogers (1975). He describes, in analytical terms, how fertility, mortality, and migration combine to determine the features and the growth of multiregional population systems. The basic tool used is matrix algebra.

Mathematical demography demonstrates how various population characteristics may be expressed in terms of observed age-specific fertility, mortality, and migration rates. The fundamental assumption underlying the models is that the age-specific rates, that is, the structural parameters, are known exactly and that they remain fixed over time. The implications of this are expressed by Keyfitz (1968, page 27): "The object [of population projection] is to understand the past rather than to predict the future; apparently the way to think effectively about an observed set of birth and death rates is to ask what it would lead to if continued".

No one truly believes that fertility, mortality, and migration schedules are measured without observation error and that they will remain unchanged for a prolonged period of time. However, variations in structural parameters have not been considered until recently (Goodman, 1969; 1971; Keyfitz, 1971; Preston, 1974).

It is the purpose of this paper to improve understanding of the impact on the population system of changes in its structural parameters. The system considered is the multiregional demographic one described in Rogers (1975). The parameters are the age-specific fertility, mortality, and migration rates. In general terms, the problem is to find how sensitive stationary population characteristics and population projections are to changes in age-specific rates. In another paper I have investigated some long-run impacts of the changes, namely the sensitivity of the stable population characteristics (Willekens, 1976).

There are two approaches to impact analysis. The first is the *simulation approach*, or the arithmetic approach as Keyfitz (1971, page 275) calls it. It is simply the computation of the life table and population projection under the old and the new rates. The differences in the output give the impact of changing the rates. Suitable tools for the simulation approach are provided by model life tables, such as those developed by Coale and Demeny (1966) for a single-region demographic system and by Rogers (1975, chapter 6) for a multiregional system. For an illustration of this approach see Rogers (1975, pages 169–172), Rogers and Willekens (1976, pages 28–30). Besides its demanding character in terms of computer time, the approach tells us nothing about the complete set of parameters on which the changes in the final results depend. It will be found useful, however, for verification of the results of the second approach, which is the *analytical approach*. This procedure derives a general formula for assessing the impact of a particular change in terms of well-known population variables. Such a formula will be designated a *sensitivity function*. Partial differentiation will be seen to be the basic ingredient in the analysis of such functions.

In this paper, impact analysis is performed using the analytical approach. It is assumed that all the functions are differentiable with respect to the variables in which the changes occur. Since multiregional demographic models are formulated in matrix terms, matrix differentiation techniques are applied; and because not much work has been done in the area of matrix calculus, the appendix reviews several relevant topics of such a calculus(1).

1 Impact of changes in age-specific rates on life-table functions

The concept of a multiregional life table as developed by Rogers (1973; 1975) is a device for exhibiting the mortality and migration history of a set of regional cohorts as they age. It is assumed that the age-specific rates describing the mortality and mobility experience of an actual population remain constant, and that the system of regions is undisturbed by external migration.

The first part of this section sets out the life-table functions. The cohorts that will be considered are birth cohorts or *radices*. Their life histories are of special interest because they provide the information required by population projection models. The life-table statistics are given by place of birth. In the second part I combine the life-table functions with the matrix differentiation techniques described in the appendix. This enables one to develop life-table sensitivity functions.

1.1 *The multiregional life table*

All the life-table functions are derived from a set of age-specific death and out-migration rates. Let $\mathbf{M}(x)$ denote the matrix of observed annual rates for the persons in the age interval from x to $x+h$. The length of the interval h is arbitrary. Without loss of generality I shall consider age intervals of five years. For an N-region system, $\mathbf{M}(x)$ is

$$\mathbf{M}(x) = \begin{bmatrix} M_{1\delta}(x) + \sum\limits_{j \neq 1}^{N} M_{1j}(x) & -M_{21}(x) & -M_{31}(x) & \cdots \\ -M_{12}(x) & M_{2\delta}(x) + \sum\limits_{j \neq 2}^{N} M_{2j}(x) & -M_{32}(x) & \cdots \\ -M_{13}(x) & -M_{23}(x) & M_{3\delta}(x) + \sum\limits_{j \neq 3}^{N} M_{3j}(x) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad (1)$$

(1) All major textbooks on matrix algebra that have been consulted lack a chapter on matrix calculus, although some scattered treatment may occur. The only unified treatment of matrix differentiation that I have found is by Dwyer and MacPhail (1948). A simplified and extended version appeared twenty years later in Dwyer (1967). The formulae given there are general enough to handle differentiation problems that occur in life-table functions and in the analysis of population projections over a finite time horizon.

where $M_{i\delta}(x)$ is the age-specific annual death rate in region i, and $M_{ij}(x)$ is the age-specific annual out-migration rate from region i to region j. This out-migration rate is estimated by dividing the annual number of out-migrants to j by the midyear population of i.

Let $\mathbf{P}(x)$ be the matrix of age-specific probabilities of dying and out-migrating:

$$\mathbf{P}(x) = \begin{bmatrix} p_{11}(x) & p_{21}(x) & \cdots \\ p_{12}(x) & p_{22}(x) & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} , \tag{2}$$

with $p_{ij}(x)$ being the probability that an individual in region i at exact age x will survive and be in region j at exact age $x+5$. The diagonal element $p_{ii}(x)$ is the probability that an individual will survive and be in region i at the end of the interval. If $q_i(x)$ is the probability that an individual in region i at age x will die before reaching age $x+5$, then the following relationship follows

$$p_{ii}(x) = 1 - q_i(x) - \sum_{j \neq i}^{N} p_{ij}(x) . \tag{3}$$

The matrix of probabilities is derived from the matrix of vital rates (Rogers and Ledent, 1976; Schoen, 1975):

$$\mathbf{P}(x) = [\mathbf{I} + \tfrac{5}{2}\mathbf{M}(x)]^{-1}[\mathbf{I} - \tfrac{5}{2}\mathbf{M}(x)] . \tag{4}$$

Note that $\mathbf{P}(x)$ is the transition matrix of an *absorbing* Markov chain.

The probability that an individual starting out in region j, that is, born in j, will be in region i at exact age x is denoted by ${}_j\hat{l}_i(x)$. The matrix containing those probabilities is

$$\hat{\mathbf{l}}(x) = \begin{bmatrix} {}_1\hat{l}_1(x) & {}_2\hat{l}_1(x) & \cdots \\ {}_1\hat{l}_2(x) & {}_2\hat{l}_2(x) & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} . \tag{5}$$

The matrix $\hat{\mathbf{l}}(x)$ describes the state of the system at time x. If one assumes that the state at time x depends only on the previous state $\hat{\mathbf{l}}(x-5)$ (that is, the Markov property), then equation (5) may be written as

$$\hat{\mathbf{l}}(x) = \mathbf{P}(x-5)\mathbf{P}(x-10) \ldots \mathbf{P}(0) = \mathbf{P}(x-5)\hat{\mathbf{l}}(x-5) . \tag{6}$$

Define

$$\mathbf{l}(x) = \hat{\mathbf{l}}(x)\mathbf{l}(0) , \tag{7}$$

where $\mathbf{l}(0)$ is a diagonal matrix of the cohorts of babies born in the N regions at a given instant in time. Typically, ${}_il_i(0)$ is called the radix of region i and is set equal to some arbitrary constant such as 100000. Then $\mathbf{l}(x)$ is the matrix of the number of people at exact age x by place of residence and by place of birth.

Another life-table function is the total number of people of age-group x, that is, aged x to $x+4$ years, in each region by place of birth:

$$\mathbf{L}(x) = \begin{bmatrix} {}_1L_1(x) & {}_2L_1(x) & \cdots \\ {}_1L_2(x) & {}_2L_2(x) & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} , \tag{8}$$

with ${}_jL_i(x)$ being the number of people in region i in age-group x who were born in region j. In addition the element ${}_jL_i(x)$ can be thought of as the total person-years lived in region i between ages x and $x+5$, by the people born in region j (Rogers, 1975, page 65). The matrix $\mathbf{L}(x)$ is given by

$$\mathbf{L}(x) = \int_0^5 \mathbf{l}(x+t)\mathrm{d}t = \left[\int_0^5 \hat{\mathbf{l}}(x+t)\mathrm{d}t\right]\mathbf{l}(0) . \tag{9}$$

With the assumption of a uniform distribution of out-migrations and deaths over the five-year age interval, we may obtain numerical values for $\mathbf{L}(x)$ by the linear interpolation

$$\mathbf{L}(x) = \tfrac{5}{2}[\mathbf{l}(x)+\mathbf{l}(x+5)] ,$$

or

$$\mathbf{L}(x) = \tfrac{5}{2}[\mathbf{I}+\mathbf{P}(x)]\mathbf{l}(x) . \tag{10}$$

Another useful expression for $\mathbf{L}(x)$ is found by noting that

$$[\mathbf{I}+\mathbf{P}(x)] = [\mathbf{I}+\tfrac{5}{2}\mathbf{M}(x)]^{-1}\Big[[\mathbf{I}+\tfrac{5}{2}\mathbf{M}(x)]+[\mathbf{I}-\tfrac{5}{2}\mathbf{M}(x)]\Big] = 2[\mathbf{I}+\tfrac{5}{2}\mathbf{M}(x)]^{-1} . \tag{11}$$

Equation (10) may therefore be rewritten as

$$\mathbf{L}(x) = 5[\mathbf{I}+\tfrac{5}{2}\mathbf{M}(x)]^{-1}\mathbf{l}(x) . \tag{12}$$

If $\mathbf{L}(x)$ is aggregated over all age groups beyond age x, the expected total number of person-years remaining to the people at exact age x can be defined as

$$\mathbf{T}(x) = \sum_{y=x}^{z} \mathbf{L}(y) , \tag{13}$$

where z is the terminal age group.

By expressing $\mathbf{T}(x)$ per individual, we get the matrix of expectation of life of an individual at exact age x:

$$\mathbf{e}(x) = \mathbf{T}(x)\mathbf{l}^{-1}(x) = \left[\sum_{y=x}^{z} \mathbf{L}(y)\right]\mathbf{l}^{-1}(x) . \tag{14}$$

The matrix $\mathbf{e}(x)$ expresses the life expectancies by place of residence. An element ${}_ie_j(x)$ denotes the expected number of years lived in region j beyond age x by an individual now residing in region i and x years of age. The expectation-of-life matrix by place of birth is given by

$$\bar{\mathbf{e}}(x) = \mathbf{T}(x)\bar{\mathbf{l}}^{-1}(x) , \tag{15}$$

where

$$\bar{\mathbf{l}}(x) = \begin{bmatrix} {}_1l.(x) & 0 & \cdots \\ 0 & {}_2l.(x) & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \tag{16}$$

is a diagonal matrix. The element ${}_il.(x)$ denotes the total number of people of exact age x who are born in region i. An element ${}_i\bar{e}_j(x)$ gives the expected number of years lived in region j beyond age x by an individual born in region i and now x years of age (his current place of residence is not considered).

A very useful life-table function is the survivorship matrix. It is an essential component of the population projection matrix. Rogers (1975, page 79) has shown that the survivorship matrix

$$\mathbf{S}(x) = \begin{bmatrix} s_{11}(x) & s_{21}(x) & \cdots \\ s_{12}(x) & s_{22}(x) & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \qquad (17)$$

is given by

$$\mathbf{S}(x) = \mathbf{L}(x+5)\mathbf{L}^{-1}(x) . \qquad (18)$$

The element $s_{ij}(x)$ denotes the proportion of individuals aged x to $x+4$ in region i who survive to be $x+5$ to $x+9$ years old five years later, and are then in region j.

The survivorship matrix may be expressed directly in terms of the observed age-specific rates[2]. Substitution of equation (12) into equation (18) gives

$$\mathbf{S}(x) = [\mathbf{I}+\tfrac{5}{2}\mathbf{M}(x+5)]^{-1}\mathbf{P}(x)\mathbf{l}(x)\mathbf{l}^{-1}(x)[\mathbf{I}+\tfrac{5}{2}\mathbf{M}(x)] ,$$

which yields, after replacing equation (4) for $\mathbf{P}(x)$ and reworking,

$$\mathbf{S}(x) = [\mathbf{I}+\tfrac{5}{2}\mathbf{M}(x+5)]^{-1}[\mathbf{I}-\tfrac{5}{2}\mathbf{M}(x)] . \qquad (19)$$

I have now set up the important life-table functions, and can proceed to the analysis of their sensitivities to changes in the underlying rates, that is, in $\mathbf{M}(x)$.

1.2 *Sensitivity analysis of life-table functions*

The fundamental question posed in this section is: what is the effect on the various life-table statistics of a change in the observed age-specific rates? To resolve this question the life-table functions are combined with the matrix differentiation techniques given in the appendix.

This section is divided into five parts, each of which starts out with a specific life-table function. The derivative of this function with respect to an element of the matrix of age-specific rates yields the corresponding sensitivity function.

1.2.1 *Sensitivity of the probabilities of dying and out-migrating.* Recall the estimating formula set out in equation (4). The probability matrix $\mathbf{P}(x)$ depends only on $\mathbf{M}(x)$. Therefore $\mathbf{P}(a)$ is not affected by a change in $\mathbf{M}(x)$ for $a \neq x$.

The derivative of $\mathbf{P}(x)$ with respect to an arbitrary element of $\mathbf{M}(x)$ is, by virtue of formulae (A13) and (A25) of the appendix,

$$\begin{aligned}\frac{\partial\mathbf{P}(x)}{\partial\langle\mathbf{M}(x)\rangle} &= \frac{\partial[\mathbf{I}+\tfrac{5}{2}\mathbf{M}(x)]^{-1}}{\partial\langle\mathbf{M}(x)\rangle}[\mathbf{I}-\tfrac{5}{2}\mathbf{M}(x)] + [\mathbf{I}+\tfrac{5}{2}\mathbf{M}(x)]^{-1}\frac{\partial[\mathbf{I}-\tfrac{5}{2}\mathbf{M}(x)]}{\partial\langle\mathbf{M}(x)\rangle} \\ &= -[\mathbf{I}+\tfrac{5}{2}\mathbf{M}(x)]^{-1}\frac{\partial[\mathbf{I}+\tfrac{5}{2}\mathbf{M}(x)]}{\partial\langle\mathbf{M}(x)\rangle}[\mathbf{I}+\tfrac{5}{2}\mathbf{M}(x)]^{-1}[\mathbf{I}-\tfrac{5}{2}\mathbf{M}(x)] \\ &\quad + [\mathbf{I}+\tfrac{5}{2}\mathbf{M}(x)]^{-1}\frac{\partial[\mathbf{I}-\tfrac{5}{2}\mathbf{M}(x)]}{\partial\langle\mathbf{M}(x)\rangle} = -[\mathbf{I}+\tfrac{5}{2}\mathbf{M}(x)]^{-1}[\tfrac{5}{2}\mathbf{J}\mathbf{P}(x)+\tfrac{5}{2}\mathbf{J}] ,\end{aligned}$$

where $\mathbf{J}$ is a matrix of the same dimension as $\mathbf{M}(x)$, with all elements zero except for a unit element in the position of the arbitrary element $\langle\mathbf{M}(x)\rangle$. (This notation is further explained in the appendix.) Therefore the sensitivity function for $\mathbf{P}(x)$ is

$$\frac{\partial\mathbf{P}(x)}{\partial\langle\mathbf{M}(x)\rangle} = -\tfrac{5}{2}[\mathbf{I}+\tfrac{5}{2}\mathbf{M}(x)]^{-1}\mathbf{J}[\mathbf{P}(x)+\mathbf{I}] . \qquad (20)$$

Substitution from equation (11) for $[\mathbf{P}(x)+\mathbf{I}]$ yields

$$\frac{\partial\mathbf{P}(x)}{\partial\langle\mathbf{M}(x)\rangle} = -5[\mathbf{I}+\tfrac{5}{2}\mathbf{M}(x)]^{-1}\mathbf{J}[\mathbf{I}+\tfrac{5}{2}\mathbf{M}(x)]^{-1} . \qquad (21)$$

[2] The author is grateful to Jacques Ledent for the derivation.

1.2.2 *Sensitivity of the number of people at exact age a.* A change in $\mathbf{M}(x)$ does not affect $\mathbf{l}(a)$ for $a \leqslant x$. Therefore we look only at the case $a > x$. Note that because of equation (6) $\mathbf{l}(a)$ may be written as

$$\mathbf{l}(a) = \mathbf{P}(a-5)\mathbf{P}(a-10) \ldots \mathbf{P}(x)\mathbf{l}(x) .$$

On recalling that $\mathbf{M}(x)$ affects only $\mathbf{P}(x)$, one may write

$$\frac{\partial \mathbf{l}(a)}{\partial \langle \mathbf{M}(x) \rangle} = \mathbf{P}(a-5)\mathbf{P}(a-10) \ldots \frac{\partial \mathbf{P}(x)}{\partial \langle \mathbf{M}(x) \rangle}\mathbf{l}(x)$$

$$= -\tfrac{5}{2}\left[\mathbf{P}(a-5)\mathbf{P}(a-10) \ldots \mathbf{P}(x+5)[\mathbf{I}+\tfrac{5}{2}\mathbf{M}(x)]^{-1}\mathbf{J}[\mathbf{P}(x)+\mathbf{I}]\mathbf{l}(x)\right] . \quad (22)$$

By inserting

$$\mathbf{I} = [\mathbf{I}-\tfrac{5}{2}\mathbf{M}(x)]\mathbf{l}(x)\mathbf{l}^{-1}(x)[\mathbf{I}-\tfrac{5}{2}\mathbf{M}(x)]^{-1}$$

into equation (22), substituting for $\mathbf{P}(x)$, and using equation (10), one obtains

$$\frac{\partial \mathbf{l}(a)}{\partial \langle \mathbf{M}(x) \rangle} = -\mathbf{l}(a)\mathbf{l}^{-1}(x)[\mathbf{I}-\tfrac{5}{2}\mathbf{M}(x)]^{-1}\mathbf{J}\mathbf{L}(x) . \quad (23)$$

Another expression is

$$\frac{\partial \mathbf{l}(a)}{\partial \langle \mathbf{M}(x) \rangle} = -\mathbf{l}(a)\mathbf{l}^{-1}(x+5)[\mathbf{I}+\tfrac{5}{2}\mathbf{M}(x)]^{-1}\mathbf{J}\mathbf{L}(x) . \quad (24)$$

For $a = x+5$, we have

$$\frac{\partial \mathbf{l}(x+5)}{\partial \langle \mathbf{M}(x) \rangle} = -\mathbf{P}(x)[\mathbf{I}-\tfrac{5}{2}\mathbf{M}(x)]^{-1}\mathbf{J}\mathbf{L}(x) = -[\mathbf{I}+\tfrac{5}{2}\mathbf{M}(x)]^{-1}\mathbf{J}\mathbf{L}(x) . \quad (25)$$

An interesting formulation of the sensitivity function is

$$\frac{\partial \mathbf{l}(a)}{\partial \langle \mathbf{M}(x) \rangle} = \mathbf{l}(a)\mathbf{l}^{-1}(x)\mathbf{P}^{-1}(x)\frac{\partial \mathbf{P}(x)}{\partial \langle \mathbf{M}(x) \rangle}\mathbf{l}(x) ,$$

or

$$\mathbf{l}^{-1}(a)\frac{\partial \mathbf{l}(a)}{\partial \langle \mathbf{M}(x) \rangle} = \mathbf{l}^{-1}(x+5)\frac{\partial \mathbf{P}(x)}{\partial \langle \mathbf{M}(x) \rangle}\mathbf{l}(x) . \quad (26)$$

This shows that the relative sensitivity of $\mathbf{l}(a)$ to changes in $\mathbf{M}(x)$ is a weighted average of the relative sensitivity of $\mathbf{P}(x)$, and is independent of a. Consider the first age group and suppose that all regions have the same radices, that is, $\mathbf{l}(0)$ is a scalar matrix—a diagonal matrix with the same diagonal elements. The relative sensitivity of any $\mathbf{l}(a)$ is then proportional to the relative sensitivity of $\mathbf{P}(0)$.

1.2.3 *Sensitivity of the number of people in age-group (a, a+4).* What is the impact of a change in $\mathbf{M}(x)$ on the number of people in age-group $(a, a+4)$ and on their spatial distribution? It is clear that $\mathbf{M}(x)$ does not affect $\mathbf{L}(a)$ for $a < x$. Therefore I shall consider here the case of $a \geqslant x$. Recall from equation (10) that

$$\mathbf{L}(a) = \tfrac{5}{2}[\mathbf{l}(a+5)+\mathbf{l}(a)] .$$

Differentiating both sides gives

$$\frac{\partial \mathbf{L}(a)}{\partial \langle \mathbf{M}(x) \rangle} = \frac{5}{2}\left[\frac{\partial \mathbf{l}(a+5)}{\partial \langle \mathbf{M}(x) \rangle} + \frac{\partial \mathbf{l}(a)}{\partial \langle \mathbf{M}(x) \rangle}\right] .$$

If $a = x$, then $\partial \mathbf{l}(a)/\partial \langle \mathbf{M}(x) \rangle = \mathbf{0}$ and we have

$$\frac{\partial \mathbf{L}(x)}{\partial \langle \mathbf{M}(x) \rangle} = \frac{5}{2}\frac{\partial \mathbf{l}(x+5)}{\partial \langle \mathbf{M}(x) \rangle} = -\tfrac{5}{2}[\mathbf{I}+\tfrac{5}{2}\mathbf{M}(x)]^{-1}\mathbf{J}\mathbf{L}(x) \quad (27)$$

which has the following alternative expression:

$$\frac{\partial \mathbf{L}(x)}{\partial \langle \mathbf{M}(x) \rangle} = -\tfrac{5}{4}[\mathbf{P}(x)+\mathbf{I}]\mathbf{J}\mathbf{L}(x) \ . \tag{28}$$

If $a > x$, we know that $\mathbf{P}(a)$ is independent of $\mathbf{M}(x)$ and therefore

$$\frac{\partial \mathbf{L}(a)}{\partial \langle \mathbf{M}(x) \rangle} = \tfrac{5}{2}[\mathbf{P}(a)+\mathbf{I}] \frac{\partial \mathbf{l}(a)}{\partial \langle \mathbf{M}(x) \rangle} = -\mathbf{L}(a)\mathbf{l}^{-1}(x)[\mathbf{I}-\tfrac{5}{2}\mathbf{M}(x)]^{-1}\mathbf{J}\mathbf{L}(x) \tag{29}$$

$$= -\mathbf{L}(a)\mathbf{l}^{-1}(x+5)[\mathbf{I}+\tfrac{5}{2}\mathbf{M}(x)]^{-1}\mathbf{J}\mathbf{L}(x) \ , \tag{30}$$

which may also be written as

$$\frac{\partial \mathbf{L}(a)}{\partial \langle \mathbf{M}(x) \rangle} = \tfrac{5}{2}[\mathbf{P}(a)+\mathbf{I}]\mathbf{l}(a)\mathbf{l}^{-1}(a)\frac{\partial \mathbf{l}(a)}{\partial \langle \mathbf{M}(x) \rangle} \ ,$$

whence, since $\tfrac{5}{2}[\mathbf{P}(a)+\mathbf{I}]\mathbf{l}(a) = \mathbf{L}(a)$,

$$\mathbf{L}^{-1}(a)\frac{\partial \mathbf{L}(a)}{\partial \langle \mathbf{M}(x) \rangle} = \mathbf{l}^{-1}(a)\frac{\partial \mathbf{l}(a)}{\partial \langle \mathbf{M}(x) \rangle} \ . \tag{31}$$

Equation (31) indicates that the relative sensitivity of the number of people in age-group $(a, a+4)$ is equal to the relative sensitivity of the number of people at exact age a for $a > x$.

1.2.4 *Sensitivity of the expectation of life at age a.* I now derive the sensitivity function of the most important life-table statistic, namely the expectation of life. I shall distinguish between the expectation of life by place of residence and the expectation of life by place of birth.

1.2.4.1 *Expectation of life by place of residence.* Consider first the sensitivity of $\mathbf{e}(x)$. Differentiating both sides of equation (14) yields

$$\frac{\partial \mathbf{e}(x)}{\partial \langle \mathbf{M}(x) \rangle} = \frac{\partial \left[\sum_{y=x}^{z} \mathbf{L}(y) \right]}{\partial \langle \mathbf{M}(x) \rangle} \mathbf{l}^{-1}(x) + \left[\sum_{y=x}^{z} \mathbf{L}(y) \right] \frac{\partial \mathbf{l}^{-1}(x)}{\partial \langle \mathbf{M}(x) \rangle} \ . \tag{32}$$

From equations (27) and (29) we see that

$$\begin{aligned} \frac{\partial \left[\sum_{y=x}^{z} \mathbf{L}(y) \right]}{\partial \langle \mathbf{M}(x) \rangle} &= -\left[\sum_{y=x+5}^{z} \mathbf{L}(y) \right] \mathbf{l}^{-1}(x)[\mathbf{I}-\tfrac{5}{2}\mathbf{M}(x)]^{-1}\mathbf{J}\mathbf{L}(x) - \tfrac{5}{2}[\mathbf{I}+\tfrac{5}{2}\mathbf{M}(x)]^{-1}\mathbf{J}\mathbf{L}(x) \\ &= -\left[\sum_{y=x+5}^{z} \mathbf{L}(y)\mathbf{l}^{-1}(x) + \tfrac{5}{2}\mathbf{P}(x) + \tfrac{5}{2}\mathbf{I} - \tfrac{5}{2}\mathbf{I} \right] [\mathbf{I}-\tfrac{5}{2}\mathbf{M}(x)]^{-1}\mathbf{J}\mathbf{L}(x) \\ &= -\left[\sum_{y=x+5}^{z} \mathbf{L}(y)\mathbf{l}^{-1}(x) + \mathbf{L}(x)\mathbf{l}^{-1}(x) - \tfrac{5}{2}\mathbf{I} \right] [\mathbf{I}-\tfrac{5}{2}\mathbf{M}(x)]^{-1}\mathbf{J}\mathbf{L}(x) \\ &= -[\mathbf{e}(x)-\tfrac{5}{2}\mathbf{I}][\mathbf{I}-\tfrac{5}{2}\mathbf{M}(x)]^{-1}\mathbf{J}\mathbf{L}(x) \ . \end{aligned} \tag{33}$$

Since $\mathbf{l}(x)$ is independent of $\mathbf{M}(x)$, equation (32) may be written as follows:

$$\frac{\partial \mathbf{e}(x)}{\partial \langle \mathbf{M}(x) \rangle} = -[\mathbf{e}(x)-\tfrac{5}{2}\mathbf{I}][\mathbf{I}-\tfrac{5}{2}\mathbf{M}(x)]^{-1}\mathbf{J}\mathbf{L}(x)\mathbf{l}^{-1}(x) \tag{34}$$

or

$$\frac{\partial \mathbf{e}(x)}{\partial \langle \mathbf{M}(x) \rangle} = -5[\mathbf{e}(x)-\tfrac{5}{2}\mathbf{I}][\mathbf{I}-\tfrac{5}{2}\mathbf{M}(x)]^{-1}\mathbf{J}[\mathbf{I}+\tfrac{5}{2}\mathbf{M}(x)]^{-1} \ . \tag{35}$$

For $a < x$ we have

$$\frac{\partial e(a)}{\partial \langle M(x)\rangle} = \frac{\partial\left[\sum_{y=x+5}^{z} L(y) + L(x) + \sum_{y=a}^{x-5} L(y)\right] l^{-1}(a)}{\partial\langle M(x)\rangle}$$

We know that

$$\frac{\partial L(y)}{\partial\langle M(x)\rangle} = 0\,, \qquad \text{for } y < x\,,$$

and

$$\frac{\partial l(a)}{\partial\langle M(x)\rangle} = 0\,, \qquad \text{for } a < x\,.$$

Therefore

$$\frac{\partial e(a)}{\partial\langle M(x)\rangle} = \frac{\partial\left[\sum_{y=x}^{z} L(y)\right] l^{-1}(a)}{\partial\langle M(x)\rangle}\,,$$

$$\frac{\partial e(a)}{\partial\langle M(x)\rangle} = \frac{\partial e(x)}{\partial\langle M(x)\rangle} l(x)\, l^{-1}(a)\,, \tag{36}$$

$$\frac{\partial e(a)}{\partial\langle M(x)\rangle} = -[e(x) - \tfrac{5}{2}I]\,[I - \tfrac{5}{2}M(x)]^{-1} J L(x)\, l^{-1}(a) \tag{37}$$

$$= -e(x)\,[I - \tfrac{5}{2}M(x)]^{-1} J L(x)\, l^{-1}(a) + \tfrac{5}{2}\,[I - \tfrac{5}{2}M(x)]^{-1} J L(x)\, l^{-1}(a)\,. \tag{38}$$

The second component of the sensitivity function follows from the linear approximation (10) of the continuous relationship (9).

Consider the continuous definition of $e(a)$

$$e(a) = \left[\int_a^{\omega} l(t)\,dt\right] l^{-1}(a)$$

where ω is the terminal age. Differentiation, for $a \leqslant x$, yields

$$\frac{\partial e(a)}{\partial\langle M(x)\rangle} = \left[\int_a^{\omega} \frac{\partial l(t)}{\partial\langle M(x)\rangle} dt\right] l^{-1}(a)$$

$$= \left[\int_x^{\omega} -l(t)\, l^{-1}(x)\,[I - \tfrac{5}{2}M(x)]^{-1} J l(x)\, dt\right] l^{-1}(a)\,.$$

Since $l(t)$ is independent of $M(x)$ for $t < x$, we have

$$\frac{\partial e(a)}{\partial\langle M(x)\rangle} = -\left[\int_x^{\omega} l(t)\,dt\right] l^{-1}(x)\,[I - \tfrac{5}{2}M(x)]^{-1} J l(x)\, l^{-1}(a)$$

$$= -e(x)\,[I - \tfrac{5}{2}M(x)]^{-1} J l(x)\, l^{-1}(a)\,, \tag{39}$$

which is equivalent to the first term of equation (38) with $l(x)$ replaced by $L(x)$ in the discrete case. The expression (39), written in terms of differentials, is similar to the sensitivity function of the expectation of life, given by Keyfitz (1971, page 276) for the single-region case as

$$de(a) = -e(x)\,[dM(x)]\, l(x)\, l^{-1}(a)\,,$$

where $e(\cdot)$, $l(\cdot)$, and $M(\cdot)$ are scalars. The term $[I - \tfrac{5}{2}M(x)]^{-1}$ in equation (39) occurs because I have considered observed rates where Keyfitz derived the formula using instantaneous rates. If $M(x)$ contained instantaneous rates, then $M(x) \doteq 0$ and $[I - \tfrac{5}{2}M(x)] \doteq I$.

For $a > x$, we have

$$\frac{\partial \mathbf{e}(a)}{\partial \langle \mathbf{M}(x) \rangle} = \frac{\partial \left[\sum_{y=a}^{z} \mathbf{L}(y) \right]}{\partial \langle \mathbf{M}(x) \rangle} \mathbf{l}^{-1}(a) + \left[\sum_{y=a}^{z} \mathbf{L}(y) \right] \frac{\partial \mathbf{l}^{-1}(a)}{\partial \langle \mathbf{M}(x) \rangle} .$$

Applying equations (23), (29), and (A25) gives

$$\frac{\partial \mathbf{e}(a)}{\partial \langle \mathbf{M}(x) \rangle} = -\mathbf{e}(a) \mathbf{l}^{-1}(x) [\mathbf{I} - \tfrac{5}{2}\mathbf{M}(x)]^{-1} \mathbf{J} \mathbf{L}(x) \mathbf{l}^{-1}(a) + \mathbf{e}(a) \mathbf{l}^{-1}(a) \mathbf{l}(a) \mathbf{l}^{-1}(x) [\mathbf{I} - \tfrac{5}{2}\mathbf{M}(x)]^{-1} \mathbf{J} \mathbf{L}(x) \mathbf{l}^{-1}(a) = 0 . \quad (40)$$

Therefore a change in $\mathbf{M}(x)$ does not affect $\mathbf{e}(a)$ for $a > x$.

1.2.4.2 *Expectation of life by place of birth.* The life expectancy by place of birth is given by formula (15). By making use of derivations analogous to the ones in section 1.2.4.1, it can be shown that for $a \leqslant x$ the sensitivity functions of $\bar{\mathbf{e}}(a)$ and $\mathbf{e}(a)$ are very similar:

$$\frac{\partial \bar{\mathbf{e}}(a)}{\partial \langle \mathbf{M}(x) \rangle} = -[\mathbf{e}(x) - \tfrac{5}{2}\mathbf{I}] [\mathbf{I} - \tfrac{5}{2}\mathbf{M}(x)]^{-1} \mathbf{J} \mathbf{L}(x) \bar{\mathbf{l}}^{-1}(a) . \quad (41)$$

Contrary to the finding of the previous section, a change in $\mathbf{M}(x)$ affects $\bar{\mathbf{e}}(a)$ for $a > x$. For $a > x$, we may write

$$\frac{\partial \bar{\mathbf{e}}(a)}{\partial \langle \mathbf{M}(x) \rangle} = \frac{\partial \left[\sum_{y=a}^{z} \mathbf{L}(y) \right]}{\partial \langle \mathbf{M}(x) \rangle} \bar{\mathbf{l}}^{-1}(a) + \left[\sum_{y=a}^{z} \mathbf{L}(y) \right] \frac{\partial \bar{\mathbf{l}}^{-1}(a)}{\partial \langle \mathbf{M}(x) \rangle} . \quad (42)$$

By equation (29) we have

$$\frac{\partial \left[\sum_{y=a}^{z} \mathbf{L}(y) \right]}{\partial \langle \mathbf{M}(x) \rangle} = -\mathbf{e}(a) \mathbf{l}^{-1}(x) [\mathbf{I} - \tfrac{5}{2}\mathbf{M}(x)]^{-1} \mathbf{J} \mathbf{L}(x) .$$

On the other hand

$$\frac{\partial \bar{\mathbf{l}}^{-1}(a)}{\partial \langle \mathbf{M}(x) \rangle} = -\mathbf{l}^{-1}(a) \frac{\partial \bar{\mathbf{l}}(a)}{\partial \langle \mathbf{M}(x) \rangle} \mathbf{l}^{-1}(a) ,$$

where

$$\frac{\partial \bar{\mathbf{l}}(a)}{\partial \langle \mathbf{M}(x) \rangle} = \mathbf{Q} \quad (43)$$

is a diagonal matrix. The diagonal consists of the elements of the vector

$$\{\mathbf{1}\}' \frac{\partial \mathbf{l}(a)}{\partial \langle \mathbf{M}(x) \rangle} = -\{\mathbf{1}\}' \mathbf{l}(a) \mathbf{l}^{-1}(x) [\mathbf{I} - \tfrac{5}{2}\mathbf{M}(x)]^{-1} \mathbf{J} \mathbf{L}(x) .$$

Introducing these results into equation (42) gives

$$\frac{\partial \bar{\mathbf{e}}(a)}{\partial \langle \mathbf{M}(x) \rangle} = -\mathbf{e}(a) \mathbf{l}^{-1}(x) [\mathbf{I} - \tfrac{5}{2}\mathbf{M}(x)]^{-1} \mathbf{J} \mathbf{L}(x) \bar{\mathbf{l}}^{-1}(a) - \bar{\mathbf{l}}^{-1}(a) \mathbf{Q} \bar{\mathbf{l}}^{-1}(a)$$
$$= \mathbf{e}(a) \mathbf{l}^{-1}(a) \frac{\partial \mathbf{l}(a)}{\partial \langle \mathbf{M}(x) \rangle} \bar{\mathbf{l}}^{-1}(a) - \bar{\mathbf{l}}^{-1}(a) \frac{\partial \bar{\mathbf{l}}(a)}{\partial \langle \mathbf{M}(x) \rangle} \bar{\mathbf{l}}^{-1}(a) . \quad (44)$$

1.2.5 *Sensitivity of the survivorship proportions.* As in the preceding sections, we treat $\mathbf{S}(a)$ for $a = x$ and for $a \neq x$ separately. The survivorship matrix is given by equation (19). Differentiating with respect to $\langle \mathbf{M}(x) \rangle$ yields

$$\frac{\partial \mathbf{S}(x)}{\partial \langle \mathbf{M}(x) \rangle} = -\tfrac{5}{2}[\mathbf{I} + \tfrac{5}{2}\mathbf{M}(x+5)]^{-1}\mathbf{J} \ , \tag{45}$$

or

$$\frac{\partial \mathbf{S}(x)}{\partial \langle \mathbf{M}(x) \rangle} = -\tfrac{5}{2}\mathbf{S}(x)[\mathbf{I} - \tfrac{5}{2}\mathbf{M}(x)]^{-1}\mathbf{J} \ . \tag{46}$$

To illustrate the dynamic relationship between the life-table statistics, one may express the sensitivity of $\mathbf{S}(x)$ in relation to the sensitivity of other statistics. For example, combination of equation (46) with equation (29) yields

$$\mathbf{S}^{-1}(x)\frac{\partial \mathbf{S}(x)}{\partial \langle \mathbf{M}(x) \rangle} = \mathbf{P}^{-1}(x)\frac{\partial \mathbf{L}(x)}{\partial \langle \mathbf{M}(x) \rangle}\mathbf{L}^{-1}(x) \ ,$$

and combination of equation (46) with equation (23) gives

$$\mathbf{S}^{-1}(x)\frac{\partial \mathbf{S}(x)}{\partial \langle \mathbf{M}(x) \rangle} = -\tfrac{5}{2}\mathbf{P}^{-1}(x)\frac{\partial \mathbf{l}(x+5)}{\partial \langle \mathbf{M}(x) \rangle}\mathbf{L}^{-1}(x) \ .$$

The relative sensitivity of $\mathbf{S}(x)$ may be regarded as a weighted measure of the sensitivities of other statistics.

I now turn to the sensitivity of $\mathbf{S}(a)$ to changes in $\mathbf{M}(x)$ for $a \neq x$. For $a > x$ and for $a < x-5$, $\mathbf{S}(a)$ is independent of a change in $\mathbf{M}(x)$. The sensitivity of $\mathbf{S}(x-5)$ to a change in $\mathbf{M}(x)$ is derived easily. Write equation (19) for $x-5$:

$$\mathbf{S}(x-5) = [\mathbf{I} + \tfrac{5}{2}\mathbf{M}(x)]^{-1}[\mathbf{I} - \tfrac{5}{2}\mathbf{M}(x-5)] \ .$$

Then

$$\begin{aligned}\frac{\partial \mathbf{S}(x-5)}{\partial \langle \mathbf{M}(x) \rangle} &= -\tfrac{5}{2}[\mathbf{I} + \tfrac{5}{2}\mathbf{M}(x)]^{-1}\mathbf{J}[\mathbf{I} + \tfrac{5}{2}\mathbf{M}(x)]^{-1}[\mathbf{I} - \tfrac{5}{2}\mathbf{M}(x-5)] \\ &= -\tfrac{5}{2}[\mathbf{I} + \tfrac{5}{2}\mathbf{M}(x)]^{-1}\mathbf{J}\mathbf{S}(x-5) \ .\end{aligned} \tag{47}$$

The relationship between the sensitivity of $\mathbf{S}(x)$ and $\mathbf{S}(x-5)$ is thus

$$\frac{\partial \mathbf{S}(x)}{\partial \langle \mathbf{M}(x) \rangle} = \mathbf{S}(x)\mathbf{P}^{-1}(x)\frac{\partial \mathbf{S}(x-5)}{\partial \langle \mathbf{M}(x) \rangle}\mathbf{S}^{-1}(x-5) \ , \tag{48}$$

and

$$\frac{\partial \mathbf{S}(x-5)}{\partial \langle \mathbf{M}(x) \rangle} = \mathbf{P}(x)\mathbf{S}^{-1}(x)\frac{\partial \mathbf{S}(x)}{\partial \langle \mathbf{M}(x) \rangle}\mathbf{S}(x-5) \ . \tag{49}$$

2 Impact of changes in age-specific rates on the population projection

Population projection is often carried out under the assumption that an observed population-growth regime will remain constant. This implies that the observed age-specific rates will not change over the projection period. (This is a crude assumption and no demographer or planner considers it to be realistic. Nevertheless it produces a useful benchmark against which to compare other alternative projections.) In this section I deal with the question of how sensitive are population projections to changes in age-specific rates. These variations may occur at any point in time. If they occur in the base year, they can be related to observation errors. The sensitivity functions developed remain exactly the same no matter what the causes of the variations are.

In the first part the population-growth model is set out as a system of first-order linear homogeneous difference equations with constant coefficients, as in Rogers

(1975, chapter 5). The second part studies the sensitivity of population growth to changes in observed age-specific rates.

2.1 The discrete model of multiregional demographic growth
Population growth may be expressed in terms of the changing level of population or in terms of the variation of the number of births over time. In demography it has been a custom to formulate the discrete model of population growth in terms of total population, whereas the continuous version describes the birth trajectory (Keyfitz, 1968; Rogers, 1975). In this paper I shall consider only the discrete population-growth model.

A multiregional growth process may be described as a matrix multiplication (Rogers, 1975, page 123):

$$\{K^{(t+1)}\} = \mathbf{G}\{K^{(t)}\} \tag{50}$$

where the vector $\{K^{(t)}\}$ describes the regional age-specific population distribution at time t, with

$$\{K^{(t)}\} = \begin{bmatrix} \{K^{(t)}(0)\} \\ \{K^{(t)}(5)\} \\ \vdots \\ \{K^{(t)}(z)\} \end{bmatrix}, \quad \text{and} \quad \{K^{(t)}(x)\} = \begin{bmatrix} K_1^{(t)}(x) \\ K_2^{(t)}(x) \\ \vdots \\ K_N^{(t)}(x) \end{bmatrix}, \tag{51}$$

z being the terminal age interval and N the number of regions.

Each element $K_i^{(t)}(x)$ denotes the number of people in region i at time t who are x to $x+4$ years old. Note that $t+1$ represents the next moment in time, that is, five years later than t; age-groups and time intervals of five years are being considered. The operator $\mathbf{G}$ is the generalized Leslie matrix

$$\mathbf{G} = \begin{bmatrix} 0 & 0 & \mathbf{B}(\alpha-5) & \dots & \mathbf{B}(\beta-5) & \dots & 0 & 0 \\ \mathbf{S}(0) & 0 & & & & & & \\ 0 & \mathbf{S}(5) & \ddots & & & & & \\ \vdots & \vdots & & & & \ddots & & \\ 0 & 0 & & & & & \mathbf{S}(z-5) & 0 \end{bmatrix}, \tag{52}$$

with $\mathbf{S}(x)$, the matrix of survivorship proportions, set out in section 1.1. The first and last ages of childbearing may be denoted by α and β, respectively, and

$$\mathbf{B}(x) = \begin{bmatrix} b_{11}(x) & b_{21}(x) & \dots \\ b_{12}(x) & b_{22}(x) & \dots \\ \vdots & \vdots & \end{bmatrix},$$

where an element $b_{ij}(x)$ denotes the average number of babies born during the unit time interval in region i and alive in region j at the end of that interval, per individual living in region i at the beginning of the interval, and aged x to $x+4$ years old.

The matrix $\mathbf{B}(x)$ may be estimated by the following expression (Rogers, 1975, pages 120–121):

$$\mathbf{B}(x) = \tfrac{1}{2}\mathbf{L}(0)\mathbf{l}^{-1}(0)\,[\mathbf{F}(x)+\mathbf{F}(x+5)\mathbf{S}(x)]\ .$$

Since

$$L(0) = \tfrac{5}{2}[l(5)+l(0)] = \tfrac{5}{2}[P(0)+I]l(0) ,$$

we may write

$$B(x) = \tfrac{5}{4}[P(0)+I][F(x)+F(x+5)S(x)] . \tag{53}$$

The quantities L(0), l(0), P(0), and S(x) have been defined in section 1.1. P(0) and S(x) are given by the life table, and F(x) is a diagonal matrix containing the annual regional birthrates of people aged x to $x+4$ years old. The number of births in year t from people aged x to $x+4$ years old at t is $F(x)\{K^{(t)}(x)\}$. The number of births during a five-year period starting at t, from people aged x to $x+4$ years old at t, is

$$\tfrac{5}{2}[F(x)\{K^{(t)}(x)\}+F(x+5)\{K^{(t+1)}(x+5)\}] = \tfrac{5}{2}[F(x)+F(x+5)S(x)]\{K^{(t)}(x)\} .$$

Of these births, a proportion $L(0)[5l(0)]^{-1}$ will be surviving in the various regions at the end of the time interval. Note that this model of the survivorship and migratory behavior of zero- to four-year old children does not incorporate the observed dependence of the migration pattern of children on that of their parents. For a critique and possible extension see Rees and Wilson (1977, pages 146–150).

Because of the particular structure of the generalized Leslie matrix, equation (50) may be written as two equation systems:

$$\{K^{(t+1)}(0)\} = \sum_{\alpha-5}^{\beta-5} B(x)\{K^{(t)}(x)\} , \tag{54}$$

$$\{K^{(t+1)}(x+5)\} = S(x)\{K^{(t)}(x)\} , \qquad \text{for } 5 \leqslant x \leqslant z-5 . \tag{55}$$

The vector $\{K^{(t)}(x)\}$ may be expressed in the form

$$\begin{aligned}\{K^{(t+x/5)}(x)\} &= [S(x-5)S(x-10)\ \dots\ S(5)S(0)]\{K^{(t)}(0)\} \\ &= A(x)\{K^{(t)}(0)\} , \qquad \text{say,}\end{aligned} \tag{56}$$

where we define

$$A(x) = \left\{\begin{matrix} I , & \text{for } x = 0 \\ \prod_{y=x-5}^{0} S(y) , & \text{for } x = 5, 10, \dots, z \end{matrix}\right\} = L(x)L^{-1}(0) , \tag{57}$$

with

$$\prod_{y=x-5}^{0} S(y) = S(x-5)S(x-10)\ \dots\ S(5)S(0) . \tag{58}$$

The element $a_{ij}(x)$ of A(x) is the proportion of individuals aged 0 to 4 years old in region i who will survive to be x to $x+4$ years old exactly x years later, and will at that time be in region j.

2.2 *Sensitivity analysis of the population projection*

Recall the population growth model defined in equation (50):

$$\{K^{(t+1)}\} = G\{K^{(t)}\} .$$

The assessment of the sensitivity of $\{K^{(t+1)}\}$ to changes in age-specific rates M(x) may be analyzed by means of a two-step process. The first step considers the sensitivity of the growth matrix to changes in age-specific rates. The second step derives a sensitivity function which describes the impact on the population distribution of a change in the growth matrix. In the sensitivity analysis of life-table statistics, we were not concerned with the time when the change in $\mathbf{M}(x)$ occurred. The time consideration was irrelevant since the life table is a static model. For the sensitivity analysis of the population growth, however, it is important to know not only the age group *where* a change in $\mathbf{M}(x)$ occurs, but also the time *when* the change occurs. I shall denote this time by t_0. The time at which the change in the population distribution is measured will be denoted by t_1.

Besides the change in $\{\boldsymbol{K}^{(t_1)}\}$ caused by a change in the age-specific rates at t_0, one may also consider the problem of how a unique change in $\{\boldsymbol{K}^{(t_0)}\}$ affects $\{\boldsymbol{K}^{(t_1)}\}$. These are two separate sensitivity problems. In the first, the parameter changes at t_0 and then remains at this new level. The second problem, however, is equivalent to a parameter change at the period preceding t_0 only. These two sensitivity problems will be treated separately.

2.2.1 *Sensitivity of the growth matrix.* The growth matrix $\mathbf{G}$ is composed of two types of submatrices, $\mathbf{S}(x)$ and $\mathbf{B}(x)$. It has been shown in section 1.2, equations (45) and (47), that a change in $\mathbf{M}(x)$ affects only $\mathbf{S}(x)$ and $\mathbf{S}(x-5)$:

$$\frac{\partial \mathbf{S}(x)}{\partial \langle \mathbf{M}(x) \rangle} = -\tfrac{5}{2}[\mathbf{I}+\tfrac{5}{2}\mathbf{M}(x+5)]^{-1}\mathbf{J} \,,$$

$$\frac{\partial \mathbf{S}(x-5)}{\partial \langle \mathbf{M}(x) \rangle} = -\tfrac{5}{2}[\mathbf{I}+\tfrac{5}{2}\mathbf{M}(x)]^{-1}\mathbf{J}\,\mathbf{S}(x-5) \,.$$

The sensitivity function of $\mathbf{B}(x)$ remains to be derived. In equation (53) $\mathbf{B}(x)$ depends on the age-specific death rates and out-migration rates through $\mathbf{S}(x)$ and $\mathbf{P}(0)$, and on the age-specific fertility rates $\mathbf{F}(x)$ and $\mathbf{F}(x+5)$. Consider the partial derivative of $\mathbf{B}(x)$ with respect to $\mathbf{M}(x)$:

$$\frac{\partial \mathbf{B}(x)}{\partial \langle \mathbf{M}(x) \rangle} = \frac{5}{4}\frac{\partial[\mathbf{P}(0)+\mathbf{I}]}{\partial \langle \mathbf{M}(x) \rangle}\mathbf{F}(x) + \frac{5}{4}\frac{\partial[\mathbf{P}(0)+\mathbf{I}]}{\partial \langle \mathbf{M}(x) \rangle}\mathbf{F}(x+5)\,\mathbf{S}(x) + \tfrac{5}{4}[\mathbf{P}(0)+\mathbf{I}]\,\mathbf{F}(x+5)\,\frac{\partial \mathbf{S}(x)}{\partial \langle \mathbf{M}(x) \rangle} \,. \qquad (59)$$

Since $\mathbf{P}(0)$ is affected by a change in $\mathbf{M}(x)$ only if $x = 0$, and because for this case $\mathbf{F}(x)$ and $\mathbf{F}(x+5)$ are $\mathbf{0}$, equation (59) reduces to

$$\frac{\partial \mathbf{B}(x)}{\partial \langle \mathbf{M}(x) \rangle} = \tfrac{5}{4}[\mathbf{P}(0)+\mathbf{I}]\,\mathbf{F}(x+5)\,\frac{\partial \mathbf{S}(x)}{\partial \langle \mathbf{M}(x) \rangle} \,, \qquad (60)$$

which, by equation (45), is

$$\frac{\partial \mathbf{B}(x)}{\partial \langle \mathbf{M}(x) \rangle} = -\tfrac{25}{8}[\mathbf{P}(0)+\mathbf{I}]\,\mathbf{F}(x+5)\,[\mathbf{I}+\tfrac{5}{2}\mathbf{M}(x+5)]^{-1}\mathbf{J} \,. \qquad (61)$$

Since a change in $\mathbf{M}(x)$ affects $\mathbf{S}(x-5)$, it also affects $\mathbf{B}(x-5)$:

$$\frac{\partial \mathbf{B}(x-5)}{\partial \langle \mathbf{M}(x) \rangle} = -\tfrac{25}{8}[\mathbf{P}(0)+\mathbf{I}]\,\mathbf{F}(x)\,[\mathbf{I}+\tfrac{5}{2}\mathbf{M}(x)]^{-1}\mathbf{J}\,\mathbf{S}(x-5) \,. \qquad (62)$$

The sensitivity of $\mathbf{B}(x)$ with respect to $\mathbf{F}(x)$ and $\mathbf{F}(x+5)$ may also be derived easily:

$$\frac{\partial \mathbf{B}(x)}{\partial \langle \mathbf{F}(x) \rangle} = \tfrac{5}{4}[\mathbf{P}(0)+\mathbf{I}]\mathbf{J} = \tfrac{5}{2}\mathbf{L}(0)[5\mathbf{l}(0)]^{-1}\mathbf{J} \,, \qquad (63)$$

and

$$\frac{\partial \mathbf{B}(x-5)}{\partial \langle \mathbf{F}(x) \rangle} = \tfrac{5}{4}[\mathbf{P}(0)+\mathbf{I}]\mathbf{J}\,\mathbf{S}(x-5) \,. \qquad (64)$$

Thus the impact of a unit change in the fertility matrix $\mathbf{F}(x)$ on the element $\mathbf{B}(x)$ is $\frac{1}{2}$ times the proportion of newborn babies that will be alive at the end of the time interval.

Having derived sensitivity functions for the elements of the growth matrix, we can now proceed to the question of how changes in the growth matrix affect the growth of the population. This is sometimes called trajectory sensitivity.

2.2.2 *Sensitivity of the population trajectory.* Recall the population growth equation (50):

$$\{K^{(t+1)}\} = \mathbf{G}\{K^{(t)}\} \, .$$

Since $\mathbf{G}$ is assumed to be constant over time, the population distribution at time t_1 is given by

$$\{K^{(t_1)}\} = \mathbf{G}^{t_1 - t_0}\{K^{(t_0)}\} \, .$$

I shall assume that the change in the growth matrix occurs at t_0. Without loss of generality, t_0 may be set equal to zero, and t_1 equal to t. Then

$$\{K^{(t)}\} = \mathbf{G}^t\{K^{(0)}\} \, .$$

The sensitivity of $\{K^{(t)}\}$ to a change in $\mathbf{G}$ is

$$\frac{\partial\{K^{(t)}\}}{\partial\langle\mathbf{G}\rangle} = \frac{\partial[\mathbf{G}^t]}{\partial\langle\mathbf{G}\rangle}\{K^{(0)}\} \, .$$

The sensitivity of $\mathbf{G}^t$ to a change in $\langle\mathbf{G}\rangle$ is given by equation (A24). Applying this result yields

$$\frac{\partial\{K^{(t)}\}}{\partial\langle\mathbf{G}\rangle} = \sum_{i=0}^{t-1} \mathbf{G}^i \mathbf{J} \mathbf{G}^{t-1-i}\{K^{(0)}\} \, . \qquad (65)$$

A problem which is related to the computation of the sensitivity of $\{K^{(t)}\}$ to changes in $\mathbf{G}$ is to find out under what conditions variations in $\mathbf{G}$ do not affect $\{K^{(t)}\}$. Such specific conditions have been derived by Tomović and Vukobratović (1972); they will not be discussed here. This and similar problems of trajectory insensitivity or invariance are receiving increasing attention in system theory and optimal-control theory. For a review of some applications in the social sciences see Erickson and Norton (1973).

The next section addresses the topic of the sensitivity of population growth to changes in the population distribution at a certain point in time. This will be called the analysis of small perturbations around the growth path.

2.2.3 *Perturbations around the population-growth path.* The impact on $\{K^{(t)}\}$ of a change in $\{K^{(0)}\}$ is very simple in the time-invariant equation system (50). Applying the results of vector differentiation set out in the appendix gives

$$\frac{\partial\{K^{(t)}\}}{\partial\{K^{(0)}\}'} = \frac{\partial[\mathbf{G}^t]\{K^{(0)}\}}{\partial\{K^{(0)}\}'} = \mathbf{G}^t \, , \qquad (66)$$

where $\{K^{(0)}\}'$ is the transpose of $\{K^{(0)}\}$.

Equation (66) relates changes in the state vector at time t to changes in the state vector at time zero. If the growth matrix is time-dependent this problem cannot be solved analytically and one must rely on simulation. An illustration of such a situation is when the model incorporates a feedback loop, that is, the growth matrix at time t depends on the state vector at time t. For an application of feedback models to urban analysis see Forrester (1969). Nelson and Kern (1971) have simulated the impact of small perturbations around the trajectory for a Forrester-type of urban model.

3 Conclusion

This paper has been devoted to the problem of sensitivity analysis in multiregional demographic systems. From mathematical demography we know that demographic change may be traced back to changes in age-specific fertility, mortality, and migration rates. To show how the mechanism works has been the subject of this paper.

A set of sensitivity functions have been derived which relate a change in demographic characteristics to a change in the vital rates. A summary of the important functions is given in table 1. The primary purpose of this study was to contribute to the knowledge of spatial population dynamics by presenting a unifying technique of impact assessments. In single-region mathematical demography, ordinary differential calculus is used to perform sensitivity analysis. In multiregional demography, where one is dealing with matrix and vector functions, the application of ordinary calculus is very complicated. Instead, matrix differentiation techniques prove to be very useful. A review of these techniques is given in the appendix. These mathematical tools have been applied to derive analytical expressions for multiregional demographic features, such as life-table statistics and population projection, which represent the impacts of changes in vital rates. The sensitivity functions reveal how each spatial demographic characteristic depends on the age-specific rates and how it reacts to changes in those rates. *Matrix differentiation techniques form a powerful tool for the analysis of structural change in multiregional systems.*

Table 1. Table of sensitivity functions.

Function $\mathbf{A}(a)$	Sensitivity function $\dfrac{\partial \mathbf{A}(a)}{\partial \langle \mathbf{M}(x) \rangle}$	
$\mathbf{M}(a)$	$\mathbf{0}$	for $a \neq x$
	$\mathbf{J}$	for $a = x$
$\mathbf{P}(a) = [\mathbf{I} + \frac{5}{2}\mathbf{M}(a)]^{-1}[\mathbf{I} - \frac{5}{2}\mathbf{M}(a)]$	$\mathbf{0}$	for $a \neq x$
	$-5[\mathbf{I} + \frac{5}{2}\mathbf{M}(x)]^{-1}\mathbf{J}[\mathbf{I} + \frac{5}{2}\mathbf{M}(x)]^{-1}$	for $a = x$
$\mathbf{l}(a) = \mathbf{P}(a-5)\mathbf{l}(a-5)$	$\mathbf{0}$	for $a \leqslant x$
	$-\mathbf{l}(a)\mathbf{l}^{-1}(x+5)[\mathbf{I} + \frac{5}{2}\mathbf{M}(x)]^{-1}\mathbf{J}\mathbf{L}(x)$	for $a > x$
$\mathbf{L}(a) = \frac{5}{2}[\mathbf{I} + \mathbf{P}(a)]\mathbf{l}(a)$	$\frac{5}{2}[\mathbf{I} + \frac{5}{2}\mathbf{M}(x)]^{-1}\mathbf{J}\mathbf{L}(x)$	for $a = x$
	$-\mathbf{L}(a)\mathbf{l}^{-1}(x+5)[\mathbf{I} + \frac{5}{2}\mathbf{M}(x)]^{-1}\mathbf{J}\mathbf{L}(x)$	for $a > x$
$\mathbf{e}(a) = \left[\sum_{y=a}^{z} \mathbf{L}(y)\right]\mathbf{l}^{-1}(a)$	$\mathbf{0}$	for $a > x$
	$-[\mathbf{e}(x) - \frac{5}{2}\mathbf{I}][\mathbf{I} - \frac{5}{2}\mathbf{M}(x)]^{-1}\mathbf{J}\mathbf{L}(x)\mathbf{l}^{-1}(a)$	for $a \leqslant x$
$\bar{\mathbf{e}}(a) = \left[\sum_{y=a}^{z} \mathbf{L}(y)\right]\bar{\mathbf{l}}^{-1}(a)$	$-\mathbf{e}(a)\mathbf{l}^{-1}(x)[\mathbf{I} - \frac{5}{2}\mathbf{M}(x)]^{-1}\mathbf{J}\mathbf{L}(x)\bar{\mathbf{l}}^{-1}(a) - \bar{\mathbf{l}}^{-1}(a)\mathbf{Q}\bar{\mathbf{l}}^{-1}(a)$	for $a > x$
	$-[\mathbf{e}(x) - \frac{5}{2}\mathbf{I}][\mathbf{I} - \frac{5}{2}\mathbf{M}(x)]^{-1}\mathbf{J}\mathbf{L}(x)\bar{\mathbf{l}}^{-1}(a)$	for $a \leqslant x$
$\mathbf{S}(x) = [\mathbf{I} + \frac{5}{2}\mathbf{M}(x+5)]^{-1}[\mathbf{I} - \frac{5}{2}\mathbf{M}(x)]$	$-\frac{5}{2}[\mathbf{I} + \frac{5}{2}\mathbf{M}(x+5)]^{-1}\mathbf{J}$	for $a = x$
	$-\frac{5}{2}[\mathbf{I} + \frac{5}{2}\mathbf{M}(x)]^{-1}\mathbf{J}\mathbf{S}(x-5)$	for $a = x-5$
	$\mathbf{0}$	for $a > x$ or $a < x-5$

Acknowledgement. This paper is based on my PhD dissertation entitled *Analytics of Multiregional Population Distribution Policy* (1976) submitted to Northwestern University, Evanston, Illinois. I am most grateful to Professor A Rogers for his guidance and close cooperation.

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APPENDIX

Matrix differentiation techniques

The purpose of this appendix is to provide the necessary mathematical tools to perform sensitivity analysis of structural change in multiregional demographic systems. The basic notion is that of matrix differentiation. Neudecker (1969, page 953) defines matrix differentiation as the procedure of finding partial derivatives of the elements of a matrix function with respect to the elements of the argument matrix. Although not much has been written on matrix differentiation and the technique is not covered in most textbooks on matrix algebra, this appendix is not intended to be complete. It covers only the techniques applied in this paper and some direct extensions. It is mainly based on the papers by Dwyer and MacPhail (1948) and Dwyer (1967).

Let $\mathbf{Y}$ be a $P \times Q$ matrix with elements y_{ij} and let $\mathbf{X}$ be an $M \times N$ matrix with elements x_{kl}. Dwyer (1967) makes a distinction between the position of an element in the matrix and its value. The symbol $\langle \mathbf{X} \rangle_{kl}$ is used to indicate a specific k,l-element of $\mathbf{X}$. Its scalar value is x_{kl}. Less formally, $\langle \mathbf{X} \rangle_{kl}$ may be replaced by $\langle \mathbf{X} \rangle$. Therefore $\langle \mathbf{X} \rangle$ is an arbitrary element of the matrix $\mathbf{X}$. As in conventional notation, $\mathbf{X}'$ denotes the transpose of $\mathbf{X}$, and $\mathbf{X}^{-1}$ is the inverse of $\mathbf{X}$.

The relevant results of matrix calculus are given below. To introduce some notation, I start out with the differentiation of a matrix with respect to its elements. This is followed by the differentiation of a matrix with respect to a scalar, and the differentiation of a scalar function with respect to a matrix. The most important scalar function is the determinant. The tools provided in the section on the differentiation of matrix products are frequently used in performing sensitivity analysis of multiregional systems. Also of great importance is the derivative of the inverse. The next section gives some chain rules of matrix differentiation. Vector calculus and matrix calculus are closely related, since a vector is a matrix with only one row or one column. The formulae for vector differentiation, however, have a different appearance and are less complex. Therefore a separate section will be devoted to vector differentiation.

A1 *Differentiation of a matrix with respect to its elements*

The derivative of a matrix $\mathbf{X}$ with respect to the element $\langle \mathbf{X} \rangle_{kl}$ is

$$\frac{\partial \mathbf{X}}{\partial \langle \mathbf{X} \rangle_{kl}} = \mathbf{J}_{kl} , \tag{A1}$$

where $\mathbf{J}_{kl}$ denotes an $M \times N$ matrix with zero elements everywhere except for a unit element in the kth row and lth column.

Similarly

$$\frac{\partial \mathbf{X}'}{\partial \langle \mathbf{X} \rangle_{kl}} = \mathbf{J}'_{lk} , \tag{A2}$$

where $\mathbf{J}'_{lk}$ is an $N \times M$ matrix with all elements zero except for a unit element in the lth row and kth column.

Instead of considering the derivative of a matrix with respect to an element, one may also consider the derivative of a matrix element with respect to the matrix:

$$\frac{\partial \langle \mathbf{Y} \rangle_{ij}}{\partial \mathbf{Y}} = \mathbf{K}_{ij} , \tag{A3}$$

where $\mathbf{K}_{ij}$ is a $P \times Q$ matrix with zeros everywhere except for a unit element in the ith row and jth column.

Similarly

$$\frac{\partial \langle \mathbf{Y} \rangle_{ij}}{\partial \mathbf{Y}'} = \mathbf{K}'_{ji} . \tag{A4}$$

For convenience the subscripts will be dropped. For example, $\langle \mathbf{X} \rangle$ will denote an arbitrary element of $\mathbf{X}$, and $\mathbf{J}$ will denote a matrix with all elements zero except for a unit element in the appropriate place, as determined by the location of $\langle \mathbf{X} \rangle$.

A2 *Differentiation of a matrix with respect to a scalar and of a scalar with respect to a matrix*

Let $\mathbf{Y}(a)$ be a matrix function of the scalar a. The derivative

$$\frac{\partial \mathbf{Y}(a)}{\partial a} \tag{A5}$$

is a matrix with elements $\partial y_{ij}/\partial a$. Each element of $\mathbf{Y}(a)$ is differentiated.

The derivative of a scalar function with respect to a matrix is denoted by

$$\frac{\partial \mathrm{f}(\mathbf{X})}{\partial \mathbf{X}} , \tag{A6}$$

and is a matrix with elements

$$\frac{\partial \mathrm{f}(\mathbf{X})}{\partial \langle \mathbf{X} \rangle_{ij}} . \tag{A7}$$

Two important scalar functions are considered: the determinant and the trace. I begin by making the assumption that $\mathbf{X}$ is a square matrix.

A2.1 *Determinant.* The determinant of the square matrix $\mathbf{X}$ can be evaluated in terms of the cofactors of the elements of the ith row (Rogers, 1971, page 81):

$$|\mathbf{X}| = x_{i1} X_{i1}^{c} + x_{i2} X_{i2}^{c} + ... + x_{iN} X_{iN}^{c} .$$

It can easily be seen that

$$\frac{\partial |\mathbf{X}|}{\partial \langle \mathbf{X} \rangle_{ij}} = X_{ij}^{c} , \tag{A8}$$

where X_{ij}^{c} is the cofactor of the element x_{ij}, and

$$\frac{\partial |\mathbf{X}|}{\partial \mathbf{X}} = \mathrm{cof}\mathbf{X} = [\mathrm{adj}\mathbf{X}]' ,$$

where $\mathrm{cof}\mathbf{X}$ is the matrix of cofactors, and $\mathrm{adj}\mathbf{X}$ is the adjoint matrix of the matrix $\mathbf{X}$. But if $\mathbf{X}$ is nonsingular,

$$\mathrm{cof}\mathbf{X} = |\mathbf{X}|[\mathbf{X}']^{-1} . \tag{A9}$$

Equation (A8) may be written as

$$\frac{\partial |\mathbf{X}|}{\partial \mathbf{X}} = |\mathbf{X}|[\mathbf{X}']^{-1} . \tag{A10}$$

This formula is well-known in matrix theory and can also be found in Bellman (1970, page 182).

It should be noted that if $\mathbf{X}$ is symmetric, then

$$\frac{\partial |\mathbf{X}|}{\partial \langle \mathbf{X} \rangle_{ij}} = \begin{cases} 2\mathbf{X}_{ij}^{c} & \text{for } i \neq j , \\ \mathbf{X}_{ij}^{c} & \text{for } i = j . \end{cases} \tag{A11}$$

A2.2 *Trace.* The trace of the square matrix **X** is the sum of its diagonal elements, and

$$\frac{\partial\, \mathrm{tr}(\mathbf{X})}{\partial \langle \mathbf{X}\rangle_{ij}} = \begin{cases} 1 & \text{for } i = j\,, \\ 0 & \text{for } i \neq j\,, \end{cases} \tag{A12}$$

with

$$\frac{\partial\, \mathrm{tr}(\mathbf{X})}{\partial \mathbf{X}} = \mathbf{I}\,,$$

where **I** is the identity matrix.

A3 *Differentiation of matrix products*
Let **U** and **V** be two matrix functions of the matrix **X**. The derivative of their product, $\mathbf{Y} = \mathbf{UV}$, with respect to $\langle \mathbf{X}\rangle$ is

$$\frac{\partial \mathbf{Y}}{\partial \langle \mathbf{X}\rangle} = \frac{\partial[\mathbf{UV}]}{\partial \langle \mathbf{X}\rangle} = \frac{\partial \mathbf{U}}{\partial \langle \mathbf{X}\rangle}\mathbf{V} + \mathbf{U}\frac{\partial \mathbf{V}}{\partial \langle \mathbf{X}\rangle}\,. \tag{A13}$$

The derivative of a product of three matrices is

$$\frac{\partial \mathbf{Y}}{\partial \langle \mathbf{X}\rangle} = \frac{\partial[\mathbf{UVW}]}{\partial \langle \mathbf{X}\rangle} = \frac{\partial \mathbf{U}}{\partial \langle \mathbf{X}\rangle}\mathbf{VW} + \mathbf{U}\frac{\partial \mathbf{V}}{\partial \langle \mathbf{X}\rangle}\mathbf{W} + \mathbf{UV}\frac{\partial \mathbf{W}}{\partial \langle \mathbf{X}\rangle}\,. \tag{A14}$$

These general formulae may be applied to various cases—some cases of interest are listed below. The matrices **A** and **B** are constant, that is, independent of **X**. The matrices **J** and **K** are as defined in section A1.

$$\mathbf{Y} = \begin{cases} \mathbf{AX}\,, \\ \mathbf{XB}\,, \\ \mathbf{X'B}\,, \\ \mathbf{XX}\,, \\ \mathbf{X'X}\,, \\ \mathbf{AXB}\,, \\ \mathbf{XXX}\,, \\ \mathbf{AXA}^{-1}\,, \end{cases} \quad \text{then} \quad \frac{\partial \mathbf{Y}}{\partial \langle \mathbf{X}\rangle} = \begin{cases} \mathbf{AJ}\,, & \text{(A15)} \\ \mathbf{JB}\,, & \text{(A16)} \\ \mathbf{J'B}\,, & \text{(A17)} \\ \mathbf{JX}+\mathbf{XJ}\,, & \text{(A18)} \\ \mathbf{J'X}+\mathbf{X'J}\,, & \text{(A19)} \\ \mathbf{AJB}\,, & \text{(A20)} \\ \mathbf{JXX}+\mathbf{XJX}+\mathbf{XXJ}\,, & \text{(A21)} \\ \mathbf{AJA}^{-1}\,. & \text{(A22)} \end{cases}$$

The derivative of the power of a square matrix can readily be computed using these formulae, thus

$$\frac{\partial[\mathbf{X}^n]}{\partial \langle \mathbf{X}\rangle} = \mathbf{JX}^{n-1} + \sum_{s=1}^{n-2} \mathbf{X}^s\mathbf{J}\mathbf{X}^{n-1-s} + \mathbf{X}^{n-1}\mathbf{J}\,, \tag{A23}$$

or, if we write $\mathbf{X}^0 = \mathbf{I}$, then

$$\frac{\partial[\mathbf{X}^n]}{\partial \langle \mathbf{X}\rangle} = \sum_{s=0}^{n-1} \mathbf{X}^s\mathbf{J}\mathbf{X}^{n-1-s}\,. \tag{A24}$$

The derivative of an inverse follows. By definition

$$\mathbf{XX}^{-1} = \mathbf{I}\,.$$

Therefore

$$\frac{\partial[\mathbf{XX}^{-1}]}{\partial \langle \mathbf{X}\rangle} = \frac{\partial \mathbf{I}}{\partial \langle \mathbf{X}\rangle} = \mathbf{0}\,,$$

but also

$$\frac{\partial[\mathbf{XX}^{-1}]}{\partial\langle\mathbf{X}\rangle} = \frac{\partial\mathbf{X}}{\partial\langle\mathbf{X}\rangle}\mathbf{X}^{-1} + \mathbf{X}\frac{\partial[\mathbf{X}^{-1}]}{\partial\langle\mathbf{X}\rangle} .$$

It follows therefore that

$$\frac{\partial[\mathbf{X}^{-1}]}{\partial\langle\mathbf{X}\rangle} = -\mathbf{X}^{-1}\mathbf{J}\mathbf{X}^{-1} . \tag{A25}$$

One application of this result is

$$\frac{\partial\mathbf{XAX}^{-1}}{\partial\langle\mathbf{X}\rangle} = \mathbf{JAX}^{-1} - \mathbf{XAX}^{-1}\mathbf{JX}^{-1} . \tag{A26}$$

So far I have considered the derivative $\partial\mathbf{Y}/\partial\langle\mathbf{X}\rangle$, where $\mathbf{Y}$ is a matrix product and $\langle\mathbf{X}\rangle$ is an arbitrary element of $\mathbf{X}$. The result is a matrix of partial derivatives. But what is the formula for $\partial\mathbf{Y}/\partial\mathbf{X}$, where $\mathbf{X}$ represents the full matrix? This question has been studied by Neudecker (1969). Its solution involves the transformation of a matrix into a vector and the use of Kronecker products. For example, let $\mathbf{Y} = \mathbf{AXB}$ and suppose that one is interested in the derivative of $\mathbf{Y}$ with respect to $\mathbf{X}$.

If $\mathbf{Y}$ is of order $P \times Q$ define the PQ column vector, vec$\mathbf{Y}$ (denoted this way to distinguish it from the vector $\{\mathbf{Y}\}$), by

$$\mathrm{vec}\,\mathbf{Y} = \begin{bmatrix} \{Y_{\cdot 1}\} \\ \{Y_{\cdot 2}\} \\ \vdots \\ \{Y_{\cdot Q}\} \end{bmatrix}$$

In a similar way one can construct vec$\mathbf{X}$. Neudecker shows that

$$\mathrm{vec}(\mathbf{AXB}) = [\mathbf{B}' \otimes \mathbf{A}]\,\mathrm{vec}\,\mathbf{X} , \tag{A27}$$

where $\otimes$ denotes the Kronecker product. Equation (A27) may be differentiated by making use of the formulae for vector differentiation:

$$\frac{\partial\,\mathrm{vec}[\mathbf{AXB}]}{\partial\,\mathrm{vec}\,\mathbf{X}} = [\mathbf{B}' \otimes \mathbf{A}]' .$$

Since the transpose of a Kronecker product is the Kronecker product of the transposes, we have[3]

$$\frac{\partial\,\mathrm{vec}[\mathbf{AXB}]}{\partial\,\mathrm{vec}\,\mathbf{X}} = \mathbf{B} \otimes \mathbf{A}' . \tag{A28}$$

I shall not explore the various formulae for $\partial\mathbf{Y}/\partial\mathbf{X}$ further since they are not explicitly used in this paper.

A4 *Chain rules of differentiation*

Let $\mathrm{f}(\mathbf{Y})$ be a scalar function of $\mathbf{Y}$ and let $\mathbf{Y}$ be a matrix function of $\mathbf{X}$. Then

$$\frac{\partial\mathrm{f}(\mathbf{Y})}{\partial\langle\mathbf{X}\rangle} = \sum_{kl}\frac{\partial\mathrm{f}(\mathbf{Y})}{\partial\langle\mathbf{Y}\rangle_{kl}}\frac{\partial\langle\mathbf{Y}\rangle_{kl}}{\partial\langle\mathbf{X}\rangle} , \tag{A29}$$

$$\frac{\partial\mathrm{f}(\mathbf{Y})}{\partial\langle\mathbf{X}\rangle} = \mathrm{tr}\left[\frac{\partial\mathrm{f}(\mathbf{Y})}{\partial\mathbf{Y}}\frac{\partial\mathbf{Y}'}{\partial\langle\mathbf{X}\rangle}\right] . \tag{A30}$$

[3] For an exposition of the properties of Kronecker products or direct products see Lancaster (1969, pages 256–259).

If $\mathbf{Y}$ is a matrix function of a scalar a, that is, $\mathbf{Y}(a)$, the formula becomes

$$\frac{\partial \mathrm{f}(\mathbf{Y})}{\partial a} = \operatorname{tr}\left[\frac{\partial \mathrm{f}(\mathbf{Y})}{\partial \mathbf{Y}} \frac{\partial \mathbf{Y}'}{\partial a}\right] . \tag{A31}$$

Consider also the derivative

$$\frac{\partial \mathrm{f}(\mathbf{Y})}{\partial \mathbf{X}} = \sum_{kl} \frac{\partial \mathrm{f}(\mathbf{Y})}{\partial \langle \mathbf{Y} \rangle_{kl}} \frac{\partial \langle \mathbf{Y} \rangle_{kl}}{\partial \mathbf{X}} . \tag{A32}$$

Several interesting applications arise. For example, let $\mathrm{f}(\mathbf{Y}) = |\mathbf{X}-\lambda\mathbf{I}|$, where $\mathbf{X}$ may be the population-growth matrix. Then

$$\frac{\partial |\mathbf{X}-\lambda\mathbf{I}|}{\partial \langle \mathbf{X} \rangle} = \operatorname{tr}\left[\frac{\partial |\mathbf{X}-\lambda\mathbf{I}|}{\partial [\mathbf{X}-\lambda\mathbf{I}]} \frac{\partial [\mathbf{X}-\lambda\mathbf{I}]'}{\partial \langle \mathbf{X} \rangle}\right] ,$$

$$\frac{\partial |\mathbf{X}-\lambda\mathbf{I}|}{\partial \langle \mathbf{X} \rangle} = |\mathbf{X}-\lambda\mathbf{I}| \operatorname{tr}\left[[[\mathbf{X}-\lambda\mathbf{I}]']^{-1}\mathbf{J}'\right] \tag{A33}$$

$$= \operatorname{tr}\left[[\operatorname{cof}(\mathbf{X}-\lambda\mathbf{I})]\mathbf{J}'\right] ,$$

and

$$\frac{\partial |\mathbf{X}-\lambda\mathbf{I}|}{\partial \mathbf{X}} = \sum_{kl} \frac{\partial |\mathbf{X}-\lambda\mathbf{I}|}{\partial \langle [\mathbf{X}-\lambda\mathbf{I}] \rangle_{kl}} \frac{\partial \langle [\mathbf{X}-\lambda\mathbf{I}]' \rangle_{kl}}{\partial \mathbf{X}} = \sum_{kl} |\mathbf{X}-\lambda\mathbf{I}| [\mathbf{X}-\lambda\mathbf{I}]_{kl}'^{-1} \mathbf{J}_{kl}' , \tag{A34}$$

$$\frac{\partial |\mathbf{X}-\lambda\mathbf{I}|}{\partial \mathbf{X}} = |\mathbf{X}-\lambda\mathbf{I}| [\mathbf{X}-\lambda\mathbf{I}]'^{-1} = \operatorname{cof}[\mathbf{X}-\lambda\mathbf{I}] , \tag{A35}$$

where $\operatorname{cof}[\mathbf{X}-\lambda\mathbf{I}]$ is the cofactor matrix of $[\mathbf{X}-\lambda\mathbf{I}]$.

If $\mathbf{Y}(r)$ is a function of the scalar r, then

$$\frac{\partial |\mathbf{Y}(r)|}{\partial r} = \operatorname{tr}\left[\frac{\partial |\mathbf{Y}(r)|}{\partial [\mathbf{Y}(r)]} \frac{\partial [\mathbf{Y}(r)]'}{\partial r}\right] = \operatorname{tr}\left[|\mathbf{Y}(r)| \left[[\mathbf{Y}(r)]'\right]^{-1} \frac{\partial [\mathbf{Y}(r)]'}{\partial r}\right] ,$$

and, since $\operatorname{tr}\mathbf{AB} = \operatorname{tr}[\mathbf{AB}]' = \operatorname{tr}\mathbf{B}'\mathbf{A}'$, then

$$\frac{\partial |\mathbf{Y}(r)|}{\partial r} = |\mathbf{Y}(r)| \operatorname{tr}\left[\frac{\partial [\mathbf{Y}(r)]}{\partial r} [\mathbf{Y}(r)]^{-1}\right] . \tag{A36}$$

Formula (A36) is not only of interest in a study of the sensitivity of the determinant of a polynomial matrix, but is also useful in the computation of the determinant, as shown by Emre and Hüseyin (1975, page 136) An application of equation (A36) that is relevant is

$$\frac{\partial |\mathbf{A}-\lambda\mathbf{I}|}{\partial \lambda} = -|\mathbf{A}-\lambda\mathbf{I}| \operatorname{tr}[\mathbf{A}-\lambda\mathbf{I}]^{-1} . \tag{A37}$$

This formula can also be found in Newbery (1974, page 1016). Finally consider the application where $\mathrm{f}(\mathbf{Y}) = \operatorname{tr}[\mathbf{AXB}]$, whence

$$\frac{\partial \mathrm{f}(\mathbf{Y})}{\partial \mathbf{X}} = \frac{\partial \operatorname{tr}[\mathbf{AXB}]}{\partial \mathbf{X}} = \mathbf{A}'\mathbf{B}' . \tag{A38}$$

A5 *Vector differentiation*

Vectors may be considered as matrices with only one row or one column, and the rules for matrix differentiation may be applied. But the derivative of a vector or of a vector equation has a simpler form than the matrix analogue. It is therefore worthwhile to list the formulae for vector differentiation separately. Two cases are considered: the derivative of a scalar function with respect to a vector and the derivative of a vector function with respect to a vector.

A5.1 *Differentiation of a scalar function with respect to a vector.* Consider the general scalar function $f(\{x\})$, where $\{x\}$ is the argument vector. Some relevant formulations of $f(\{x\})$ and their derivatives are listed below.

$$f(\{x\}) = \begin{cases} \{a\}'\{x\} , \\ \{x\}'\{x\} , \\ \{x\}'A\{x\} , \end{cases} \qquad \text{then } \frac{\partial f(\{x\})}{\partial\{x\}} = \begin{cases} \{a\} , & \text{(A39)} \\ 2\{x\} , & \text{(A40)} \\ A\{x\} + A'\{x\} . & \text{(A41)} \end{cases}$$

A5.2 *Differentiation of a vector function with respect to a vector.* Let $\{f(\{x\})\}$ denote a column vector of scalar functions $f_i(\{x\})$, where $\{x\}$ is the argument vector and $\{f(\{x\})\}$ represents a system of equations. For example, let $\{f(\{x\})\}$ be a system of linear equations in $\{x\}$, that is, $\{f(\{x\})\} = A\{x\}$, then

$$\frac{\partial A\{x\}}{\partial\langle\{x\}\rangle_i} = \{a_i\} , \qquad \text{(A42)}$$

where $\{a_i\}$ is the ith column of A.

The derivatives of $\{f(\{x\})\}$ with respect to all the elements of the argument vector form a matrix if the argument vector is a row vector. For example

$$\frac{\partial A\{x\}}{\partial\{x\}'} = A . \qquad \text{(A43)}$$

The determinant $|\partial\{f(\{x\})\}/\partial\{x\}'|$ is known as the Jacobian or functional determinant.

Corresponding to the chain rule of matrix differentiation, one may formulate the chain rule of vector differentiation. Let $\{x\}$, $\{y\}$, and $\{z\}$ be vectors. It can be shown that

$$\frac{\partial\{y\}}{\partial\{x\}'} = \frac{\partial\{y\}}{\partial\{z\}'}\,\frac{\partial\{z\}}{\partial\{x\}'} . \qquad \text{(A44)}$$

Environment and Planning A, 1976, volume 8, pages 515-541

Shrinking large-scale population-projection models by aggregation and decomposition

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Received 12 December 1975

Abstract. During the past two decades social scientists have come to model dynamic socioeconomic systems of growing size and complexity. Despite a heavy reliance on ever more sophisticated high-speed digital computers, however, computer capacity for handling such systems has not kept pace with the growing demands for more detailed information. Consequently, it is becoming ever more important to identify those aspects of a system which permit one to deal with parts of it independently from the rest or to treat relationships *among* particular subsystems as though they were independent of the relationships *within* those subsystems. These questions are, respectively, those of decomposition and aggregation, and their application toward 'shrinking' large-scale population projection models is the focus of this paper.

1 Introduction

During the past two decades social scientists have come to model dynamic socioeconomic systems of growing size and complexity. However, despite a heavy reliance on ever more sophisticated high-speed digital computers, their ability to handle such systems has not kept pace with the growing demands for more detailed information.

> "As a consequence, it is becoming more and more important to secure information on the nature of those aspects of a system which, when present, enable us to treat a part of it separately from the rest or to deal with the relationship among particular subsystems as though it were independent of the structures within those subsystems. The latter question is that of aggregation, while the former is ... one of partition ..." (Ando and Fisher, 1963, page 92).

An increasing number of social scientists currently find themselves in the somewhat frustrating position of being asked to provide accurate projections at very fine levels of detail with resources that are scarcely sufficient for carrying out such projections at much more aggregate levels of resolution. Prominent among them are demographers who are called upon to produce consistent projections of regional populations disaggregated by age, color, race, sex, and such indicators of class and welfare as employment category and income. Since the computational requirements of this task are staggering, the need for developing improved methods for 'shrinking' population-projection models by reducing their dimensionality is an urgent one, and the two most obvious methods for effecting such a reduction are *aggregation* and partitioning or, more appropriately, *decomposition*.

1.1 *Aggregation*

The need to use aggregates arises because most areas of research in the social sciences involve large systems. Both theoretical abstract reasoning and numerical empirical computation rely on the conceptual clarity and efficient manipulation of variables afforded by aggregation. In economic modeling, for example, producers and consumers of a national or regional economy are aggregated into a relatively small number of sectors, and the interaction among these sectors is then studied as though it were free of influences arising from intrasectoral interaction. A typical example of this occurs in input–output analysis; indeed, it was the increasing worldwide numerical application of such models that first stimulated much of the interest in aggregation among social scientists (for example, Ara, 1959; Fisher, 1969; Rogers, 1969).

Aggregation generally introduces inconsistencies between the outputs of the disaggregated and the aggregated models. The conditions for aggregation without such inconsistencies, that is, for *perfect aggregation*, are very severe and therefore are almost never met in practice. However, since any model is at best only an approximate description of reality, we remain interested in establishing the conditions under which perfect aggregation may be carried out. These conditions suggest the criteria, or rules, for selecting which variables to aggregate and help to identify the circumstances under which such an aggregation will yield results that are consistent with those of the original disaggregated model.

Aggregation of large-scale problems, therefore, has two fundamental aspects. The first is the process of consolidation itself, in which the two sets of variables that are connected by a system of relations are grouped into aggregates, and a new smaller system of relations that connects the two sets of aggregates is developed. The second fundamental aspect of the aggregation process is the selection of the consolidation scheme that most closely satisfies the conditions necessary for perfect aggregation, while at the same time meeting whatever informational requirements and additional constraints that may have been specified a priori. In short, consolidation is an operation that expresses a set of 'new' variables as weighted averages of the set of original 'old' variables such that there are fewer new variables than old ones. Criteria for perfect aggregation, on the other hand, are rules that indicate which variables to consolidate: for example, the rule that variables which always move together may be consolidated into a single variable without introducing an aggregation error.

Two particular forms of aggregation are frequently employed in demographic analysis. The first is a consolidation across age groups. When carried out over *all* age groups, this form of consolidation transforms a cohort-survival model into a components-of-change model (Rogers, 1971, chapter 1). I shall, therefore, refer to aggregations of this sort as *components-of-change* aggregations. Such aggregations retain the geographical areal units of the original cohort-survival model but sacrifice all age-specific details.

The second form of aggregation that is frequently used is the division of a multiregional population system into two regions: a particular region under study and 'the rest of the world'. In this paper such consolidations will be called ***biregional aggregations***. They sacrifice considerable geographical information but preserve details about age compositions. However, if applied in sequence to each and every region of a multiregional system, they permit a *collection* of aggregated projections to preserve completely the levels of detail found in the original unconsolidated projection.

1.2 *Decomposition*

The idea of decomposing a large and complex problem into several smaller subproblems in order to simplify its solution is not new and has been used for well over a century in the physical and social sciences, as well as in engineering. However, during the past two decades the development and use of high-speed computers to solve these problems have focused interest on the application of decomposition techniques to such various fields as process control, structural engineering, systems optimization, electrical network theory, and a wide variety of seemingly unrelated problems in economics, mathematics, design, and operations research (for example, Himmelblau, 1973; Rose and Willoughby, 1972; Tewarson, 1973; Theil, 1972).

The central principle of decomposition analysis is that the solution of a problem that contains a large number of systems, and that involves many interacting elements, can often be broken up and expressed in terms of the solutions of relatively independent subsystem problems of lower dimensionality. The solutions of the subsystem problems can then be combined and, if necessary, modified to yield the solution of the original problem. A well-known illustration of this approach is provided by the Dantzig and Wolfe (1960) decomposition algorithm in mathematical programming. This algorithm breaks up a large linear-programming problem into several smaller linear-programming problems and imposes additional constraints on each of the latter in order to ensure that their solutions combine to yield the optimal solution for the large-scale problem.

In general, decompositions of large-scale problems proceed in two stages. First, there is the *partitioning* stage in which a large system of variables and relations is rearranged and reordered in a search for *disjoint* subsystems; that is, subsets of relations which do not contain any common variables. If such subsystems exist, then each one can be treated independently of the rest. In this way the relational structure of the original large-scale problem can be exploited to produce a more efficient method of solution.

Systems that can be partitioned into independent (disjoint) subsystems are said to be *completely decomposable*, and their matrix expression can be transformed into what is known as a *block-diagonal* form. The rearrangement and reordering of the relations to identify and to delineate the disjoint subsystems is called *permutation*, and the actual separation of the large system into disjoint subsystems is called *partitioning*.

Obviously, partitioning of a large system into disjoint subsystems cannot be accomplished if each relation in the system contains every variable. Such systems are said to be *indecomposable*. Fortunately, the relations in most mathematical models of socioeconomic phenomena contain only a few common variables. Moreover, when complete decomposition cannot be achieved, a *partial decomposition* that rearranges and reorders the relations into a *block-triangular* form may still be possible.

A block-triangular structure defines an information flow that is serial and without loops. Causal sequences in such systems, therefore, run one-way and permit feedbacks in the triangular hierarchy in a single direction only. An example of such a structure is afforded by a hierarchy of migration flows in which people migrate only to larger urban regions. If in the process of projecting population the regions are ordered according to their size, then the growth matrix assumes a block-triangular form.

Once a large system of variables and relations has been either completely or partially decomposed into indecomposable subsystems, further simplification of the problem can only be achieved by a process called *tearing*. This is the second stage of the decomposition procedure and consists of deleting variables from one or more of the relations in which they appear. Thus tearing represents an attempt to solve a system problem by a 'forced' partitioning of that system into supposedly disjoint subsystems. The partitioning is forced because the subsystems are not truly disjoint and are rendered so only through a disregard of certain connecting relationships which are held to be insignificant. If the impacts of these connecting relationships are not completely disregarded but are allowed somehow to affect the solution of the system problem, then we have an instance of *compensated tearing*.

1.3 *Numerical illustrations*

Imagine a multiregional population distributed among four regions called, respectively, the North, South, East, and West regions. Assume that the multiregional population is a closed system which experiences internal migration but is undisturbed by external migration flows. Moreover, assume that every year half of the populations of the North and South regions and three-quarters of the populations of the East and West regions, respectively, out-migrate in equal proportions to the remaining three regions.

Finally, to simplify matters further, let the number of births equal the number of deaths in each region, so that natural increase is zero in each region.

Starting with an initial multiregional population of 480 individuals distributed equally among the four regions, the above regime of growth and change would produce the following population distribution one year later:

$$\text{North:}\quad 140 = \overbrace{\tfrac{1}{2}(120)}^{\text{non-migrants}} + \overbrace{\tfrac{1}{6}(120) + \tfrac{1}{4}(120) + \tfrac{1}{4}(120)}^{\text{in-migrants}} \,,$$

$$\text{South:}\quad 140 = \tfrac{1}{6}(120) + \overbrace{\tfrac{1}{2}(120)}^{\text{non-migrants}} + \tfrac{1}{4}(120) + \tfrac{1}{4}(120) \,,$$

$$\text{East:}\quad 100 = \tfrac{1}{6}(120) + \tfrac{1}{6}(120) + \overbrace{\tfrac{1}{4}(120)}^{\text{non-migrants}} + \tfrac{1}{4}(120) \,,$$

$$\text{West:}\quad 100 = \tfrac{1}{6}(120) + \tfrac{1}{6}(120) + \tfrac{1}{4}(120) + \overbrace{\tfrac{1}{4}(120)}^{\text{non-migrants}} \,.$$

This projection process can be expressed conveniently in matrix form as follows:

$$\begin{bmatrix} 140 \\ 140 \\ 100 \\ 100 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{6} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 120 \\ 120 \\ 120 \\ 120 \end{bmatrix} . \tag{1}$$

Let us now 'shrink' our components-of-change population-projection model to a fourth of its original size by aggregating the North and South regions into one region and the East and West regions into another. The corresponding consolidation of equation (1) then yields

$$\begin{bmatrix} 280 \\ 200 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 240 \\ 240 \end{bmatrix} . \tag{2}$$

An alternative consolidation scheme is to treat one region as interacting with the rest of the system. For example, a focus on the interaction between the North region and the aggregate of all other regions gives

$$\begin{bmatrix} 140 \\ 340 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{2}{9} \\ \frac{1}{2} & \frac{7}{9} \end{bmatrix} \begin{bmatrix} 120 \\ 360 \end{bmatrix} . \tag{3}$$

Note that this particular spatial consolidation is an example of biregional aggregation, and observe that by repeating this procedure with each of the original four regions we can obtain a population projection for every one of them.

By using the growth models given in equations (1), (2), and (3), another round of projections reveals that the first consolidation is an example of perfect aggregation inasmuch as it forecasts the same total population as the original unconsolidated model in equation (1). However, the biregional consolidation in equation (3) is an example of imperfect aggregation and projects a slightly higher population for the North region than the one generated by the unconsolidated model. The first consolidation satisfies the sufficient condition for perfect aggregation, which asserts that two populations exhibiting identical rates of birth, death, and out-migration to the rest of the multiregional system may be consolidated without introducing an error into the projection process (Rogers, 1969).

Assume now that the migration flows from the North and South regions to the East and West regions, and the corresponding flows in the reverse direction, are ignored. The projection matrix in equation (1) then becomes completely decomposable and assumes a block-diagonal form:

$$\begin{bmatrix} 80 \\ 80 \\ \hline 60 \\ 60 \end{bmatrix} = \left[\begin{array}{cc|cc} \frac{1}{2} & \frac{1}{6} & 0 & 0 \\ \frac{1}{6} & \frac{1}{2} & 0 & 0 \\ \hline 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} \end{array}\right] \begin{bmatrix} 120 \\ 120 \\ \hline 120 \\ 120 \end{bmatrix} . \tag{4}$$

The resulting population projection becomes an example of uncompensated tearing and, of course, produces an erroneous population forecast. Consequently we may wish to introduce an adjustment to the model, in the form of *net* migration rates, by including the migration flows that were ignored in the diagonal elements of the projection matrix, thereby illustrating the process of compensated tearing. This gives

$$\begin{bmatrix} 140 \\ 140 \\ \hline 100 \\ 100 \end{bmatrix} = \left[\begin{array}{cc|cc} 1 & \frac{1}{6} & 0 & 0 \\ \frac{1}{6} & 1 & 0 & 0 \\ \hline 0 & 0 & \frac{7}{12} & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & \frac{7}{12} \end{array}\right] \begin{bmatrix} 120 \\ 120 \\ \hline 120 \\ 120 \end{bmatrix} . \tag{5}$$

The advantage of a block-diagonal decomposition of the kind set out in equation (5) is the shrinking that it achieves. The larger system projection can be partitioned and torn into independent subsystems, each of which can then be projected separately. For example, in place of the 'large-scale' population projection described in equation (1), we may, instead, carry out the two 'smaller' projections:

$$\begin{bmatrix} 140 \\ 140 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{6} \\ \frac{1}{6} & 1 \end{bmatrix} \begin{bmatrix} 120 \\ 120 \end{bmatrix} , \tag{6}$$

and

$$\begin{bmatrix} 100 \\ 100 \end{bmatrix} = \begin{bmatrix} \frac{7}{12} & \frac{1}{4} \\ \frac{1}{4} & \frac{7}{12} \end{bmatrix} \begin{bmatrix} 120 \\ 120 \end{bmatrix} , \tag{7}$$

respectively.

For our final numerical illustration of decomposition, let us now, instead, ignore only the out-migration flows from the North and South regions to the East and West regions, respectively. The projection matrix in equation (1) then becomes partially decomposable and assumes a block-triangular form:

$$\begin{bmatrix} 140 \\ 140 \\ \hline 60 \\ 60 \end{bmatrix} = \left[\begin{array}{cc|cc} \frac{1}{2} & \frac{1}{6} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \hline 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} \end{array}\right] \begin{bmatrix} 120 \\ 120 \\ \hline 120 \\ 120 \end{bmatrix} . \tag{8}$$

Modifying this projection matrix to take into account the flows that were ignored, we obtain

$$\begin{bmatrix} 140 \\ 140 \\ \hline 100 \\ 100 \end{bmatrix} = \left[\begin{array}{cc|cc} \frac{1}{2} & \frac{1}{6} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \hline 0 & 0 & \frac{7}{12} & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & \frac{7}{12} \end{array}\right] \begin{bmatrix} 120 \\ 120 \\ \hline 120 \\ 120 \end{bmatrix} . \tag{9}$$

Observe that the block-triangular decomposition in this equation also permits some shrinking of the original 'large-scale' model, and note that decomposition with tearing, like aggregation, generally introduces errors into the projection process.

Figure 1 summarizes the principal points of the numerical examples by illustrating the structures of the various projection matrices used in them.

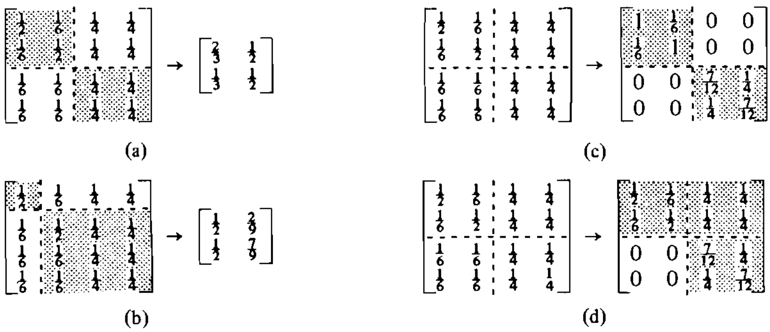

Figure 1. An illustration of the principal means of shrinking population projection matrices. (a) An arbitrary aggregation, (b) a biregional aggregation, (c) complete (compensated) decomposition into block-diagonal form, (d) partial (compensated) decomposition into (upper) block-triangular form.

2 Shrinking by aggregation

Aggregation in demographic analysis may be carried out by consolidating

(1) *population characteristics*; for example, combining several groups: sex, color, or age;

(2) *time units*; for example, dealing with five-year intervals of time rather than yearly intervals; and

(3) *spatial units*; for example, aggregating the fifty states of the US into its nine census divisions.

In each case the consolidated projection produces results that are coarser with regard to levels of detail than those provided by the original unconsolidated model.

Consider, for example, the two multiregional population systems illustrated in figure 2: the nine census divisions of the US and the corresponding four census regions. Spatial expectations of life at birth and migration levels for the nine-region population system are set out in table 1, and a cohort-survival population projection, carried out by using five-year age groups, produces the aggregate results that appear in table 2. A spatial consolidation of the nine census divisions into the four census regions permits a considerable shrinkage of the original model, but the process introduces some aggregation error and, what is more important, leads to population projections that are less detailed geographically than those obtained from the unconsolidated model. This can be seen by examining tables 3 and 4, which give the four-region counterparts of the nine-region results set out in tables 1 and 2, respectively.

Collectively, the four tables illustrate the following important features of aggregation. First, aggregated demographic measures are weighted averages of the corresponding disaggregated measures. Second, spatial aggregation necessarily reduces the level of interregional migration, since a part of what was previously defined to be interregional migration becomes intraregional migration in the consolidated model. Last, aggregation normally introduces an aggregation bias or error into the consolidated population projections.

These three features may be illustrated with the numerical data set out in tables 1-4. For example, table 1 shows that a baby born in the New England division of the US and subjected to the multiregional regime of mortality and migration that prevailed in 1958 would have a life expectancy of 70 years $[{}_1e(0) = 70\cdot00]$, over a third of which would be lived outside of the division of birth $\left(\sum_{j \neq 1} {}_1\theta_j = 0\cdot3607\right)$. The corresponding life expectancy of a baby born in the Middle Atlantic division is 69·68 years. Aggregation of the nine divisions into the four regions consolidates these two cohorts of babies, according them an average life expectancy of 69·76 years (table 3).

The levels of interregional migration in the nine-region system may be measured by summing the off-diagonal elements in each row of the matrix in table 1. These sums define, for each regional cohort, the average fraction of a lifetime that is expected to be lived outside the region of birth. Such a summation results in values of 0·3607 and 0·3009, respectively, for the New England and Middle Atlantic divisions of the US, for example. However, the same computation for the larger Northeastern region gives the lower value of 0·2705.

Finally, a comparison of the population projections summarized in tables 2 and 4 indicates the magnitudes of the aggregation errors that are introduced by the consolidation of the nine divisions into the four regions. For the US as a whole one finds, for example, that a fifty-year projection of the 1958 population to the year 2008, on the assumption of an unchanging growth regime, produces an overprojection of almost 400000 people. But, curiously enough, a further projection of the same population until stability is achieved does not appreciably alter the intrinsic rate of growth, r, of the multiregional system. A difference of 0·00008 is all that distinguishes the intrinsic rate of growth of the four-region projection from that of the nine-region projection.

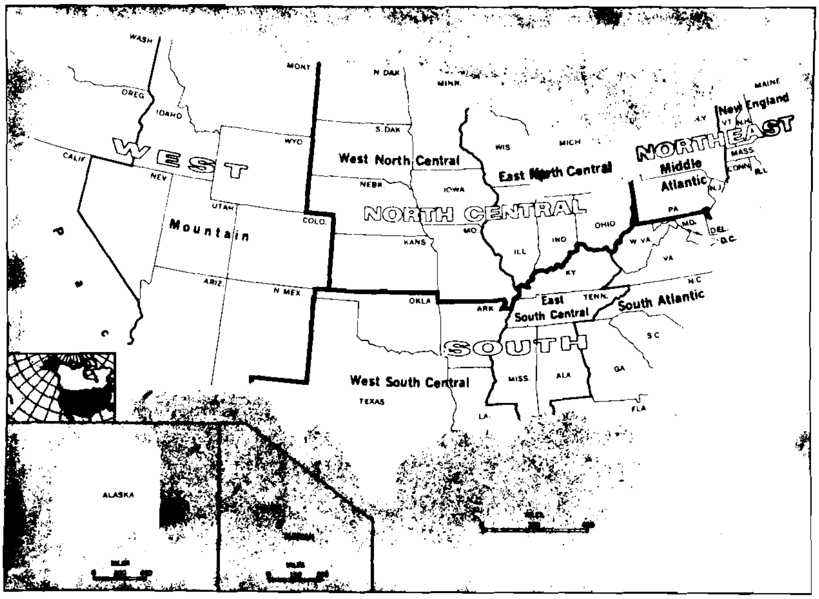

Figure 2. Regions and geographic divisions of the US (source: US Bureau of the Census).

Table 1. Expectations of life at birth, and migration levels by division of residence and division of birth: US total population, 1958.

Division of birth	Division of residence									Total
	1	2	3	4	5	6	7	8	9	
Expectations of life at birth: ${}_ie_j(0)$										
1 New England	44·75	6·16	3·03	1·04	6·46	0·82	1·52	1·16	5·06	70·00
2 Middle Atlantic	2·50	48·71	3·58	0·89	6·70	0·87	1·31	1·05	4·07	69·68
3 East North Central	0·89	2·56	47·14	2·61	5·16	2·05	2·08	1·85	5·82	70·17
4 West North Central	0·79	1·75	6·32	39·56	3·45	1·20	3·98	4·13	9·57	70·75
5 South Atlantic	1·58	5·16	4·82	1·28	45·39	2·57	2·31	1·23	4·46	68·81
6 East South Central	0·77	2·27	8·94	1·68	8·36	37·48	3·81	1·28	4·25	68·83
7 West South Central	0·76	1·76	3·85	3·16	3·98	2·25	41·90	3·39	8·48	69·54
8 Mountain	0·97	2·00	3·87	3·89	3·47	1·17	5·28	33·22	15·90	69·78
9 Pacific	1·03	2·10	3·35	2·55	3·72	1·08	3·56	4·19	48·65	70·21
Migration levels: ${}_i\theta_j$										
1 New England	0·6393	0·0880	0·0433	0·0149	0·0923	0·0117	0·0217	0·0166	0·0723	1·00
2 Middle Atlantic	0·0357	0·6991	0·0514	0·0128	0·0962	0·0125	0·0188	0·0151	0·0584	1·00
3 East North Central	0·0127	0·0365	0·6718	0·0372	0·0735	0·0292	0·0296	0·0264	0·0829	1·00
4 West North Central	0·0112	0·0248	0·0893	0·5592	0·0488	0·0170	0·0563	0·0584	0·1353	1·00
5 South Atlantic	0·0230	0·0750	0·0700	0·0186	0·6596	0·0373	0·0336	0·0179	0·0648	1·00
6 East South Central	0·0112	0·0330	0·1299	0·0244	0·1215	0·5445	0·0554	0·0186	0·0617	1·00
7 West South Central	0·0109	0·0253	0·0554	0·0454	0·0572	0·0324	0·6025	0·0487	0·1219	1·00
8 Mountain	0·0139	0·0287	0·0555	0·0557	0·0497	0·0168	0·0757	0·4761	0·2279	1·00
9 Pacific	0·0147	0·0299	0·0477	0·0363	0·0530	0·0154	0·0507	0·0597	0·6929	1·00

Table 2. Multiregional projections to stability: US total population, 1958, nine-region projection.

Projections and parameters of stable growth	Division of residence									Total
	(1) New England	(2) Middle Atlantic	(3) East North Central	(4) West North Central	(5) South Atlantic	(6) East South Central	(7) West South Central	(8) Mountain	(9) Pacific	
K (1958)	9911000	33181000	35763000	15114000	24749000	11769000	16177000	6349000	19141000	172154000
% (1958)	0·0576	0·1927	0·2077	0·0878	0·1438	0·0684	0·0940	0·0369	0·1112	1·0000
K (2008)	21644039	59187140	80761069	31173278	68283065	24394274	40446886	22805818	73166573	421862143
% (2008)	0·0513	0·1403	0·1914	0·0739	0·1619	0·0578	0·0959	0·0541	0·1734	1·0000
r (∞)	–	–	–	–	0·02184	–	–	–	–	0·02184
% (∞)	0·0447	0·1013	0·1719	0·0727	0·1535	0·0492	0·1024	0·0680	0·2362	1·0000

Table 3. Expectations of life at birth and migration levels by region of residence and region of birth: US total population, 1958, four-region projection.

Region of birth	Region of residence				Total	Region of residence				Total
	1	2	3	4		1	2	3	4	
	Expectations of life at birth: ${}_ie_j(0)$					*Migration levels:* ${}_i\theta_j$				
1 Northeast	50·90	4·49	8·88	5·50	69·76	0·7295	0·0643	0·1273	0·0788	1·00
2 North Central	3·18	48·45	9·10	9·60	70·32	0·0452	0·6889	0·1294	0·1365	1·00
3 South	4·58	7·52	49·21	7·67	68·98	0·0664	0·1091	0·7134	0·1111	1·00
4 West	3·18	6·60	8·95	51·22	69·94	0·0454	0·0944	0·1279	0·7322	1·00

$$
\begin{array}{c}
1959 \\
\begin{bmatrix}
10022829 \\
33457706 \\
36216395 \\
15249522 \\
25261427 \\
11892775 \\
16429159 \\
6518501 \\
19678904
\end{bmatrix}
\end{array}
=
\begin{bmatrix}
1{\cdot}001728 & 0{\cdot}001205 & 0{\cdot}000342 & 0{\cdot}000284 & 0{\cdot}000775 & 0{\cdot}000270 & 0{\cdot}000289 & 0{\cdot}000444 & 0{\cdot}000436 \\
0{\cdot}002935 & 1{\cdot}002820 & 0{\cdot}001049 & 0{\cdot}000580 & 0{\cdot}002710 & 0{\cdot}000812 & 0{\cdot}000625 & 0{\cdot}000823 & 0{\cdot}000827 \\
0{\cdot}001106 & 0{\cdot}001430 & 1{\cdot}004586 & 0{\cdot}003253 & 0{\cdot}002303 & 0{\cdot}005235 & 0{\cdot}001613 & 0{\cdot}001838 & 0{\cdot}001328 \\
0{\cdot}000349 & 0{\cdot}000268 & 0{\cdot}001297 & 0{\cdot}999266 & 0{\cdot}000507 & 0{\cdot}000714 & 0{\cdot}001712 & 0{\cdot}002556 & 0{\cdot}001202 \\
0{\cdot}003269 & 0{\cdot}003430 & 0{\cdot}002605 & 0{\cdot}001362 & 1{\cdot}005136 & 0{\cdot}004931 & 0{\cdot}001740 & 0{\cdot}001499 & 0{\cdot}001549 \\
0{\cdot}000253 & 0{\cdot}000279 & 0{\cdot}001073 & 0{\cdot}000486 & 0{\cdot}001524 & 0{\cdot}999640 & 0{\cdot}001315 & 0{\cdot}000530 & 0{\cdot}000427 \\
0{\cdot}000511 & 0{\cdot}000408 & 0{\cdot}000792 & 0{\cdot}001978 & 0{\cdot}001053 & 0{\cdot}002181 & 1{\cdot}004362 & 0{\cdot}003391 & 0{\cdot}001647 \\
0{\cdot}000426 & 0{\cdot}000395 & 0{\cdot}000887 & 0{\cdot}002591 & 0{\cdot}000507 & 0{\cdot}000471 & 0{\cdot}002060 & 0{\cdot}996787 & 0{\cdot}002626 \\
0{\cdot}002081 & 0{\cdot}001548 & 0{\cdot}002471 & 0{\cdot}004574 & 0{\cdot}001823 & 0{\cdot}001545 & 0{\cdot}004025 & 0{\cdot}010701 & 1{\cdot}005854
\end{bmatrix}
\begin{array}{c}
1958 \\
\begin{bmatrix}
9911000 \\
33181000 \\
35763000 \\
15114000 \\
24749000 \\
11769000 \\
16177000 \\
6349000 \\
19141000
\end{bmatrix}
\end{array}
$$

Figure 3. The multiregional components-of-change population-projection model: US total population, 1958, nine-region projection.

Aggregation over regions preserves age-specific details at the expense of geographic details. If the latter are of greater interest than the former, one may instead consolidate all age groups into a single variable and retain the original set of geographical areas. The application of such an aggregation to the cohort-survival model associated with tables 1 and 2 yields the components-of-change projection process illustrated in figure 3 and produces the multiregional projections shown in table 5.

Table 5 reveals that a components-of-change aggregation of the original cohort-survival model leads to a substantial underprojection of total population growth, but to a relatively accurate projection of the spatial distribution of that growth. The total US population in the year 2008, for example, is underprojected by over fifty-one million people, and the intrinsic rate of growth is underprojected by more than six per thousand. Yet the Pacific division is allocated approximately seventeen percent of the total population in the year 2008 by both models.

The divergence between the projections in tables 2 and 5 increases exponentially over time. Figure 4 shows that the two models project similar population totals

Table 4. Multiregional projections to stability and associated parameters: US total population, 1958, four-region projection.

Projections and parameters of stable growth	Region of residence				Total
	(1) Northeast	(2) North Central	(3) South	(4) West	
K (1958)	43092000	50877000	52695000	25490000	172154000
% (1958)	0·2503	0·2955	0·3061	0·1481	1·0000
K (2008)	80383757	112077195	132843209	96955108	422259268
% (2008)	0·1904	0·2654	0·3146	0·2296	1·0000
r (∞)		0·02192			0·02192
% (∞)	0·1431	0·2491	0·3046	0·3032	1·0000

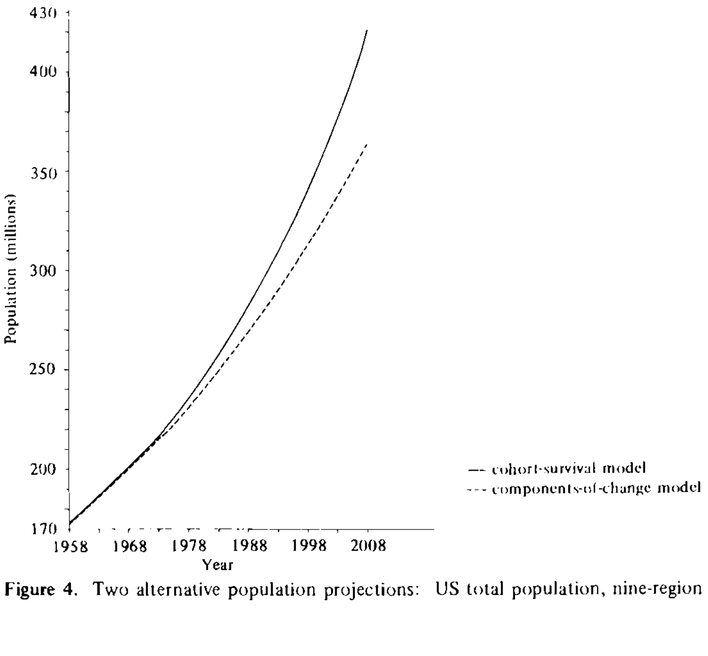

Figure 4. Two alternative population projections: US total population, nine-region projection.

Table 5. Multiregional projections to stability: US total population, 1958, nine-region components-of-change projections.

Projections and parameters of stable growth	Division of residence									Total
	(1) New England	(2) Middle Atlantic	(3) East North Central	(4) West North Central	(5) South Atlantic	(6) East South Central	(7) West South Central	(8) Mountain	(9) Pacific	
K (1958)	9911000	33181000	35763000	15114000	24749000	11769000	16177000	6349000	19141000	172154000
% (1958)	0·0576	0·1927	0·2077	0·0878	0·1438	0·0684	0·0940	0·0369	0·1112	1·0000
K (2008)	17927349	53159821	68434148	25822107	62159432	21199129	35493951	19076175	61336572	364608685
% (2008)	0·0492	0·1458	0·1877	0·0708	0·1705	0·0581	0·0973	0·0523	0·1682	1·0000
r (∞)	–	–	–	–	0·01554	–	–	–	–	0·01554
% (∞)	0·0360	0·0897	0·1516	0·0631	0·1748	0·0490	0·1107	0·0717	0·2533	1·0000

Table 6. Expectations of life at birth and migration levels by place of residence and place of birth: US total population, 1958.

Place of birth	Place of residence 1	2	Place of birth	Place of residence 1	2	Place of birth	Place of residence 1	2
Expectations of life at birth: ${}_ie_j(0)$								
1 New England	44·70	25·28	1 West North Central	39·33	31·10	1 West South Central	41·64	27·67
2 Rest of the US	1·36	68·07	2 Rest of the US	2·03	67·32	2 Rest of the US	2·62	66·88
1 Middle Atlantic	48·55	21·14	1 South Atlantic	45·39	23·37	1 Mountain	32·68	36·74
2 Rest of the US	3·12	66·38	2 Rest of the US	5·60	64·04	2 Rest of the US	2·16	67·34
1 East North Central	47·13	22·90	1 East South Central	37·36	31·39	1 Pacific	47·96	22·09
2 Rest of the US	4·86	64·51	2 Rest of the US	1·69	67·84	2 Rest of the US	6·31	63·18
Migration levels: ${}_i\theta_j$								
1 New England	0·6388	0·3612	1 West North Central	0·5584	0·4416	1 West South Central	0·6008	0·3992
2 Rest of the US	0·0196	0·9804	2 Rest of the US	0·0293	0·9707	2 Rest of the US	0·0378	0·9622
1 Middle Atlantic	0·6967	0·3033	1 South Atlantic	0·6601	0·3399	1 Mountain	0·4708	0·5292
2 Rest of the US	0·0449	0·9551	2 Rest of the US	0·0804	0·9196	2 Rest of the US	0·0311	0·9689
1 East North Central	0·6730	0·3270	1 East South Central	0·5435	0·4565	1 Pacific	0·6847	0·3153
2 Rest of the US	0·0700	0·9300	2 Rest of the US	0·0243	0·9757	2 Rest of the US	0·0909	0·9091

during the first decade, start to diverge shortly thereafter, and then grow increasingly further apart. This suggests that shrinking by components-of-change aggregation is most effective for short-run projections.

We have seen that aggregation is generally accompanied by loss of detail. This, however, need not always be the case. One can, for example, obtain a biregionally aggregated population projection for every region of a multiregional system and thereby retain the same level of detail in the resulting collection of consolidated projections as originally existed in the single unconsolidated model. By way of illustration, consider the nine sets of 2 x 2 regional life expectancies and migration levels that appear in table 6. They were obtained with the aid of nine biregional aggregations of the data set that produced table 1. The projection model that produced table 2 was similarly aggregated, and the collection of nine biregional projections yielded the results set out in table 7. A comparison of the projections in table 7 with those in table 2 suggests that an exhaustive collection of biregional aggregations is a reasonably accurate substitute for a large-scale population-projection model.

Although biregional aggregations may be applied with some success to shrink a large model, they can be computationally demanding if the number of times they must be applied is as great as the number of regions in a multiregional system. In such instances, a more efficient and effective shrinking technique can often be developed by using decomposition methods.

3 Shrinking by decomposition

Decomposition procedures have often been used in demographic analysis, although they have not been specifically identified by that name. Perhaps their most common application is manifested in representations of multiregional population systems by collections of single-region models, which assume that each regional population is undisturbed by migration. Such an assumption is, of course, equivalent to the premise that the multiregional population system is completely decomposable into independent single-region subsystems arranged in block-diagonal form. A modification of this assumption is often introduced into the single-region model by including the impact of *net* migration in the survivorship proportions, that is, by treating an out-migrant as a 'death' and an in-migrant as a replacement for a death. Such a modification of the complete single-region decomposition was adopted to derive the projections in table 8.

This table summarizes the results of nine single-region cohort-survival population projections. The regions are those delineated in figure 2, and the results correspond to the ones set out earlier in table 2. Thus table 8 may be viewed as the output produced by a particular shrinking of the 'large-scale' population-projection model associated with table 2. The discrepancies between the two sets of results may be attributed largely to the representation of interregional migration as net migration in the decomposed model.

Table 8 reveals that the representation of internal migration as a net flow can introduce serious errors into the process of projecting population. Net migration is defined with respect to the particular regional population being projected. If that population is currently experiencing an excess of in-migrants over out-migrants, this feature will be built in as part of the projection process, and its effects will multiply and increase cumulatively over time. The converse applies, of course, to regions experiencing net out-migration. In short, regional populations with a positive net migration rate are likely to be overprojected and those with a negative net migration rate are likely to be underprojected. The projections in table 8 support this argument.

Table 7. Multiregional projections to stability: US total population, 1958, nine biregional projections.

Projections and parameters of stable growth	Division of residence									Total
	(1) New England	(2) Middle Atlantic	(3) East North Central	(4) West North Central	(5) South Atlantic	(6) East South Central	(7) West South Central	(8) Mountain	(9) Pacific	
K (1958)	9911000	33181000	35763000	15114000	24749000	11769000	16177000	6349000	19141000	172154000
% (1958)	0·0576	0·1927	0·2077	0·0878	0·1438	0·0684	0·0940	0·0369	0·1112	1·0000
K (2008)	22420802	60240640	83052968	31136660	70878872	24837796	40472448	22355426	73141824	428537436
% (2008)	0·0523	0·1406	0·1938	0·0727	0·1654	0·0580	0·0944	0·0522	0·1707	1·0000
r (∞)	0·02157	0·02181	0·02157	0·02154	0·02155	0·02155	0·02157	0·02162	0·02159	–
% (∞)	0·0513	0·1070	0·1890	0·0663	0·1737	0·0513	0·0933	0·0565	0·2118	1·0000

Table 8. Multiregional projections to stability: US total population, 1958, nine single-region decompositions with net migration.

Projections and parameters of stable growth	Division of residence									Total
	(1) New England	(2) Middle Atlantic	(3) East North Central	(4) West North Central	(5) South Atlantic	(6) East South Central	(7) West South Central	(8) Mountain	(9) Pacific	
K (1958)	9911000	33181000	35763000	15114000	24749000	11769000	16177000	6349000	19141000	172154000
% (1958)	0·0576	0·1927	0·2077	0·0878	0·1438	0·0684	0·0940	0·0369	0·1112	1·0000
K (2008)	21361806	54784164	80574344	27888196	72708288	21538842	38569232	27877196	105479992	450782060
% (2008)	0·0474	0·1215	0·1787	0·0619	0·1613	0·0478	0·0856	0·0618	0·2340	1·0000
r (∞)	0·02027	0·01451	0·02049	0·01638	0·02379	0·01400	0·02034	0·03207	0·03907	–
% (∞)	0·0543	0·1856	0·2025	0·0861	0·1549	0·0750	0·0976	0·0393	0·1047	1·0000

Only the populations of the three census divisions that experienced a positive net migration in 1958 are overprojected in the year 2008 (that is, the South Atlantic, the Mountain, and the Pacific divisions); the populations of the remaining six census divisions are underprojected.

The original nine-region population-projection model and its complete single-region decomposition represent opposite extremes of the decomposition spectrum. A large number of alternatives lie in between, two of which appear in figure 5.

This figure describes two complete decompositions of the nine-region population system. Both decompositions reflect the particular structure of interregional migration levels described in table 1, and both were defined by an essentially arbitrary decision to delete interregional linkages that exhibited migration levels below eight percent. Since in both cases this procedure still did not produce a complete decomposition, four additional migration levels (those lying outside of the block-diagonal submatrices in figure 5) were also deleted in each decomposition.

Figure 5(a) illustrates a decomposition of the nine-region population model into three smaller multiregional models containing two, four, and three regions, respectively. Internal migration is treated as a place-to-place flow among regions within each diagonal block and as a net flow elsewhere. Thus we have here an example of compensated tearing in which the conceptual approaches at both extremes of the decomposition spectrum are represented. Table 9 summarizes the multiregional population projections produced by this particular model.

Figure 5(b) depicts an alternative decomposition. In this instance, a permutation of the rows and columns of the migration-level matrix and a decision to delete a different set of four migration levels yield a different connectivity structure and associated decomposition. This decomposition partitions the nine-region system into three, three-region subsystems, and results in the projections set out in table 10.

The two alternative decompositions both overproject the total US population in 2008. The individual regional shares of this total population follow the general pattern exhibited by the single-region decomposition of table 8. That is, regional populations that experienced positive net migration in 1958 are accorded a larger regional share than warranted and vice versa. This pattern arises from the particular method of compensated tearing used in the projections, namely, compensation by means of net migration, and reflects the same biases that were found in the single-region decomposition.

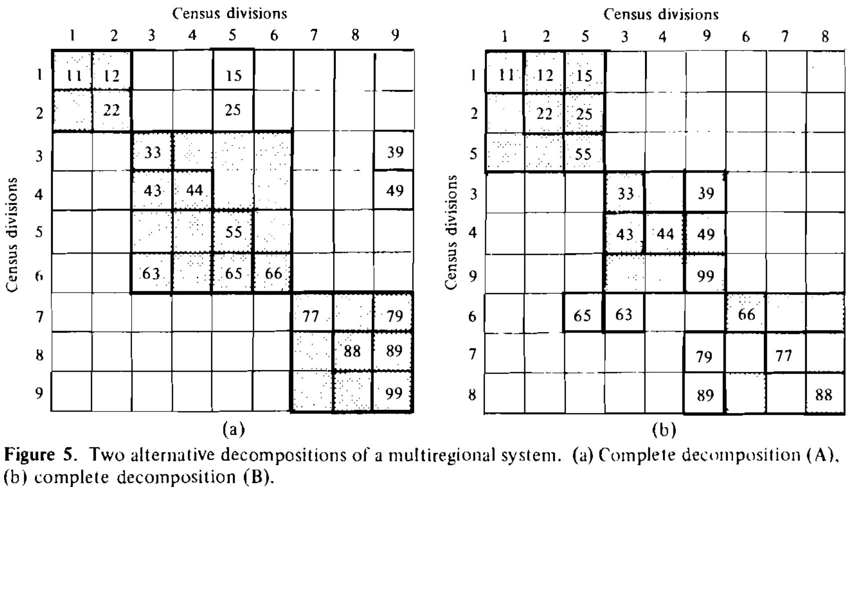

Figure 5. Two alternative decompositions of a multiregional system. (a) Complete decomposition (A), (b) complete decomposition (B).

Table 9. Multiregional projections to stability: US total population, 1958, decomposition A.

Projections and parameters of stable growth	Division of residence									Total
	(1) New England	(2) Middle Atlantic	(3) East North Central	(4) West North Central	(5) South Atlantic	(6) East South Central	(7) West South Central	(8) Mountain	(9) Pacific	
K (1958)	9911000	33181000	35763000	15114000	24749000	11769000	16177000	6349000	19141000	172154000
% (1958)	0·0576	0·1927	0·2077	0·0878	0·1438	0·0684	0·0940	0·0369	0·1112	1·0000
K (2008)	20818862	55406756	79776664	28969902	69440440	23452330	42158288	27528572	93899880	441451694
% (2008)	0·0472	0·1255	0·1807	0·0656	0·1573	0·0531	0·0955	0·0624	0·2127	1·0000
r (∞)	0·01664	0·01664	0·02026	0·02026	0·02026	0·02026	0·03289	0·03289	0·03289	–
% (∞)	0·0979	0·1299	0·2036	0·0588	0·2301	0·0544	0·0312	0·0372	0·1570	1·0000

Table 10. Multiregional projections to stability: US total population, 1958, decomposition B.

Projections and parameters of stable growth	Division of residence									Total
	(1) New England	(2) Middle Atlantic	(3) East North Central	(4) West North Central	(5) South Atlantic	(6) East South Central	(7) West South Central	(8) Mountain	(9) Pacific	
K (1958)	9911000	33181000	35763000	15114000	24749000	11769000	16177000	6349000	19141000	172154000
% (1958)	0·0576	0·1927	0·2077	0·0878	0·1438	0·0684	0·0940	0·0369	0·1112	1·0000
K (2008)	21162692	57420652	82082112	30588244	69149768	22266124	39261660	25469752	87833784	435234788
% (2008)	0·0486	0·1319	0·1886	0·0703	0·1589	0·0512	0·0902	0·0585	0·2018	1·0000
r (∞)	0·02018	0·02018	0·02900	0·02900	0·02018	0·02555	0·02555	0·02555	0·02900	–
% (∞)	0·0637	0·1385	0·0566	0·0298	0·2924	0·0111	0·0481	0·0943	0·2656	1·0000

Another contributor to the discrepancies between the results of the two decomposed models and those of the original model is the insufficiently weak degree of connectivity between the various sets of multiregional subsystems. Recall that, for illustrative purposes, we arbitrarily deleted internal migration flows associated with migration levels below eight percent. It is likely that this is much too high a value for a threshold level, and its adoption undoubtedly contributed something to the overall projection error. That contribution, however, is surely small compared to the one introduced by the representation of internal migration as a net flow. Both sources of error are, of course, interrelated. The level of compensation which is required in the form of net migration is intimately related to the amount of net migration which is to be treated in that way, and this amount in turn depends on the volume of migration that falls below the threshold level.

Aggregation and decomposition techniques are not mutually exclusive methods of shrinking a large-scale population model. They can, of course, be combined in various ways to reduce the dimensionality of such a model without incurring a major sacrifice in accuracy or in the level of detail in the process. We now turn to an examination of one of the more obvious ways in which they may be combined and compare its empirical performance with that of an equally obvious alternative.

4 Aggregation and decomposition combined

The idea that it might be useful to model different parts of a large system at different levels of detail received one of its first formal mathematical treatments two decades ago in a seminal paper read by Herbert Simon and Albert Ando at the meeting of the Econometric Society in December of 1956 and subsequently published in *Econometrica* five years later (Simon and Ando, 1961)[1]. The essence of their basic argument is neatly captured by the following physical illustration:

> "Consider a building whose outside walls provide perfect thermal insulation from the environment. The building is divided into a large number of rooms, the walls between them being good, but not perfect, insulators. Each room is divided into a number of offices by partitions. The partitions are poor insulators. A thermometer hangs in each of the offices. Suppose that at time t_0 the various offices within the building are in a state of thermal disequilibrium—there is a wide variation in temperature from office to office and from room to room. When we take new temperature readings at time t_1, several hours after t_0, what will we find? At t_1 there will be very little variation in temperature among the offices within each single room, but there may still be large temperature variations *among* rooms. When we take readings again at time t_2, several days after t_1, we find an almost uniform temperature throughout the building; the temperature differences among rooms have virtually disappeared.
>
> A temperature equilibrium within each room will be reached rather rapidly, while a temperature equilibrium *among* rooms will be reached only slowly, ... as long as we are not interested in the rapid fluctuations in temperature among offices in the same room, we can learn all we want to know about the dynamics of this system by placing a single thermometer in each room—it is unnecessary to place a thermometer in each office," (Simon and Ando, 1961, pages 70–71).

4.1 *The Simon–Ando theorem*

Recognizing that complete decomposability is relatively rare in socioeconomic systems, Simon and Ando (1961) examine the behavior of linear dynamic systems with 'nearly' completely decomposable subsystems. They show that, in the short-run,

(1) A recent revival of interest in this fundamental idea has produced several interesting articles, one of which specifically suggests an application to migration modeling (Batty and Masser, 1975).

such systems behave almost as though they were in fact completely decomposable and that, in the middle run, their behavior can be studied by consolidating the variables of each subsystem into a single variable and ignoring the interrelationships within each subsystem[2].

The crux of the Simon-Ando theorem is the assertion that the equilibrium of a nearly completely decomposable dynamic linear system may be viewed as a composite growth process which evolves in three temporal phases. During the first phase, the variables in each subsystem arrive at equilibrium positions determined by the completely decomposed system. After a longer time period the system enters its second phase, at which point the variables of each subsystem, maintaining their proportional relationships, move together *as a block* toward equilibrium values established by the third phase of the growth process. In this final phase all variables approach the rate of growth defined by the largest characteristic root of the matrix associated with the original nearly completely decomposable system.

The Simon-Ando theorem suggests a shrinking procedure for large-scale population-projection models that combines aggregation and decomposition in a particularly appealing way. One begins by partitioning the large multiregional system-projection model into smaller submodels in a way that effectively exploits any weak interdependencies revealed by indices such as migration levels. The growth of the original multiregional system may then be projected by appropriately combining (a), the results of disaggregated intrasubsystem projections in which *within*-subsystem interactions are represented at a relatively fine level of detail, with (b), the results of aggregate intersubsystem projections in which the *between*-subsystem interactions are modeled at a relatively coarse level of detail. For example, within each multiregional subsystem, the projection model could focus on the full age composition of every regional population and examine its evolution over time; between each multiregional subsystem the projection model would suppress the regional age compositions and would deal only with total populations. In the short run, the within-subsystem interactions would dominate the behavior of the system; in the long run, the between-subsystem interactions would become increasingly important and, ultimately, would determine the behavior of the entire system.

4.2 *A numerical illustration*

The above discussion can be illuminated with the aid of a simple numerical example drawn from the Simon and Ando paper. Recall the four-region numerical illustration in section 1.3, and assume that the projection matrix of that multiregional system is now taken to be the nearly completely decomposable matrix, say

$$\mathbf{G} = \left[\begin{array}{cc:cc} 0\cdot9700 & 0\cdot0200 & 0 & 0\cdot0002 \\ 0\cdot0295 & 0\cdot9800 & 0 & 0\cdot0002 \\ \hdashline 0\cdot0005 & 0 & 0\cdot9600 & 0\cdot0396 \\ 0 & 0 & 0\cdot0400 & 0\cdot9600 \end{array}\right] . \tag{10}$$

Let the corresponding completely decomposable matrix be

$$\mathbf{G}_\mathrm{d} = \left[\begin{array}{cc:cc} 0\cdot9700 & 0\cdot0200 & 0 & 0 \\ 0\cdot0300 & 0\cdot9800 & 0 & 0 \\ \hdashline 0 & 0 & 0\cdot9600 & 0\cdot0400 \\ 0 & 0 & 0\cdot0400 & 0\cdot9600 \end{array}\right] = \left[\begin{array}{c:c} \mathbf{G}_1 & \mathbf{0} \\ \hdashline \mathbf{0} & \mathbf{G}_2 \end{array}\right] . \tag{11}$$

[2] In a subsequent paper, Ando and Fisher (1963) extend the Simon-Ando theorem to nearly block-triangular (that is, nearly partially decomposable) linear systems. Although I do not consider such systems in the rest of this paper, it should be clear that my exposition could be appropriately expanded to cover this more general case of near decomposability.

Note that G_1 is the disaggregated projection matrix for the North-South subsystem, and G_2 is the disaggregated projection matrix for the East-West subsystem[3]. The original projection matrix G may be consolidated to give the aggregated projection matrix needed for modeling the interrelated growth of the following two subsystems[4]:

$$\hat{G} = CGD = \begin{bmatrix} 0\cdot9998 & 0\cdot0002 \\ 0\cdot0002 & 0\cdot9998 \end{bmatrix}, \tag{12}$$

where

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 0\cdot4 & 0 \\ 0\cdot6 & 0 \\ 0 & 0\cdot5 \\ 0 & 0\cdot5 \end{bmatrix}.$$

The long-run behavior of this particular system can be studied by examining the behavior of the elements of the matrix G as it is raised to higher powers. It is a simple exercise on a digital computer to show that

$$G^{128} = \left[\begin{array}{cc:cc} 0\cdot390089 & 0\cdot392503 & 0\cdot009465 & 0\cdot011385 \\ 0\cdot579037 & 0\cdot586246 & 0\cdot013138 & 0\cdot015999 \\ \hdashline 0\cdot016631 & 0\cdot011831 & 0\cdot487509 & 0\cdot485107 \\ 0\cdot014244 & 0\cdot009419 & 0\cdot489888 & 0\cdot487509 \end{array}\right], \tag{13}$$

and that

$$G^{(128)^2} = G^{128} \times G^{128} = \left[\begin{array}{cc:cc} 0\cdot200776 & 0\cdot200782 & 0\cdot200222 & 0\cdot200225 \\ 0\cdot298656 & 0\cdot298664 & 0\cdot297829 & 0\cdot297833 \\ \hdashline 0\cdot250286 & 0\cdot250279 & 0\cdot250973 & 0\cdot250970 \\ 0\cdot250282 & 0\cdot250275 & 0\cdot250976 & 0\cdot250973 \end{array}\right]. \tag{14}$$

Observe that the elements in the *diagonal submatrices* maintain the same proportion over the rows, independently of the columns within each submatrix, while moving toward their equilibrium values. That is, in both of these equations the proportional *within*-subsystem allocation is one of 0·4 to 0·6 in the upper-diagonal submatrix and one of 0·5 to 0·5 in the lower-diagonal submatrix. The same within-subsystem allocations are also defined by the completely decomposable system, that is,

$$\begin{bmatrix} 0\cdot4 \\ 0\cdot6 \end{bmatrix} = \begin{bmatrix} 0\cdot9700 & 0\cdot0200 \\ 0\cdot0300 & 0\cdot9800 \end{bmatrix} \begin{bmatrix} 0\cdot4 \\ 0\cdot6 \end{bmatrix}, \tag{15}$$

and

$$\begin{bmatrix} 0\cdot5 \\ 0\cdot5 \end{bmatrix} = \begin{bmatrix} 0\cdot9600 & 0\cdot0400 \\ 0\cdot0400 & 0\cdot9600 \end{bmatrix} \begin{bmatrix} 0\cdot5 \\ 0\cdot5 \end{bmatrix}. \tag{16}$$

(3) Note that in Simon and Ando's numerical illustration the compensation for tearing is introduced in the off-diagonal elements. For example, the element 0·0005 in equation (10) is added to 0·0295 to give the 0·0300 in equation (11). Our compensation procedure would instead have added it to 0·9700.

(4) The weights in the D matrix are those used by Simon and Ando. They are the proportions defined by the characteristic vector associated with the largest characteristic root of the G matrix. In most applications it is much more convenient to use the proportions defined by the observed population distribution, because such a procedure avoids the necessity of calculating the largest characteristic root and its associated characteristic vector. A compromise solution is to use the roots and vectors of the individual submatrices, which, in this particular illustration, leads to practically the same numerical results. (Note that the largest characteristic root of every G matrix in this illustration is unity.)

The *between*-subsystem allocations are defined by the characteristic vector associated with the largest characteristic root of **G** in equation (12) and may be shown to be equal to each other:

$$\begin{bmatrix} 0{\cdot}5 \\ 0{\cdot}5 \end{bmatrix} = \begin{bmatrix} 0{\cdot}9998 & 0{\cdot}0002 \\ 0{\cdot}0002 & 0{\cdot}9998 \end{bmatrix} \begin{bmatrix} 0{\cdot}5 \\ 0{\cdot}5 \end{bmatrix} . \tag{17}$$

Combining the information on within-subsystem allocations with that of between-subsystem allocations, we define the completely decomposable approximation of equation (14) to be the matrix

$$\left[\begin{array}{cc:cc} 0{\cdot}20 & 0{\cdot}20 & 0{\cdot}20 & 0{\cdot}20 \\ 0{\cdot}30 & 0{\cdot}30 & 0{\cdot}30 & 0{\cdot}30 \\ \hdashline 0{\cdot}25 & 0{\cdot}25 & 0{\cdot}25 & 0{\cdot}25 \\ 0{\cdot}25 & 0{\cdot}25 & 0{\cdot}25 & 0{\cdot}25 \end{array}\right] . \tag{18}$$

Note that the column proportions in equation (18) indicate that at equilibrium (that is, during stable growth), the multiregional population of 480 individuals will be distributed among the four regions according to the following allocations: 96 individuals in the North, 144 in the South, 120 in the East, and another 120 in the West.

4.3 *Simple shrinking by aggregation and decomposition*

The Simon and Ando theorem suggests the following simple method for shrinking large-scale population-projection models. One begins by partitioning a multiregional system into its constituent single regions and projecting their growth and change as if they were independent, closed population subsystems undisturbed by migration. The first stage, therefore, corresponds to a single-region decomposition with *zero* net migration. We then suppress all age-specific details and project the multiregional population by using a components-of-change model. The results of the latter stage determine the total multiregional population and its spatial distribution; the results of the first stage define the individual regional age compositions. In this way, *within*-subsystem interactions (that is, changes in age structure) are modeled at a fine level of detail, whereas *between*-subsystem interactions (that is, changes in spatial structure) are modeled at a course level of detail. If the original multiregional system is sufficiently close to being nearly decomposable, the approximate (two-stage) projection should produce a reasonably accurate multiregional population projection.

The shrinking procedure described above may be applied to the 'large-scale' nine-region population model of table 2. Table 11 sets out the principal results generated by such a shrinking of the original model. The growth of the total population and its spatial allocation are taken from the projection in table 5; the individual regional age compositions (consolidated into three age groups for ease of presentation) were obtained by recomputing the single-region projections of table 8, with net migration set equal to zero. The combined results indicate that regional age compositions and regional shares are projected moderately well, but that the total multiregional population is seriously underprojected. (The latter is no surprise since it already was observed and discussed in connection with table 5.)

In applying the above shrinking procedure I adopted the regional age compositions of the single-region (no-migration) projections and the regional shares of the components-of-change projection. For the total multiregional population the level projected by the latter (364608685) was chosen; it would have been much better to have used that of the former (419173278). In the remainder of this paper, therefore

Table 11. Multiregional projections to stability: US total population, 1958, nine single-region (no-migration) decompositions with components-of-change aggregation.

Projections and parameters of stable growth	Division of residence									Total
	(1) New England	(2) Middle Atlantic	(3) East North Central	(4) West North Central	(5) South Atlantic	(6) East South Central	(7) West South Central	(8) Mountain	(9) Pacific	
K (1958)	9911000	33181000	35763000	15114000	24749000	11769000	16177000	6349000	19141000	172154000
% (1958)	0·0576	0·1927	0·2077	0·0878	0·1438	0·0684	0·0940	0·0369	0·1112	1·0000
K (2008)	17927349	53159821	68434148	25822107	62159432	21199129	35493951	19076175	61336572	364608685
% (2008)	0·0492	0·1458	0·1877	0·0708	0·1705	0·0581	0·0973	0·0523	0·1682	1·0000
r (∞)	–	–	–	–	0·01554	–	–	–	–	0·01554
% (∞)	0·0360	0·0897	0·1516	0·0631	0·1748	0·0490	0·1107	0·0717	0·2533	1·0000
2008: approximate projection										
0–14	0·3544	0·3378	0·3678	0·3690	0·3546	0·3655	0·3742	0·3728	0·3560	–
15–64	0·5889	0·6004	0·5778	0·5751	0·5879	0·5814	0·5725	0·5740	0·5836	–
65+	0·0567	0·0618	0·0544	0·0559	0·0575	0·0532	0·0533	0·0532	0·0604	–
2008: original projection										
0–14	0·3560	0·3367	0·3642	0·3664	0·3513	0·3621	0·3709	0·3740	0·3587	0·3581
15–64	0·5873	0·5988	0·5802	0·5713	0·5840	0·5765	0·5696	0·5719	0·5865	0·5825
65+	0·0567	0·0644	0·0557	0·0623	0·0647	0·0614	0·0595	0·0541	0·0548	0·0594

Table 12. Multiregional projections to stability: US total population, 1958, decomposition B with biregional aggregation.

Projections and parameters of stable growth	Division of residence									Total
	(1) New England	(2) Middle Atlantic	(3) East North Central	(4) West North Central	(5) South Atlantic	(6) East South Central	(7) West South Central	(8) Mountain	(9) Pacific	
K (1958)	9911000	33181000	35763000	15114000	24749000	11769000	16177000	6349000	19141000	172154000
% (1958)	0·0576	0·1927	0·2077	0·0878	0·1438	0·0684	0·0940	0·0369	0·1112	1·0000
K (2008)	21737424	59648676	81625712	30941760	70046704	24580972	40261452	22187936	72139368	423170004
% (2008)	0·0514	0·1410	0·1929	0·0731	0·1655	0·0581	0·0951	0·0524	0·1705	1·0000
r (∞)	0·02186	0·02186	0·02160	0·02160	0·02186	0·02160	0·02160	0·02160	0·02160	–
% (∞)	0·0451	0·1060	0·1870	0·0696	0·1684	0·0523	0·0969	0·0590	0·2158	1·0000
2008: approximate projection										
0–14	0·3560	0·3369	0·3647	0·3665	0·3517	0·3627	0·3706	0·3730	0·3586	–
15–64	0·5874	0·5990	0·5803	0·5713	0·5843	0·5766	0·5697	0·5719	0·5864	–
65+	0·0566	0·0641	0·0550	0·0623	0·0640	0·0607	0·0597	0·0551	0·0550	–
2008: original projection										
0–14	0·3560	0·3367	0·3642	0·3664	0·3513	0·3621	0·3709	0·3740	0·3587	0·3581
15–64	0·5873	0·5988	0·5802	0·5713	0·5840	0·5765	0·5696	0·5719	0·5865	0·5825
65+	0·0567	0·0644	0·0557	0·0623	0·0647	0·0614	0·0595	0·0541	0·0548	0·0594

Table 13. Alternative projections of the total population and its regional distribution in the year 2008: US total population, 1958.

Alternative[a]	Divisional shares of total population (2008)									Total	Total population
	(1) New England	(2) Middle Atlantic	(3) East North Central	(4) West North Central	(5) South Atlantic	(6) East South Central	(7) West South Central	(8) Mountain	(9) Pacific		
Table 2	0·0513	0·1403	0·1914	0·0739	0·1619	0·0578	0·0959	0·0541	0·1734	1·0000	421862143
Table 7	0·0523	0·1406	0·1938	0·0727	0·1654	0·0580	0·0944	0·0522	0·1707	1·0000	428537436
Table 8	0·0474	0·1215	0·1787	0·0619	0·1613	0·0478	0·0856	0·0618	0·2340	1·0000	450782060
Table 9	0·0472	0·1255	0·1807	0·0656	0·1573	0·0531	0·0955	0·0624	0·2127	1·0000	441451694
Table 10	0·0486	0·1319	0·1886	0·0703	0·1589	0·0512	0·0902	0·0585	0·2018	1·0000	435234788
Table 11	0·0492	0·1458	0·1877	0·0708	0·1705	0·0581	0·0973	0·0523	0·1682	1·0000	(419173278)
Table 12	0·0514	0·1410	0·1929	0·0731	0·1655	0·0581	0·0951	0·0524	0·1705	1·0000	423170004

[a] Table 2: original unconsolidated model.
Table 7: biregional aggregations.
Table 8: single-region decompositions (with net migration).
Table 9: decomposition A.
Table 10: decomposition B.
Table 11: single-region (no-migration) decompositions with components-of-change aggregation (cohort-components shrinking).
Table 12: decomposition B with biregional aggregation (cohort-biregional shrinking).

I shall modify the shrinking procedure accordingly and define the resulting modified version to be *the cohort-components method of simple shrinking.* This method adopts the regional age compositions and total multiregional population projected by a collection of single-region cohort-survival models that ignore migration, and then spatially allocates this total population according to the regional shares projected by a components-of-change model.

The accuracy with which the biregionally aggregated models of table 7 approximated the original projection in table 2 suggests another method of simple shrinking, one which I shall call the *cohort-biregional method of simple shrinking.* In this method the tearing occasioned by complete decompositions of the kind defined in figure 5 is compensated not by net migration but by biregional aggregation. Specifically, each multiregional subsystem is augmented by an additional 'rest-of-the-world' region which serves as the destination of all migration out of the subsystem and as the source of all migration into the subsystem. Table 12 presents the results produced by the application of such a method of shrinking to the projection model of table 2. The particular decomposition scheme adopted was that of figure 5(b).

According to table 12, cohort-biregional shrinking is a more accurate method of shrinking than cohort-components shrinking, at least with regard to the particular data set examined in this paper. The former projects regional age compositions that

Table 14. Alternative projections of the age composition of the Pacific division in the year 2008: US total population, 1958.

Age	Alternative[a]						
	Table 2	Table 7	Table 8	Table 9	Table 10	Table 11	Table 12
0-4	0·1352	0·1403	0·1417	0·1389	0·1387	0·1335	0·1351
5-9	0·1182	0·1234	0·1218	0·1202	0·1202	0·1176	0·1182
10-14	0·1053	0·1094	0·1067	0·1058	0·1062	0·1049	0·1053
15-19	0·0955	0·0956	0·0982	0·0968	0·0970	0·0940	0·0954
20-24	0·0861	0·0838	0·0910	0·0889	0·0888	0·0846	0·0860
25-29	0·0768	0·0745	0·0818	0·0798	0·0797	0·0758	0·0768
30-34	0·0686	0·0664	0·0723	0·0710	0·0706	0·0672	0·0686
35-39	0·0603	0·0582	0·0619	0·0615	0·0610	0·0590	0·0603
40-44	0·0523	0·0504	0·0515	0·0521	0·0517	0·0518	0·0523
45-49	0·0462	0·0445	0·0439	0·0448	0·0449	0·0473	0·0463
50-54	0·0395	0·0386	0·0356	0·0371	0·0375	0·0406	0·0395
55-59	0·0339	0·0332	0·0298	0·0314	0·0317	0·0351	0·0339
60-64	0·0272	0·0269	0·0231	0·0247	0·0248	0·0282	0·0273
65-69	0·0190	0·0192	0·0150	0·0166	0·0166	0·0201	0·0191
70-74	0·0136	0·0137	0·0106	0·0118	0·0118	0·0157	0·0137
75-79	0·0102	0·0101	0·0077	0·0087	0·0087	0·0123	0·0102
80-84	0·0067	0·0067	0·0050	0·0056	0·0057	0·0085	0·0068
85+	0·0052	0·0052	0·0022	0·0043	0·0044	0·0039	0·0052
Total	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000
Total population	73166573	73141824	105479992	93899880	87833784	(70515742)	72139368

[a] Table 2: original unconsolidated model.
Table 7: biregional aggregations.
Table 8: single-region decompositions (with net migration).
Table 9: decomposition A.
Table 10: decomposition B.
Table 11: single-region (no-migration) decompositions with components-of-change aggregation (cohort-components shrinking).
Table 12: decomposition B with biregional aggregation (cohort-biregional shrinking).

are virtually identical to those projected by the original large-scale model. The total multiregional population and its regional distribution are somewhat less accurately approximated, but nevertheless are, in general, closer approximations than those advanced by the cohort-components method of table 11. Finally, the cohort-biregional shrinking can be more readily transformed into a method for approximating the intrinsic rate of growth and related stable growth parameters of the multiregional population system.

The cohort-components and the cohort-biregional methods of simple shrinking appear to be the most desirable shrinking methods among those examined in this paper. Table 13 indicates that they are the most accurate in projecting the total multiregional population. With the possible exception of the less efficient biregional aggregation method of shrinking, they also appear to be the most accurate in projecting the regional shares and age compositions of the multiregional population. The accuracy with which the cohort-biregional method projects regional age compositions is especially remarkable and is well illustrated in table 14, which presents the alternative projections of the age composition of the Pacific division by way of example.

5 Conclusion

Imagine a demographer faced with the problem of projecting, in a consistent manner and in age-specific detail, the future populations of the 265 Standard Metropolitan Statistical Areas (SMSAs) of the contemporary United States. Such a large-scale multiregional cohort-survival model is beyond the data processing capabilities of his digital computer and, moreover, would be needlessly cumbersome in the light of certain observed weak connectivities between several subsystems of SMSAs. What findings and what approaches does this paper present that might be useful to him as he proceeds to design a population-projection model?

The principal findings of this paper revolve around the various ways of shrinking a large-scale population-projection model and may be summarized as follows:

(1) Components-of-change models are unreliable generators of middle- and long-run projections of population totals, but seem to be reasonably accurate in projecting regional shares of such totals (see table 5).

(2) Biregional aggregation is an effective and relatively efficient method for shrinking projection models of a small to modest scale and may be used in situations where only gross out-migration and in-migration data are available for each region (that is, without reference to place of destination) (see table 7).

(3) Modeling internal migration as a net flow can introduce serious biases into the projection process (see table 8). (Such biases are inevitably introduced in treating immigration and emigration as a net flow, but in many countries they tend to be relatively small.)

(4) Effective decompositions are not unique and may be difficult to identify in large systems (see tables 9 and 10). Consequently algorithms such as those discussed in Tewarson (1973) need to be adapted and applied in searches for decompositions that are in some sense 'optimal'.

(5) The simple cohort-components method of shrinking is a reasonably accurate procedure, is easy to apply, and has the distinct advantage of not requiring *age-specific* migration flow data for its implementation (see tables 11, 13, and 14). It is, therefore, the obvious choice for shrinking large-scale projection models of population systems for which such data are either unavailable or too costly to obtain.

(6) The simple cohort-biregional method of shrinking appears to be very accurate and seems to be an effective compromise between biregional aggregation and single-region decomposition, combining the best features of each (see tables 12, 13, and 14).

It is especially well suited for shrinking large-scale projection models of population systems that are made up of several weakly connected subsystems.

The two principal approaches for shrinking examined in this paper have been aggregation and decomposition. They have been combined to define two fundamental methods of shrinking, both of which reflect the proposition that strongly interconnected regions should be modeled as separate closed subsystems by using the cohort-survival model. The two methods differ in the way that they connect these subsystems together. The *cohort-components* method uses a components-of-change model to establish such connections; the *cohort-biregional* method relies instead on a residual 'rest-of-the-world' region. Each alternative differs with respect to data inputs and outputs, computational efficiencies, and accuracy of projections. Yet little can be said about the trade-offs between these attributes in general because they depend so much on the specifics of each empirical situation. The particular connectivity structure of an observed multiregional population, the particular data availability with regard to age-specific migration flows, the particular purposes for which the projections are being generated, all are important considerations in a rational choice between the two alternatives. Yet such considerations will vary from one situation to another, and will combine in different ways to suggest the superiority of one alternative over the other. In consequence, each particular situation requires a specific evaluation.

This paper represents a first, and therefore preliminary, examination of shrinking large-scale population-projection models. Consequently it only outlines the fundamental problem and identifies what appear to be fruitful means for dealing with it. Much more remains to be done. For example, it is likely that further research could establish conditions for 'perfect decomposition' akin to those already established for perfect aggregation (Rogers, 1969; 1975). The relative computational efficiencies of the two alternative methods in shrinking certain prototype connectivity structures could be examined profitably. More complex hierarchical extensions of the simple shrinking methods could be investigated, such as the extension of the simple cohort-components method to include several multiregional (no-migration) cohort-survival submodels, and the disaggregation of the 'rest-of-the-world' region in the simple cohort-biregional method. Efficient algorithms for approximating the intrinsic rate of growth and other related stable growth measures with the use of shrinking methods appear to be another promising direction for research. The assumption of no interregional differentials in fertility and mortality has been used before to shrink a large-scale population-projection model and deserves to be reconsidered in the context of this paper (Rogers, 1968, chapter 3). Finally, the possibility of shrinking data input requirements by means of 'model' schedules also merits careful examination (United Nations, 1967; Rogers, 1975, chapter 6).

Acknowledgements. I am indebted to Frans Willekens for performing the computations required for this paper and to Richard Raquillet for assembling the empirical data. Andrew Whinston first brought the Simon and Ando paper to my attention.

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Demometrics of Migration and Settlement

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Introduction

The 'population problem' in most parts of the world has two distinct dimensions: growth and spatial distribution. Concern about rapid population growth has focused attention on fertility reduction and has fostered family planning programs in dozens of countries. The issue of population distribution, on the other hand, has only recently received serious consideration. A notable example appears in the work of the US Commission on Population Growth and the American Future, which commissioned papers that directly addressed issues and problems of internal migration and human settlement:

> "Major national attention and the Commission's primary focus has been on national population growth. But national growth implies local growth as additional population is distributed in the rural areas, small towns, cities and suburbs across the country. And choices we make about national population growth cannot help but have important meaning for local areas....
>
> Where people move inevitably affects the distribution of the population and the growth of local areas. As a result, *any national distribution policy will, to some degree, try to intervene in the migration process* by encouraging people to move to one place rather than another or not to move at all" (US Commission on Population Growth and the American Future, 1972, pages xiv–xv, emphasis added).

Despite a general recognition that migration processes and settlement patterns are intimately related and merit serious study, one nevertheless finds that the dynamics of their interrelationships are not at all well understood. An important reason for this lack of understanding is that demographers have in the past accorded migration a status subservient to fertility and mortality and have generally neglected the spatial dimension of population growth. Thus, whereas problems of fertility and mortality long ago stimulated a rich and scholarly literature, studies of migration have only recently begun to flourish. In consequence, one finds today a rather large and growing body of scholarly work on migration awaiting a systematic synthesis, for example the recent bibliographies of Greenwood (1975), Price and Sikes (1975), and Shaw (1975). The contributions of sociologists in identifying migration differentials (the 'who' of migration), of geographers in analyzing directional migration streams (the 'where' of migration), and of economists in examining the determinants and consequences of geographical mobility (the 'why' and 'so what' of migration) still have not been molded into a unified and general theory of internal migration.

Out of the recently growing literature on migration, scholars at the International Institute for Applied Systems Analysis have identified and isolated four related research subtasks that are of particular relevance to IIASA's long-term research interests in human settlement systems. They are: the study of spatial population *dynamics*; the elaboration of a new research area called *demometrics* and its application to migration analysis and spatial population forecasting; the analysis and design of migration and settlement *policy*; and a *comparative study* of national migration and settlement patterns and policies.

This paper focuses on the second of the four subtasks—demometrics—and examines the fundamental role that *theory* plays in the development of quantitative models of spatial population growth and distribution.

Theorizing about theory building: the theoretical transition

The new social science

The past two decades have seen the emergence of a profound transformation in the ways in which social scientists have come to deal with data, theory, and quantitative modelling. This transformation has been variously referred to as the Quantitative Revolution or the Systems Analysis Approach, and some of its effects have recently become manifested in the explosion of new research areas bearing the term *metrics*, for example, politimetrics (Gurr, 1972), planometrics (Zauberman, 1967), and cliometrics (Holden, 1974). Whether these new areas ever achieve the respectability currently accorded to econometrics, biometrics, or psychometrics, however, remains to be seen.

An important contributing factor to the recent quest for theoretical rigor and empirical quantification in the social sciences has undoubtedly been the development of large-scale computational facilities which have made the new approaches feasible. Another contributor to this transformation has been the postwar growth of interdisciplinary fields such as operations research, regional science, public policy analysis, urban planning, and management science. Finally, as today's social and environmental problems have dramatically moved to the forefront of public concern, it has become increasingly evident that energy, natural and human resources, economic development, and the quality of life are interrelated components that need to be dealt with holistically, with a proper recognition of their system-wide impacts. This has fostered a multidimensional mode of analysis commonly referred to as *interdisciplinary* research.

Interdisciplinary and metadisciplinary research

Because social and environmental problems do not fall neatly within the boundaries of traditional academic disciplines, it is not surprising that interdisciplinary collaboration is often held to be the appropriate approach for analyzing such problems. Since any nontrivial social or environmental problem will usually involve physical, social, ecological, economic, and political aspects,

> "what is more sensible than to say, 'Let us then get an expert in each of these fields, have the experts meet, present their special viewpoints and competencies, weigh the pros and cons as reasonable men, and as a body arrive at recommendations which are influenced by each of the contributing disciplines on each of the component problems'?" (Alonso, 1971, page 169).

William Alonso argues convincingly that what at first glance seems like plain common sense is in fact an inappropriate method for assembling a research team to study most social and environmental problems. He identifies four principal stumbling blocks to effective interdisciplinary research. *First,* a scientist is not a standard product capable of delivering his discipline's viewpoint on every subject. For example, the particular problem at hand may require the expertise of a benefit-cost economist, but the economist on the team might be an economic historian. *Second,* the system of institutional incentives and rewards in academia makes it very likely that the scientist will be either a mediocrity, a seniority, or an eccentric, because there are few scholars at the top of their field who are willing to take the time to contribute to an ad hoc interdisciplinary effort. *Third,* the indivisibility of scientific inputs to a team made up of representatives from each field contributes a certain degree of inflexibility to projects that may call for more work from one discipline and less from the others at different points in the project's lifetime. The typical consequence of this indivisibility is that one man is overworked while others may be idle. *Finally,* different intellectual species tend to use words differently and to attribute different degrees of importance to the various components of an overall study. They also hold different views about their own competencies and interests and those of their collaborators.

Alonso's alternative to the *interdisciplinary* team is a group of *metadisciplinary* individuals who "share a defined range of topics, a body of techniques, and certain standards of validation. They share, to a large degree, a technical language and competence, and they read much of the same literature" (Alonso, 1971, page 171). The principal distinction between interdisciplinary and metadisciplinary teams of scientists is that members of the former are assembled because of their *diversity*, whereas the latter are brought together because of their *commonality*.

Systems simulation modelling

A central element of many applied interdisciplinary research efforts has been the large-scale systems simulation model. Such models seem to be particularly appealing in studies of public investment decisions involving broad social goals, wide external system effects, and long-range planning horizons. Urban highway and mass-transit proposals, land-use plans, public works projects, demographic-economic development programs, environmental impact statements, and urban-renewal analyses all have fostered computer simulation studies. It is argued that, because the complexities inherent in such sociophysical systems almost always produce nonlinear relationships and feedbacks, traditional analytical methods are ineffective and recourse must therefore be made to the vast computational capabilities of the digital computer. And, since the behavior of such systems depends on an intimate interplay of social, economic, political, and engineering considerations, prospects for their proper incorporation are said to be enhanced by an early adoption of a large-scale systems simulation model that interconnects their separate contributions in the forms of linked 'modules' or submodels. The underlying philosophy seems to be that if the behavior of parts is known by disciplinary experts then the behavior of the whole can be understood by appropriately linking the diverse individual components together into a larger (interdisciplinary) ensemble. Although the idea sounds as plausible as the rationale for interdisciplinary collaboration, it is equally flawed.

The fallacy that the unpredictable future behavior of a complex system can be adequately simulated, *without a theory*, by joining submodels that describe the generally predictable behavior of a system's component interacting parts seems to have an expected lifetime of about ten years. Apparently it takes about a decade for those newly engaged in simulation modelling to discover the disappointing truth that without a theory of system behavior there can be no firm conclusions about such behavior. A mere linking of statistically established empirical regularities simply will not realistically simulate system behavior *under changed conditions.*

The theoretical transition

Many of the applied systems analysis problems that have fostered simulation modelling seem to go through a methodological-theoretical progression that is strikingly similar in character—a progression that in this paper will be referred to as *the theoretical transition.* The theoretical transition begins when a few highly motivated scientists become converted to the belief that research efforts in their particular area of concern would benefit from the adoption of the more rigorous and quantified style characteristic of the physical sciences. Mathematical curve-fitting exercises become popular, regression and factor analysis are discovered, and a heightened sense of expectancy fuels the movement. This is the *preconditions stage* of the transition, and it logically leads into the second stage, which is the takeoff stage. At this point the simulation paradigm is discovered and widely adopted, and limitations and extensions follow rapidly. Conferences are held, curricula are suitably modified and expanded, and, occasionally, new journals are established.

Then comes the crash, and disillusionment sets in. The models do not predict well, their costs escalate astronomically as they are continually expanded to serve a wider range of purposes and patched-up to produce 'reasonable' outputs, little is contributed by them to theory building or to policy evaluation, and their magic recedes mirage-like. It is concluded that systems modelling without the overall guidance of a credible theory is fraught with difficulties, and the movement becomes stalled until a new and different set of preconditions for takeoff is satisfied. These new preconditions revolve around the emergence of one or more theoretical paradigms that command the support of a large enough body of scientists for the modelling movement to proceed once again (for example, the Keynesian paradigm in macroeconomic modelling). At this point the movement enters the final stage of the theoretical transition—the *metrics stage*.

Not all applied social systems analysis areas have experienced, or will experience, all of the above stages from preconditions to metrics, but a surprisingly large number of them either have or seem to be in the process of doing so. Two areas in particular offer instructive lessons: land-use simulation modelling and economic-demographic simulation modelling.

Land-use simulation modelling

A land-use simulation model is a mechanism that allocates people and jobs to land located in the subareas or zones that together define a particular study region. The population may be diaggregated by age and income, and employment may be differentiated by industrial sector. The model begins with a base-year description of the system and proceeds to simulate its spatial dynamics by locating increments and decrements of population and employment on the basis of some set of rules such as the gravity law of social physics.

First-generation land-use simulation models were born in North America in the early 1960s and effectively died there before the end of that decade. The preconditions stage occurred in the late 1950s, when pioneering studies such as the Chicago Area Transportation Study (1960) first collected and summarized relatively large quantities of land-use data by means of electronic computing hardware and statistical techniques. Entry into the takeoff stage was signalled by the development and implementation of models such as the Lowry model (Lowry, 1964) and the Chapel Hill model (Donnelly et al, 1964). The crash occurred sometime before 1970, and the movement (at least in the US) then turned to fundamental theory building of the kind exemplified by the works of Beckmann (1969), Mills (1967), Muth (1969), and Wheaton (1974a; 1974b).

A sure sign that the crash stage of the theoretical transition has been reached is an attack on the movement by one of its own. In land-use simulation modelling this occurred in 1973 with the publication of D B Lee's well-known article "Requiem for large-scale models" in which he asserts:

> "these models were begun in the early 1960s and largely abandoned by the end of the 1960s. Considerable effort was expended on them, and a good deal was learned. Contrary to what has often been claimed, what was learned had almost nothing to do with urban spatial structure; the knowledge that was increased was our understanding of model building and its relationship to policy analysis" (D B Lee, 1973, page 163).

Economic-demographic simulation modelling

Although different economic-demographic simulation models exhibit somewhat different peculiarities and emphases, they generally share certain distinct features. In all, the process of population growth and economic development proceeds according to model dynamics that have remained essentially unchanged for over two decades. A population disaggregated by age, sex, location, and other attributes such as income and education, is survived forward in time by the appropriate application of rates that either are assumed to vary in a certain way, or are linked to variations in 'explanatory' socioeconomic variables, some of which are internally generated by the model. Underlying most of the economic submodels is either a Cobb-Douglas production function or a dynamic Leontieff input-output system. Various behavioral relationships are built in, and numerous feedback loops connect the population submodel with the economic submodel and vice versa.

The preconditions stage of economic-demographic simulation modelling occurred in the late 1950s, at which time contributions such as the pioneering 1958 Coale-Hoover study (Coale and Hoover, 1958) provided the fundamental conceptual framework that spawned a host of successors in the 1960s. The takeoff stage was entered in the mid-1960s, when simulation models such as General Electric's TEMPO-I model (TEMPO, 1968) became available. In the early 1970s the movement was in full flower and produced a second generation of much larger models such as TEMPO-II (Brown, 1974), the International Labor Organization models BACHUE-1 and BACHUE-2 (Blandy and Wéry, 1973; Wéry et al, 1974) and the Rogers-Walz West Virgina model (Rogers and Walz, 1973). The crash stage either has already occurred or is due shortly. The preconditions for a second takeoff appear to be forming as economists are establishing a firm intellectual beachhead in population economics, with a particularly spectacular recent spurt of theoretical work on the economics of fertility (Leibenstein, 1974; T P Schultz, 1973; T W Schultz, 1973; 1974) and migration (Bowles, 1970; David, 1974; Todaro, 1969).

A reliable indicator that the crash stage of the theoretical transition has already occurred in economic-demographic simulation modelling is the recent appearance of a sharp critique of the movement. Arguing along lines that in many ways resemble those of D B Lee (1973), Arthur and McNicoll (1975) conclude that large-scale economic-demographic simulation models are of little use to planners because they are

> "...large, structurally inflexible, overdressed in side issues, and beyond proper validation. Largeness and algebraic complexity are not, in themselves, crimes: complex problems call for complex models, and simulation is well suited to such analysis. But the complexity of development problems puts a premium on the relevance and validity of model specifications and the wise use of economic theory. A model in which much is irrelevant to a particular issue, is likely to oversimplify and distort the issue. And a model that is concerned more with its own behavior than with the degree to which its constituent parts capture reality is not a trustworthy guide to policy" (Arthur and McNicoll, 1975, page 262).

A summing up

The birth and development of the 'new social science' in the postwar years has fostered a more rigorous, analytical approach in studies of a wide variety of social and environmental problems. Early interdisciplinary analyses of such problems often produced large and complex computer simulation models which failed to meet most of the goals that brought them into being. Growing recognition of the fundamental importance of theory and a gradual evolution of metadisciplinary scientists have combined to channel research activities away from a simple linking-up of statistically estimated empirical regularities toward a form of theory building that is characterized by its successful marriage of theory, mathematics, and statistical inference. This final stage of the theoretical transition may be called the metrics stage of a field's development, and an obvious example of a social science that is well into this stage is economics. Other fields are at the point of entry; still others are either in the preconditions stage or have long ago entered the takeoff stage. Demography appears to be poised to enter the final stage—one identifiable as *demometrics*.

Demographic theory and demometrics

The resolution of several major population policy issues hinges on the answers that theory can give to fundamental questions about national and regional development. Will the per capita income of a poor country increase faster if its fertility is reduced? Should such a country invest heavily in family planning programs, or can it safely assume that fertility will decline rapidly once the income of its population is raised sufficiently beyond a subsistence level? Are certain major cities becoming excessively large, and should their in-migration rates be reduced? Or are large cities 'engines of development' that generate benefits which increase with city size more rapidly than do costs, and is population decentralization and dispersal an automatic ultimate consequence of affluence and modernization? Is rural-to-urban migration a determinant or a consequence of economic growth? Should countries such as India continue to invest a large proportion of their resources in the industrial sector, or should some of this investment be redirected toward agricultural development programs that might keep more of the rural population in the villages and away from the squatter settlements of major urban centers?

Answers to questions such as these will need to draw on a theory that as yet does not exist. The absence of a convincing theoretical analysis of these questions may explain much of the disagreements that seem increasingly to characterize international debates on such matters, for example, the discussions at the 1974 World Population Conference in Bucharest (World Population Conference, 1975). So long as the relationships between population dynamics and socioeconomic development are shrouded in mystery, policymakers are unlikely to act against their intuition and self-interest in establishing population policies.

Demographic theory

Following the precedent set by economics, the emerging concepts, generalizations, and paradigms that collectively might be viewed as the first bits and pieces of a demographic theory fall into two principal categories: macro and micro. *Macrodemographic theory* examines the behavior of global aggregates, for example, the relationships between various populations defined with reference to age, sex, and location and indicators of industrialization, modernization, and economic well-being. *Microdemographic theory* studies the behavior of decisionmaking units such as the individual or the family.

A starting point in the macrotheory of population is the so-called theory of *demographic transition*, which elaborates the principal theme that in today's developed countries a decline in death rates was followed, after a certain lag, by a decline in birth rates. The search for an explanation that could satisfactorily relate the decline in mortality and fertility to specific socioeconomic changes has fostered a vast literature but little in the way of theory.

Development, wherever it occurs, seems eventually to bring about a lowering of the rate of population growth. This has been the historical record in the developed world, and there are indications that it also is becoming true of certain less developed countries. The transition begins with a decline in mortality, which may be attributed to advances in medicine, public health measures, and improved living conditions. It ends with a delayed but matching decline in fertility, the reasons for which are the subject of some dispute among economists and sociologists. The dela produces a sudden spurt in population growth.

A geographer recently asserted that demography has only two theoretical paradigms: the theory of demographic transition and the laws of migration (Zelinsky, 1971). He elaborates on this idea and comes up with an intriguir hypothesis which he calls the mobility transition:

> "there are definite, patterned regularities in the growth of personal mobility through space-time during recent history, and these regularities comprise an essential component of the modernization process" (Zelinsky, 1971, pages 221-222).

The hypothesis of the mobility transition seems to be in accord with recently available empirical evidence. Larry Long, for example, finds that the expected number of lifetime moves for an American has been around thirteen since 1960, whereas the corresponding measure for a Japanese has increased from about four moves in 1960 to over seven moves in 1970 (Long, 1970; Long and Boertlein, 1976). The lifetime quota for the British seems to have stabilized around eight moves per capita while that of the Irish is less than half of that total. A simple plot of these migration expectancies against such proxies of development and modernization as per capita income indicates a decidedly positive association.

Studies currently in progress at IIASA suggest that the spatial dimension of the mobility transition hypothesis also has considerable validity. Specifically, it appears to be the case that the transformation of a low-income, principally agrarian economy into a high-income industrialized modern economy may be characterized in terms of spatial migration expectancies that disaggregate expected moves and lifetimes by places of birth and residence (Rogers, 1975b; 1976a). Such indicators of geographical mobility may be used to show that rural-urban migration grows and dominates during the early stages of development, then levels off and ultimately declines as interurban migration becomes the principal form of interregional migration. More recent trends indicate that the last stage in this transition is one of deconcentration and dispersal (Long and Boertlein, 1976).

Patterns that seem to be clear at the *macro* level often become somewhat blurred at the *micro* level. For example, the most common generalization suggested by the accumulating census and survey data on fertility is that lower birth rates tend to be associated with higher economic status. Per capita income is negatively correlated with fertility. Yet the early empirical studies of fertility and development show a *positive* association (Adelman, 1963; Weintraub, 1962) and common-sense considerations argue that increased affluence leads people to consume more rather than less of most goods. What is the answer? The new microeconomic theory of fertility provides an interesting explanation. It argues

> "...that increased affluence causes people to buy more of most things, the exceptions being labeled inferior goods. Since no one considers children inferior goods, many argue that children and income 'really' are positively related, but the relation is concealed by the intervention of other factors. The better-off have access to contraceptives of which

the poor are ignorant; the better-off have higher quality (that is, more expensive) children, and so can afford fewer of them" (Keyfitz, 1975, page 280).

Although this general model of the determinants of human fertility is not fully persuasive to some noneconomists (who might argue, for example, that the major contributing factors to changes in fertility behavior are changes in tastes, attitudes, and nuptiality patterns), it does nevertheless demonstrate the importance of a microtheoretical assessment of empirical trends. Similar illustrations of this sort may be found in some of the recent efforts to develop a microtheoretic explanation of migration by using the concepts of imperfect information, job search, and investment in human capital (David, 1974; Miron, 1977; Phelps, 1970).

Although the effects of development on population growth are reasonably clear and the links of causation seem to be in general accord with the empirical evidence, the same cannot be said with regard to the inverse question concerning the effects of population growth on development. Here there is much confusion and controversy.

The negative consequences of rapid population growth on development are several. Rapidly growing populations have proportionately many more children and higher dependency ratios. Children have to be fed and educated, and this diverts resources that otherwise could be applied towards industrial development programs. Larger populations, it has been argued, give rise to diminishing returns to fixed factors of production—resource and capital limitations combining to guarantee that more people will have fewer resources per capita to work with. Finally, excessively large flows of rural to urban migrants in developing countries create rapid rates of urbanization and increasing levels of urban unemployment. The phenomenal growth of urban areas has strained the urban infrastructure and has fostered congestion, pollution, and an assortment of other human and social ills.

But these simple neoclassical, Malthusian, and physical planning paradigms are increasingly being challenged in today's development literature (Kelley, 1974; Leibenstein, 1975; Ohlin, 1976). The impacts of population growth on the quality of the labor force, on the rate of technological progress, on savings behavior, and on the amount of technical change embodied in new capital goods all are important considerations that have not been adequately incorporated into the theoretical discourse. And the question of optimal human settlement patterns and hierarchies is a hotly contested issue which badly needs the kind of illumination that a good theory brings (Alonso, 1972; Berry, 1973; von Böventer, 1973). Thus, despite the large and growing literature on the relationships between population dynamics and socioeconomic change, our understanding of these matters is still woefully inadequate with the result that

> "There is an embarrassing gap between the confident assertions by prominent statesmen and international organisations which blame population growth for most of the evils of the world, and the hesitant and circumspect positions taken by those economists and demographers who have not turned crusaders... . In fact, one is tempted to say that the more rigorous the analysis and the more scrupulous the examination of the evidence, the smaller is the role attributed to population as an independent source of economic problems" (Ohlin, 1976, page 3).

Ohlin attributes the underdeveloped status of demographic–economic theory to the fallacious tendency to "distinguish between the study of the determinants of fertility or population growth, which was left to sociologists, and the consequences of population growth, which were supposed to be explained by economists" (Ohlin, 1976, page 4). Irrespective of whether he is right or wrong, it is clear that an adequate demographic theory will have to recognize and interact with the broader themes of socioeconomic growth and development. In consequence, the rate of theoretical progress is likely to be very slow, because to

> "appreciate the impact of population growth implies an adequate theory of economic development. As long as most of the variance in economic growth remains unexplained, there is no reason for us to expect to understand very much more when we consider the impact of an additional complex nonstandardized macro variable" (Leibenstein, 1975, page 233).

The importance of theory

Despite the popularly held belief that "the facts speak for themselves", it is nevertheless true that the most important causal relationships in demography cannot be established without an underlying theory. Indeed the dictum "no theory—no conclusions" is the central fact of scientific research.

Consider, for example, the economist's proposition that migration is a response to differential economic opportunities—notably in employment. Is this proposition in conformity with the frequently observed positive correlation between a region's in-migration and out-migration rates (Miller, 1967; Stone, 1971)?

> "Areas with the highest in-migration rates also had the highest out-migration rates. ...This is clearly not in agreement with the hypothesis that in- and out-migration *per capita* are inversely related; it suggests that most migrants move from a position of economic strength rather than weakness, and consequently imposes significant reservations on push–pull approaches to migration" (Cordey-Hayes, 1975, page 806).

Does the positive correlation between out-migration and in-migration rates invalidate the economic push–pull theory of migration? Decidedly not. Cross-sectional analyses prove nothing concerning longitudinal changes.

Simon (1969), for example, has demonstrated that the negative association between fertility and income, often observed in cross-sectional data, is not inconsistent with the positive association observed in (short-run) time series data. His arguments are equally applicable to migration.

Several demographers have put forward the view that migration may be a self-generating process in which today's in-migrants are tomorrow's out-migrants. This focus on migration-prone individuals appears, for example, in the work of Goldstein (1964), Lee (1974), and Morrison (1971). If such a view is valid, then it can be readily shown that a positive correlation between out-migration and in-migration rates is not inconsistent with the push–pull hypothesis. Indeed, the sign of the correlation depends on the interactions of certain 'hidden' variables such as age and employment. To see this more clearly, one must engage in a little bit of theorizing. For ease of exposition we shall restrict our attention to the simplest static models and will always omit the error term at the end of each equation.

Demographers have long recognized that migration is strongly age-selective. Most migrants are young, and typically over half of the individuals in any internal migration stream are under twenty-five years of age (Rogers, 1976b). Hence a region with a relatively young population is likely also to have a relatively mobile population. Thus if O is a region's out-migration rate and C is the proportion of its population that is under twenty-five years of age, then we might assume that

$$O = a_1 + a_2 C, \qquad a_2 > 0. \tag{1}$$

Because most migrants are young, it seems reasonable to suppose that

$$C = b_1 + b_2 I, \qquad b_2 > 0, \tag{2}$$

where I is the region's in-migration rate.

Substituting from equation (2) into equation (1) gives

$$\begin{aligned} O &= (a_1 + a_2 b_1) + a_2 b_2 I \\ &= A_1 + A_2 I, \qquad A_2 > 0, \end{aligned} \tag{3}$$

and yields a positive correlation between out-migration and in-migration.

Now assume that demographers and economists are both correct, and suppose that a region's out-migration rate, O, may be expressed as a linear function of its in-migration rate, I, and unemployment rate, U. We shall frame our arguments around the unemployment rate as an indicator of the economic health and attractiveness of a regional economy. This may not be the best indicator, but any other would not appreciably change our principal arguments and conclusions.

$$O = a_1 + a_2 I + a_3 U, \qquad a_2 > 0. \tag{4}$$

Let the in-migration rate be a simple linear function, with a negative slope, of the region's unemployment rate:

$$I = b_1 - b_2 U, \qquad b_2 > 0. \tag{5}$$

Using equation (5) to eliminate U in equation (4) gives

$$\begin{aligned} O &= \left(\frac{a_1 b_2 + a_3 b_1}{b_2}\right) + \left(\frac{a_2 b_2 - a_3}{b_2}\right) I \\ &= D_1 + D_2 I. \end{aligned} \tag{6}$$

It is readily apparent that the correlation between O and I will be positive if $a_2 b_2 > a_3$ and negative if the inequality sign is reversed. As it stands, therefore, equation (6) alone can tell us nothing about the validity of the push–pull hypothesis.

It may be instructive, at this point, to expand our simple example to recognize possible reciprocal causations. Let us adopt the perspective of an antitheoretical empirically minded social scientist who throws in a large number of potential explanatory variables into a regression equation with the idea of selecting that subset of variables which accounts for the largest proportion of the variance. Such an approach might begin with the following pair of equations:

$$O = a_1 + a_2 I + a_3 C + a_4 U, \tag{7}$$

$$I = b_1 + b_2 O + b_3 C - b_4 U. \tag{8}$$

Observe that, once again, a positive correlation between O and I reveals nothing about the push–pull hypothesis, since in using equation (8) to eliminate C in equation (7), for example, we return to the relationship set out earlier in equation (4) and arrive at the same basic indeterminacy that appeared in equation (6).

But equations (7) and (8) also illustrate a more general fundamental problem—one which econometricians have defined to be the problem of *identification* (Fisher, 1966).

When an econometrician is engaged in estimating the parameters of a behavioral or technological relationship put forward by a mathematical economic theory, he is engaged in what is known as *structural estimation*. Very frequently, the structural equation to be estimated is part of a system of such equations, all of which are assumed to hold simultaneously. In such a case the parameters of any single equation cannot logically be determined on the basis of empirical data alone. Some a priori assumptions are required in order to reduce the number of unknown items of information so that the 'true' equation can be distinguished (that is, identified) from all of the mathematically equivalent alternatives that imply the same empirical results. The logical source for such a priori information is a theory.

> "Indeed what makes an equation 'structural' is the existence of a theory which predicts a relationship among variables which appear therein. That theory provides necessary a priori information without which the very existence of a structure to be estimated would not be perceived... it does not suffice to know that the equation to be estimated contains precisely a specified list of variables. It is also necessary to know what variables are contained in other simultaneously holding equations or to have other information about the equation in question. Without such additional information, structural estimation is a logical impossibility. One literally cannot hope to know the parameters of the equation in question on the basis of empirical observations alone, *no matter how extensive and complete these observations may be*" (Fisher, 1966, pages 1-2).

The heart of the problem lies in the inability of data alone to distinguish between the 'true' system of relationships and the many other systems of relationships that can generate the same observations. For example, if one takes λ times equation (8) and adds the result to equation (7), the resulting equation, when solved for O will be indistinguishable from equation (7) so far as the data are concerned. Similarly, if one multiplies equation (7) by μ and adds the result to equation (8), the resulting equation, when solved for I will be indistinguishable from equation (8). The 'false' equations will be

$$\begin{aligned} O &= \left(\frac{a_1+\lambda b_1}{1-\lambda b_2}\right) + \left(\frac{a_2-\lambda}{1-\lambda b_2}\right) I + \left(\frac{a_3-\lambda b_3}{1-\lambda b_2}\right) C + \left(\frac{a_4-\lambda b_4}{1-\lambda b_2}\right) U \\ &= A_1 + A_2 I + A_3 C + A_4 U, \end{aligned} \tag{9}$$

and

$$\begin{aligned} I &= \left(\frac{b_1+\mu a_1}{1-\mu a_2}\right) + \left(\frac{b_2-\mu}{1-\mu a_2}\right) O + \left(\frac{b_3+\mu a_3}{1-\mu a_2}\right) C - \left(\frac{b_4-\mu a_4}{1-\mu a_2}\right) U \\ &= B_1 + B_2 O + B_3 C - B_4 U. \end{aligned} \tag{10}$$

Without further information, so long as $\lambda b_2 \neq 1 \neq \mu a_2$ and equations (9) and (10) are independent, the 'true' equations in (7) and (8) cannot be distinguished from any other pair of equations given by (9) and (10) and various values of λ and μ.

Suppose, however, that we have a priori information given to us by a theory that allows us to ignore the influence of unemployment in equation (7) and that of age composition in equation (8). Then

$$O = a_1 + a_2 I + a_3 C, \tag{11}$$

$$I = b_1 + b_2 O - b_3 U, \tag{12}$$

and the two equations are identifiable. Solving for the 'reduced form' of the system in which each endogenous variable is a function only of the exogenous variables, we find

$$O = \left(\frac{a_1 + a_2 b_1}{1 - a_2 b_2}\right) + \left(\frac{a_3}{1 - a_2 b_2}\right) C - \left(\frac{a_2 b_3}{1 - a_2 b_2}\right) U$$
$$= A_1 + A_2 C - A_3 U, \tag{13}$$

$$I = \left(\frac{b_1 + a_1 b_2}{1 - a_2 b_2}\right) + \left(\frac{a_3 b_2}{1 - a_2 b_2}\right) C - \left(\frac{b_3}{1 - a_2 b_2}\right) U$$
$$= B_1 + B_2 C - B_3 U. \tag{14}$$

Now we can identify equations (11) and (12). Given consistent estimates of A_1, A_2, A_3, B_1, B_2, and B_3, we can 'recover' the consistent estimates of a_1, a_2, a_3, b_1, b_2, and b_3. For example, given A_2 and B_2, we can solve for b_2:

$$b_2 = \frac{B_2}{A_2},$$

and, given A_3 and B_3, we can derive

$$a_2 = \frac{A_3}{B_3}.$$

The question that now may occur is whether structural estimation is really so important as to warrant such careful attention. If theory is unavailable or is of questionable validity, might not it be better simply to observe historical correlations among a set of variables and then predict the future by assuming a continuation of past relationships? Why not avoid making any prior assumptions altogether and simply use the least-squares estimates that come from the fitting of equations (7) and (8) [or, equivalently, equations (9) and (10)] to observed data?

The answer is that this does not provide us with an adequate *explanation* of the relationships. So long as the basic situation does not change and if simple short-run predictions are the principal goal of the exercise, a statistical-correlational extrapolation may indeed be adequate. But, if anything happens to alter the basic situation (that is, if a 'turning point' occurs), theoretical knowledge about structural relationships is indispensable. Without it we can have no idea what to expect from a sudden change in the value of a structural parameter or of an exogenous variable. For example, what would be the effects on a region's migration rates if fertility were suddenly to rise, increasing the value of C in the model described in equations (7), (8), (9), and (10)? Since the 'true' effect given by a_3 and b_3 is confounded with other influences in the estimates A_3 and B_3, we cannot anticipate the consequences.

But there is also a very practical objection to simple correlational predictions. Theory building is very often undertaken in order to aid policymakers in evaluating the probable consequences of alternative interventions into social and environmental systems. To foresee such consequences one must have reasonably accurate estimates of the system's structural parameters.

Take the case of the 'chicken-egg' controversy in the recent migration literature (Mazek and Chang, 1972; Muth, 1975). What is the relationship between migration and employment growth? Are differential rates of migration induced by differential rates of growth in employment, or is it the other way around? Proponents of the first view—the demand view—argue that it is the external ('export') demand for a region's outputs that creates jobs and these new jobs induce labor-force in-migration. Supporters of the second view—the supply view—argue that it is local labor-market dynamics that chiefly determine a region's growth in jobs. Recent evidence suggests that migration and employment growth each affect and are affected by the other, with the former effect dominating the latter (Greenwood, 1973; Kalindaga, 1974). Such a conclusion cannot be established without a convincing theory, and policies directed toward improving conditions in underdeveloped and declining regions, for example, cannot be truly effective without the understanding provided by such a theory.

Demometrics

A growing dissatisfaction with the qualitative-deductive character of economics led to the founding in 1930 of the Econometric Society with the avowed purpose of relating theory to observed data by transforming the discipline into a quantitative-empirical one. Growing interest and achievements in population research suggest that an analogous situation could well arise in demography during the next decade, even perhaps to the extent of the founding of a Demometric Society and the establishment of a journal entitled *Demometrica.*

Literally, demometrics means demographic measurement, and measurement forms an important part of demometrics. But purely descriptive statistics about population compositions or growth rates and nonmathematical theorizing are not demometrics.

The essence of demometrics is the union of demographic theory, mathematics, and statistics. Mathematical demographic theory studies the relationships between demographic and socioeconomic variables in algebraic terms—these relationships become part of demometrics when they take on numerical values that are estimated from observations. Statistical methods and techniques deal with relationships between variables, but, unless these variables include variables from demographic theory, the results are not part of demometrics. Demometrics is distinguished by its *fusion* of the deductive approach of mathematics, the inductive approach of statistics, and the causal approach of demographic theory. Its principal objective is to establish quantitative statements regarding major demographic variables that either *explain* the past behavior of such variables or *forecast* (that is, predict) their future behavior.

In striving to explain the past behavior of demographic variables, demometrics necessarily deals with the formulation and empirical determination of demographic hypotheses and with the specification and estimation of systems of relationships. Thus it plays a pivotal role in demographic theory building, using numerical data to verify the existence and define the form of relationships such as those postulated in the hypotheses of the demographic and mobility transitions, the fertility-income function, the push-pull and the chicken-egg arguments of migration, and other such 'laws' of population growth and change.

Demometrics also has an important role to play in demographic forecasting. When in 1938 the US National Resources Committee carried out a major demographic projection of the future US population, it adopted a set of 'reasonable' assumptions with regard to future fertility, mortality, and net immigration, and then projected the total US population in 1980 to be 158 million (US National Resources Committee, 1938). The US population passed the 158 million mark less than fifteen years later and today exceeds 210 million individuals.

It is difficult to fault such projections, for it is unlikely that any competent demographer, faced with the same situation, would have come up with radically different results. How then, can demometrics improve the accuracy of such exercises in social prediction?

The projection by the US National Resources Committee, like most population projections, did not link demographic variables with economic variables. Until very recently, this has been a standard practice in both disciplines. That is, demographers typically have given economic variables only cursory treatment in their models, and economists have accorded demographic variables a similar status. In the words of Hoover:

> "Purely demographic and purely economic models... are multitudinous and often highly complex. This makes even more striking the relatively primitive state of the art that prevails in the linking of demographic and economic variables" (Hoover, 1971, page 73).

Much of the future work in demometrics, therefore, will undoubtedly be directed toward advancing the state of the art in consistent demo-economic forecasting. It is likely that this work will borrow extensively from the successful example set by econometrics, and macrodemometric forecasting models will probably reflect many of the characteristics of macroeconometric models of national and regional economies.

Macrodemometric models are systems of equations that represent the fundamental relationships between, and the behavior over time of, such major demo-economic variables as birth rates, migration rates, labor force participation rates, unemployment rates, employment, output, investment, and population. Such models may be used for forecasting and also for policy analysis.

The variables in a 'reduced form' macrodemometric model belong to two different classes: those that appear on the left-hand side of the equations and those that do not. The former are called *endogenous* variables, and their values are determined by the model (the number of equations in a macrodemometric model, therefore, is equal to the number of endogenous variables). The magnitudes of the latter variables, the *predetermined* variables, are set outside the model. Predetermined variables consist of exogenous variables and lagged (previously) endogenous variables. A model-generated solution for the endogenous variables, called a *forecast*, is associated with each set of predicted values for the predetermined variables.

There are two important reasons for studying relationships among demo-economic variables: the forecasting reason and the policy analysis reason. Policymakers need accurate population forecasts in order to scale investment decisions made in response to, or in anticipation of, population-generated demands. They also need a reliable tool for pretesting the probable consequences of alternative courses of action. These two objectives are not necessarily incompatible, and it should be possible to construct macrodemometric models that satisfy both.

Migration and settlement

The central focus of this paper has been demographic theory building and demometrics. In the introduction we noted that such research constituted one of four interrelated subtasks currently being carried out within the Migration and Settlement Study at IIASA. We shall now conclude the paper by briefly describing the broad outlines of the other three subtasks in order to identify more clearly the role of demometrics in the overall study. The three other subtasks are the dynamics, the policy, and the comparative study subtasks.

Dynamics

The unanticipated postwar baby boom had a salutary influence on demographic research. Extrapolations of past trends appropriately adjusted for expected changes in the age, sex, and marital composition of the population were very much wide of the mark. So long as trends were stable, demographic projections prospered; but, when a 'turning point' occurred, the projections floundered. The net result was increased pressure to consider the complex interrelationships between fertility behavior and socioeconomic variables.

But the poor predictive performance also had another important effect —it stimulated research in improved methods for *measuring* fertility and for understanding the *dynamics* by which it, together with mortality, determines the age composition of a population (Coale, 1972; Ryder, 1966). Inasmuch as attention was principally directed at national population growth, measurement of internal migration and the *spatial* dynamics through which it affects a national settlement pattern were neglected. This neglect led Dudley Kirk (1960) to conclude, in his 1960 Presidential address to the Population Association of America, that the study of migration was the stepchild of demography. Sixteen years later, Sidney Goldstein echoed the same theme in *his* Presidential address to the same body:

> "the improvement in the quantity and quality of our information on population movement has not kept pace with the increasing significance of movement itself as a component of demographic change.... Redistribution has suffered far too long from neglect within the profession.... It behooves us to rectify this situation in this last quarter of the twentieth century, when redistribution in all its facets will undoubtedly constitute a major and increasingly important component of demographic change..." (Goldstein, 1976, pages 19–21).

Improved methods for measuring migration and understanding its important role in human population dynamics is a central research focus of the Migration and Settlement Study at IIASA. The search for improved methods for measuring migration has stimulated our research on the construction of multiregional life tables, and the need for a better understanding of spatial population processes has fostered a study of the fundamental 'laws of motion' of spatial population growth and distribution.

Multiregional life tables are members of a special class of life tables known as increment-decrement life tables (Rogers and Ledent, 1976; Schoen, 1975). They view in-migration as a form of increment and treat out-migration and death as forms of decrement. Such life tables describe the evolution of several regional cohorts of babies, all born at a given moment and exposed to an unchanging *multiregional* age-specific schedule of mortality and migration. For each regional birth cohort, they provide various probabilities of dying, surviving, and migrating, while simultaneously deriving regional expectations of life at various ages. These expectations of life are disaggregated both by place of birth and by place of residence, and reflect, therefore, the influences both of mortality and of migration. Thus they may be used as indicators of levels of internal migration, in addition to carrying out their traditional role as indicators of levels of mortality.

Ordinary single-region life tables normally are computed with the use of observed data on age-specific death rates. In countries lacking reliable data on death rates, however, recourse is often made to inferential methods that rely on model life tables such as those published by the United Nations (Coale and Demeny, 1967). These tables are entered with empirically determined survivorship proportions to obtain the particular expectation of life at birth (and corresponding life table) that best matches the levels of mortality implied by the observed proportions.

The inferential procedures of the single-region model may be extended to the multiregional case (Rogers, 1975a). Such an extension requires the availability of *model multiregional life tables* and uses a set of initial estimates of survivorship and migration proportions to identify the particular combination of regional expectations of life, disaggregated by region of birth and region of residence, that best matches the levels of motality and migration implied by these observed proportions.

Model multiregional life tables approximate the mortality and migration schedules of a particular multiregional population system by drawing on the regularities observed in the mortality and migration experiences of other comparable populations (Rogers and Castro, 1976). To construct such tables, we are currently summarizing the principal empirical regularities exhibited by observed age-specific patterns of migration in a number of IIASA member nations. These will be used to generate model tables that will provide demographers with a means for systematically approximating the migration schedules of populations lacking migration data. Our aim, in short, is to accomplish in the area of migration analysis what the United Nations model life tables contributed to the analysis of mortality.

The evolution of every spatial human population is governed by the interactions of births, deaths, and migration. Individuals are born into a population, age with the passage of time, reproduce, and ultimately leave the population because of death or out-migration. These events and flows enter into an accounting relationship in which the growth of a regional population is determined by the combined effects of rates of natural increase (birth rates minus death rates) and rates of net migration (in-migration rates minus out-migration rates).

A change in any one of these component rates affects the dynamics of the spatial demographic system, but it occasionally does so in ways that are not immediately self-evident. Our studies of such sensitivity analyses have led us to develop sensitivity functions that relate a change in a particular component rate to the corresponding changes in various spatial demographic statistics (Willekens, 1976a). In this analysis we have used matrix differentiation techniques to derive analytical expressions that establish the impacts of changing rates on multiregional life table statistics. These sensitivity functions reveal how each spatial demographic characteristic depends on age-specific rates and how it reacts to changes in those rates.

Increasing concern about the sizes and growth rates of national populations has generated a vast literature dealing with a particular form of sensitivity analysis, namely the demographic consequences of a reduction of fertility to replacement levels and the consequent evolution of national populations to a zero growth condition (Frejka, 1973; Ryder, 1974). But where people choose to live in the future presents issues and problems that are potentially as serious as those posed by the number of children they choose to have. Yet the spatial implications of reduced fertility have received relatively little attention and we are, in consequence, ill-equipped to develop adequate responses to questions about the ways in which stabilization of a national population is likely to affect migration and local growth. A notable exception is the work of Peter Morrison (1972).

We have considered some of the redistributional consequences of an immediate reduction of fertility to bare replacement levels and have found that stabilization of a multiregional population system will alter the relative contributions of natural increase and migration to regional population growth (Rogers and Willekens, 1976). The redistributional effects of stabilization will depend in a very direct way on the redistributional pattern of total births that is occasioned by fertility reduction. Regional age compositions will also be affected, and in ways that are strongly influenced by the age patterns of migration. Retirement havens, for example, will receive proportionately higher flows of in-migrants as a national population increases in average age, whereas destinations that previously attracted mostly younger migrants will receive proportionately fewer in-migrants.

Finally, as demographers have come to model dynamic socioeconomic systems of growing size and complexity they have been forced to rely on ever more sophisticated high-speed digital computers. However, their capacity for handling large-scale systems has not kept pace with the growing demands for more detailed information. Consequently, it is becoming especially important to identify those aspects of a system which permit one to deal with parts of it independently from the rest or to treat relationships *among* particular subsystems as though they were independent of the relationships *within* those subsystems. These questions are those of aggregation and decomposition, respectively, and their application toward 'shrinking' large-scale population projection models is an important element of our spatial population dynamics subtask (Rogers, 1976a).

We have adopted a shrinking procedure for large-scale population projection models that combines aggregation and decomposition in a particularly appealing way. One begins by partitioning the large multiregional system projection model into smaller submodels in a way that effectively exploits any weak interdependencies revealed by indices such as migration levels. The growth of the original multiregional system then is projected by appropriately combining (1) the results of disaggregated *intra*subsystem projections, in which *within*-subsystem interactions are represented at a relatively fine level of detail, with (2) the results of aggregated *inter*subsystem projections, in which the *between*-subsystem interactions are modeled at a relatively coarse level of detail. In the short run, the *within*-subsystem interactions dominate the behavior of the system; in the long run, the *between*-subsystem interactions become increasingly important and ultimately determine the behavior of the entire system. In this manner a large-scale population projection process can be modeled with a considerable saving in computer time and storage space.

Policy

If the principal purpose of the *dynamics* subtask is to understand the fundamental 'demographic processes' that govern the evolution of human settlement patterns, and that of the *demometrics* subtask is to explain the causal determinants of past and future patterns, then the major goal of the *policy* subtask is to develop a deeper appreciation of the impact of policy variables on population processes and of population processes on human welfare.

Social concern with population processes arises when the demographic acts of individuals affect the welfare of others and combine in ways that produce a sharp divergence between the sum of individual (private) preferences and social well-being. In such instances, population processes properly become the subject of public debate and the object of public policy.

Population policies are actions undertaken by public bodies with the aim of affecting processes of demographic growth and change. Family planning programs, investments in health-care facilities and services, and government-assisted migration are examples of public actions taken, respectively, to reduce fertility levels, to promote health and longevity, and to foster personal betterment through geographical mobility.

Among national population policies, the problem of fertility reduction has been of paramount importance. The negative consequences of rapid population growth for socioeconomic development are becoming widely recognized and this has led many countries to undertake serious efforts to control fertility. Since manipulating mortality levels is obviously not a feasible policy, the concern with rapid national population growth necessarily has been a concern about high levels of fertility.

Spatial population policies, on the other hand, tend to focus primarily on internal migration and its contribution to human settlement growth and structure. The perceived negative consequences of rapid rates of urban growth on socioeconomic development have led to the adoption of policies to curtail growth in certain localities, while at the same time stimulating it in others. Generally, such national urbanization or human settlement policies have been defended on the grounds either of national efficiency or of regional equity, and their principal arguments often have been framed in terms of an underlying conceptual framework known as 'growth-center theory' (Moseley, 1974).

Growth centers, it is commonly argued, generate, intercept, and attract migrants. They may encourage some underemployed people in the center's hinterland to migrate and to shift to more productive occupations. They can be used to divert migrants away from major overcrowded metropolitan areas. And they make it possible for an economically depressed region to attract the skilled and professional manpower that it needs for its growth and development.

But migration has both individual and societal consequences. The experience of migration, in general, affects favorably the personal well-being and satisfaction of the migrant (Lansing and Mueller, 1967; Morrison, 1973).

Migration, as a mechanism for transferring labor from labor surplus areas to areas with a labor deficit, moves the national economy toward greater efficiency. But this adjustment of the national labor market has local consequences with regard to equity. And it is these negative consequences that often fall on those 'left behind', since it is the most productive members of the labor force that are the ones who move away, leaving behind localities increasingly unattractive for industrial investment (Morrison, 1973).

The various individual and societal consequences of internal migration have broad implications for national policies dealing with migration and settlement. The built-in conflict between the goals of national efficiency and regional equity is a fundamental aspect of such policies, one that ultimately can only be resolved in the political arena. A potentially useful tool for illuminating some of the trade-offs that arise is offered by the formal theory of economic policy, first proposed by Jan Tinbergen (1952) in the field of economic planning.

The Tinbergen paradigm focuses on the problem of using available means to achieve desired ends in an optimal manner. It begins by adopting a quantitative empirical (econometric) model and divides variables into *endogenous* variables, that is, those determined within the model, and *predetermined* variables, that is, those determined outside the model and lagged previous endogenous variables. A further distinction is introduced within these two categories of variables. Endogenous variables are disaggregated into *target* variables, which are of direct interest to policy-makers, and *irrelevant* variables, which are not. Exogenous variables are composed of *instrument* variables, which are subject to direct control by policy bodies, and *data* variables, which are beyond their control. The latter include exogenously predetermined variables, uncontrollable variables such as the weather, and lagged endogenous variables.

The policy problem, as formulated by Tinbergen, is to choose an appropriate set of values for the instrumental variables so as to render the values of the target values equal to desired values previously established by an objective function called a welfare function. Thus the basic ingredients of the Tinbergen paradigm are a welfare function that is a function of various target variables and instrument variables; a quantitative empirical model that links target variables to instrument variables; and a set of boundary conditions or constraints which restrict the range of values that can be assumed by the different variables in the model (Fox et al, 1972).

It is important to keep in mind the fundamental difference between the manner in which the variables are related to one another in the models of the dynamics and demometrics subtasks and the way in which they are interconnected in the models of the policy subtask. In the former, the values of instrumental variables are specified and the analysis seeks to determine their effects. In the latter, the desired effects are given and the analysis is instead directed toward establishing the values that have to be assumed by the instrumental variables in order for the specified effects to be attained (Willekens, 1976b).

Despite their fundamentally different perspectives, both kinds of models are necessary in the formulation of enlightened population policies. For, as Paul Demeny recently observed,

> "... a proper formulation of population policies can be said to require the following essential elements: (a) an understanding of demographic processes in a descriptive sense; (b) an understanding of the antecedents of demographic behavior...; (c) an understanding of the impact of population processes... on ... welfare; (d) an evaluation of the welfare significance... of conceivable policy interventions..." (Demeny, 1975, page 153).

Demeny's first two essential elements are being examined in the dynamics and demometrics subtasks of the Migration and Settlement Study; the other two will be receiving considerable attention in the future work of the policy subtask. National case studies dealing with all four elements currently form the focus of the comparative study subtask.

Comparative study

The World Population Conference held in Bucharest in 1974 recognized the importance of the migration and settlement component of national population policies, called for a better coordination of migration policies and the absorptive capacities of major urban centers, and argued for the proper integration of these policies into plans and programs aimed at social and economic development (World Population Conference, 1975). But demographic and developmental processes are manifested in diverse ways in different national settings, and a meaningful analysis of their interaction must take into account important national differences. Yet certain regularities persist, and there are grounds for expecting a comparative study of migration and settlement to contribute to the state of our knowledge about the causal interrelationships between migration, urbanization, and development:

> "... in order to advance on both the theoretical and the applied levels, we must have comparative research on population movement, especially in relation to urbanization in preindustrial, industrial, and post-industrial settings. Only through such comparisons can we come to understand the varied forms which movement takes..." (Goldstein, 1976, pages 15–16).

The comparative study of migration and settlement at IIASA aims to contribute to our understanding of the relationships between geographical mobility, urbanization, and national development by assembling, summarizing, and analyzing data on migration and spatial population growth in a number of developed and developing countries. For this activity it has adopted the general framework of two recently published studies that have been carried out in a closely related area. Specifically, the comparative study of human *migration* and *redistribution* is being carried out in a manner that is analogous to the procedures used by two studies of human *mortality–fertility* and *reproduction*, namely the book by Keyfitz and Flieger (1971) entitled *Population: Facts and Methods of Demography* and the book edited by Berelson (1974) entitled *Population Policy in Developed Countries*.

The Keyfitz and Flieger book focuses on observed age- and sex-specific mortality and fertility schedules and projects the evolution of the populations exposed to these schedules. In order to examine the population trends of the present day, the authors collect together a data bank of population statistics from more than ninety countries and subject these data to a standardized analytical process.

If national population growth is the primary *focus* of the Keyfitz and Flieger study, its principal *approach* for examining such growth is embodied in a collection of computer programs that provide the means for analyzing population growth in a consistent and uniform manner. These programs and the mathematical models that underlie them are also included in the published study findings.

Finally, the major *contribution* of the Keyfitz and Flieger study is the uniform application of a consistent methodology to a vast amount of data in order to trace population growth trends in a large number of countries.

The focus, approach, and contribution of the Keyfitz and Flieger book have much in common with those of the comparative study of migration and settlement. The focus of the latter also is population growth, but *spatial* population growth. The approach also relies on a uniform set of computer programs, but these embody the models of *multiregional* mathematical demography (Rogers, 1975a). And the expected contribution also is that of linking data with theory, but the data and theory that are linked are *spatial* in character.

There are several important differences between the two study formats, however.

1 A primary concern of the Keyfitz and Flieger study is population *reproduction* and the *demographic transition* from high to low birth and death rates. An important focus of the comparative migration and settlement study is population *redistribution* and the *mobility transition* from low to high migration rates.
2 The Keyfitz and Flieger study is the product of two authors; the comparative migration and settlement study is combining the collaborative efforts of an international team of scholars residing in various member and nonmember nations.

3 The Keyfitz and Flieger study identifies trends and the numerical consequences of the continuation of such trends into the future; the comparative migration and settlement study is, in addition, striving to link national trends with explanatory variables.
4 Although chapter 4 of their book is entitled "Policy dilemmas and the future", the Keyfitz and Flieger study does not deal with national policies. (Their chapter 4 is only three pages long.) The comparative migration and settlement study, however, is explicitly considering the national migration and settlement policies of each country represented. In this respect the study resembles more the study of population policies coordinated by Bernard Berelson.

The book edited by Bernard Berelson is a review of population policies in twenty-four developed countries. The individual chapters were written by collaborating scholars residing in the particular countries. Thus, for example, Professor Charles Westoff of Princeton's Office of Population Research wrote the chapter on population policy in the USA, and Professor Dimitri Valentei of Moscow State University's Population Center authored the chapter on population policy in the USSR.

According to Berelson, "the collaborators were given a common outline as a guide to the topics to be addressed, but each author was free to prepare his report in his own manner". It is therefore not surprising that different authors elected to emphasize different aspects of population policy and drew on different kinds of demographic data to develop their presentations. Thus the book is somewhat uneven in its exposition and in the data and indicators that are put forward by the various authors.

The migration and settlement study aims to marry the Berelson approach with the Keyfitz-Flieger approach in order to capture the best features of each. Every national analysis in the comparative study of migration and settlement is, as in the Berelson study, being carried out in collaboration with scholars residing in the countries being studied. However, most of the data, projections, and indicators which form the foundation of the analysis, are being processed, as in the Keyfitz-Flieger study, by a common set of computer programs. These data and programs will be published together with the study's findings.

Conclusion

Internal migration and human settlement patterns are increasingly becoming subjects of governmental concern, both in developed countries and in the developing nations of the Third World. Whether the problem is that of ensuring an adequate supply of labor in Siberia or one of stemming the vast flood of migrants to the overcrowded major cities of Latin America, the need for a well-developed understanding of the relationships between spatial population dynamics and socioeconomic development is clear. A key to such understanding is a convincing theory of population and development.

Recent efforts at theory building in the social sciences suggest that theoretical advances often follow along a path that might be called the *theoretical transition*. Demography seems to be entering the *metrics* stage of its theoretical development and should profit from the insights and conclusions that demometrics research is likely to generate.

Progress in research on the demometrics of migration and settlement would be furthered by the availability of improved methods for measuring migration and for gauging its redistributional impacts in spatial population dynamics. The practical utility of demometrics research would be enhanced by an improved understanding of the various individual and social costs and benefits of different patterns of migration behavior and settlement processes. Finally, our knowledge about the determinants and consequences of demographic change could be much improved and broadened through a careful assessment of the population dynamics, demometrics, and policies that prevail in various countries of the world today.

Acknowledgements. The author is grateful to John Miron and Frans Willekens for careful readings of an earlier draft and for a number of useful comments.

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