On Optimal Labor Allocation Policy for Technological Followers

Sergey Aseev and Arkady Kryazhimskiy
International Institute for Applied Systems Analysis

Viennese Vintage Workshop , 26-27 November 2007

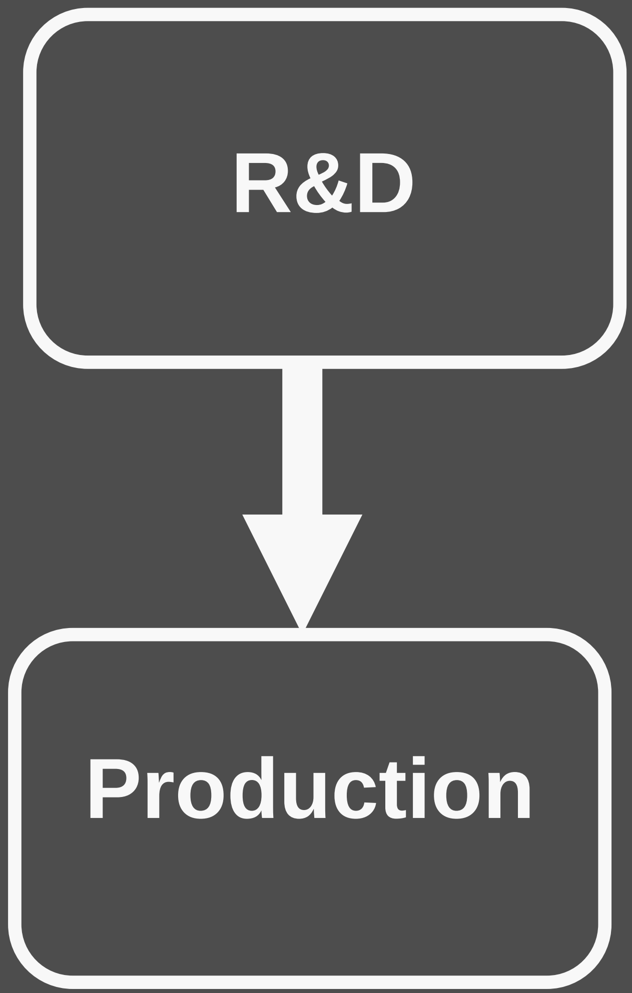
Knowledge absorption

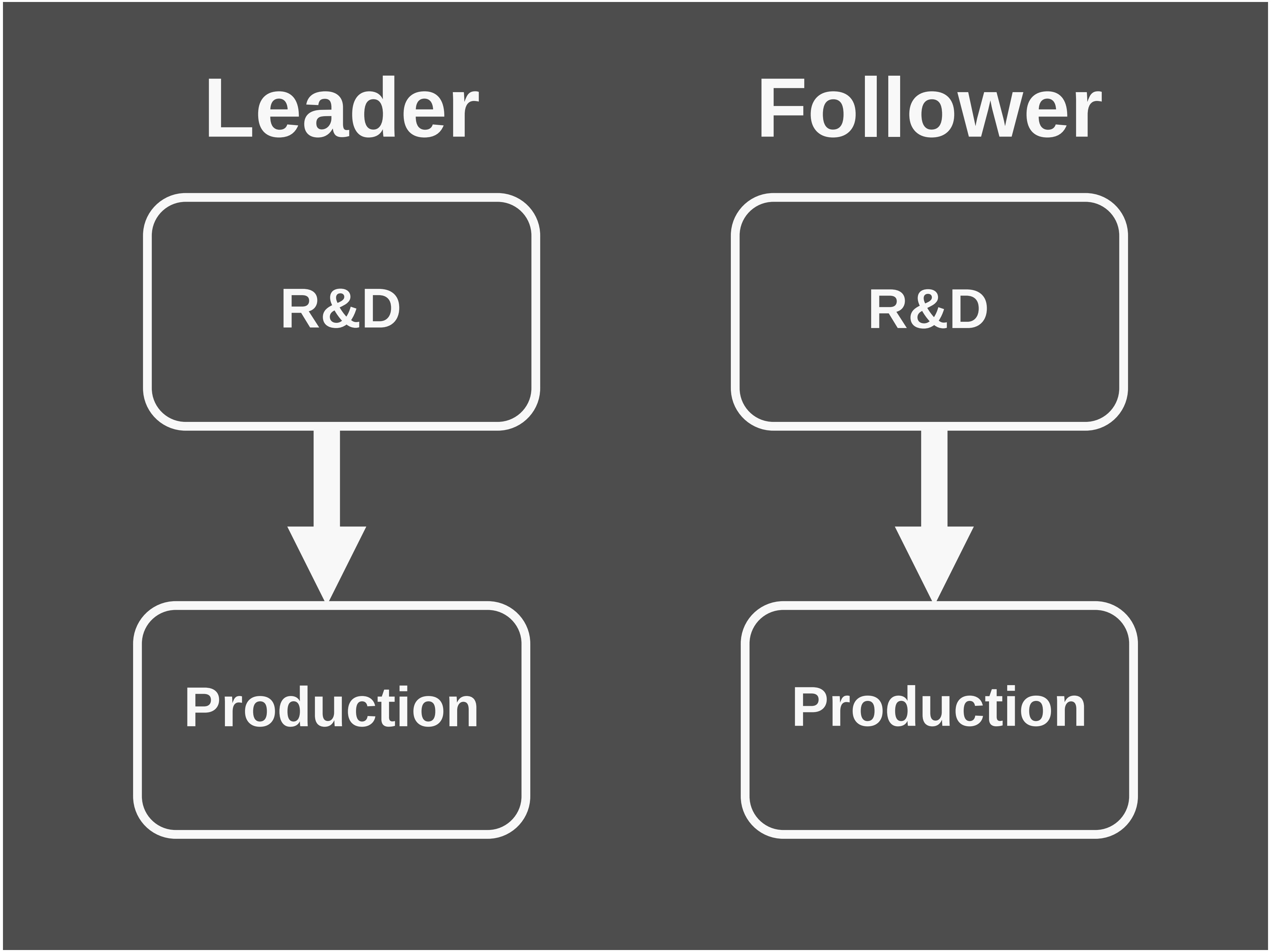
Optimal labor allocation

Catching up the leader

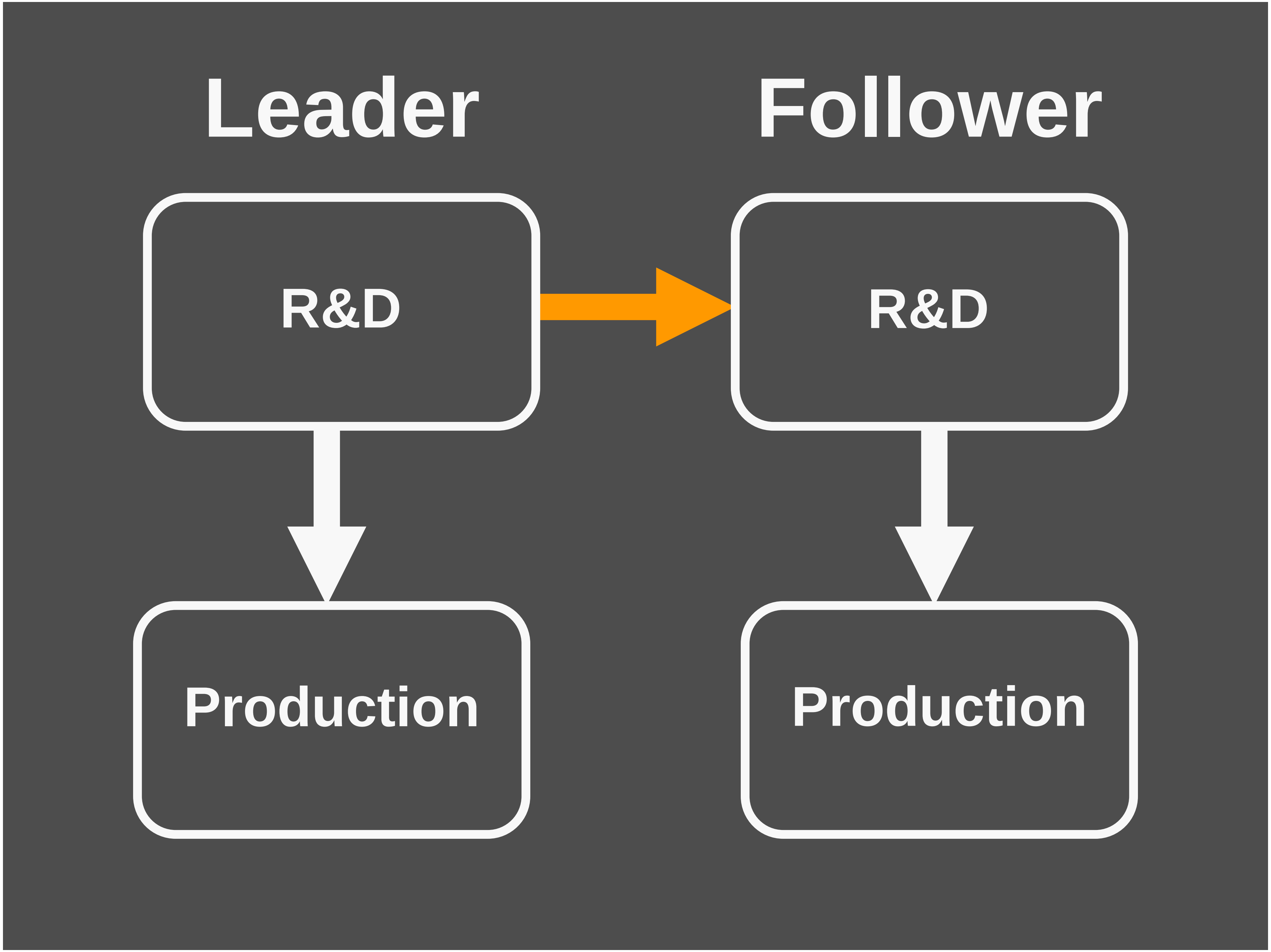
Overtaking the leader

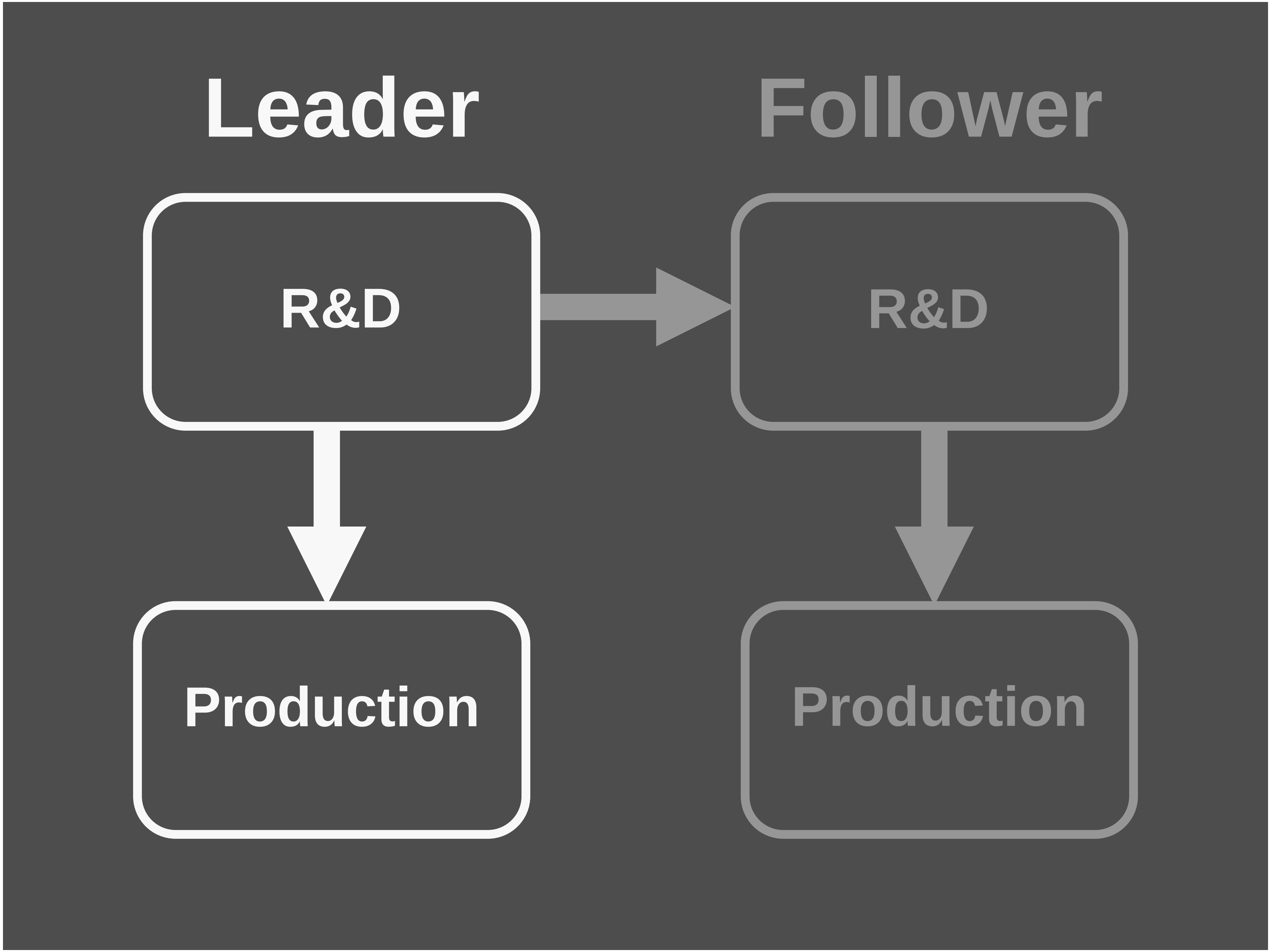
Leader





Leader
R&D
Production
Follower
R&D
Production


Leader
Follower
R&D
R&D
Production
Production

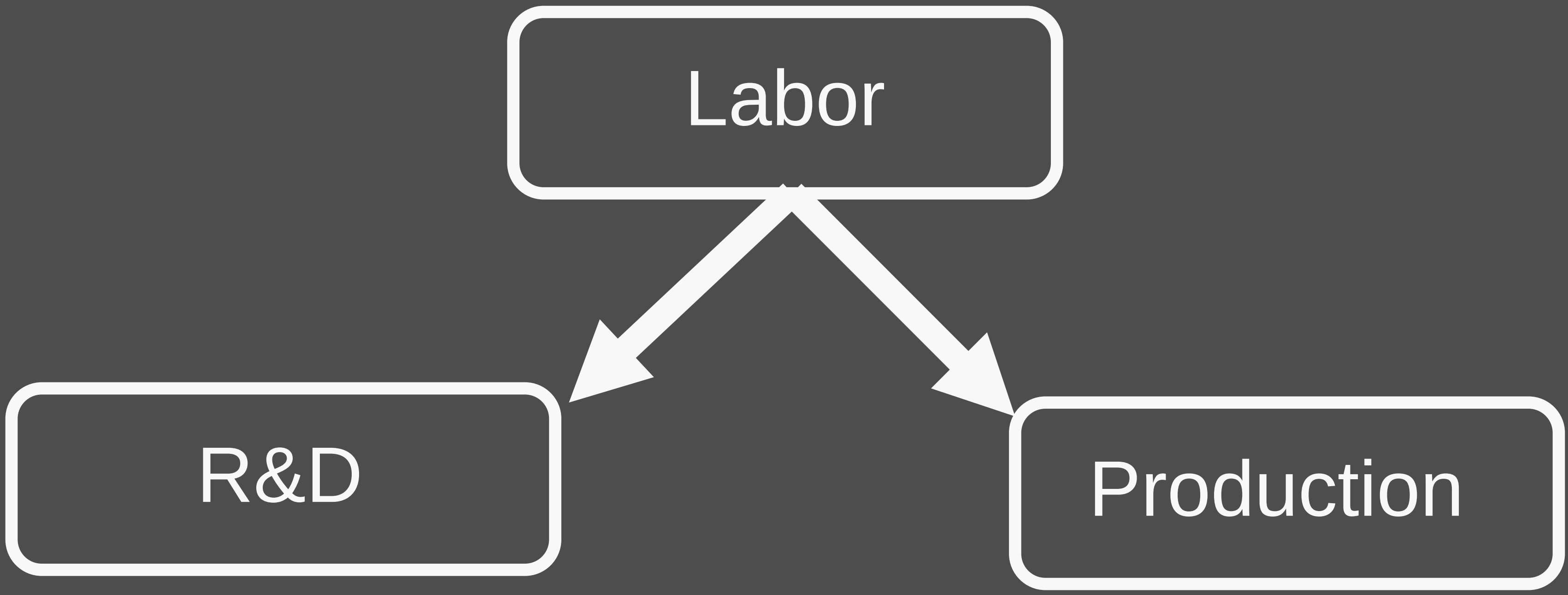

Leader
Follower
R&D
R&D
Production
Production

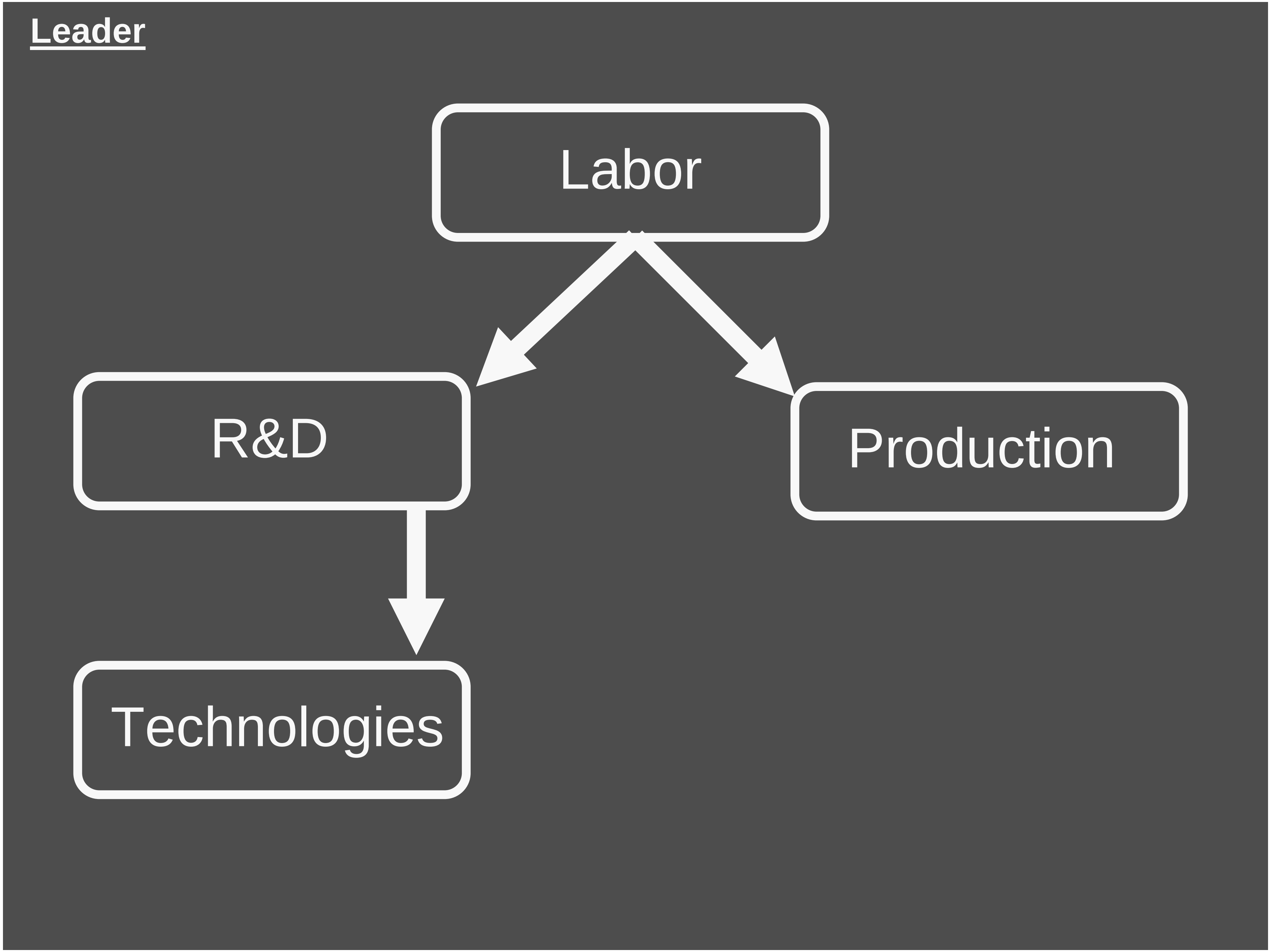
Leader

Leader

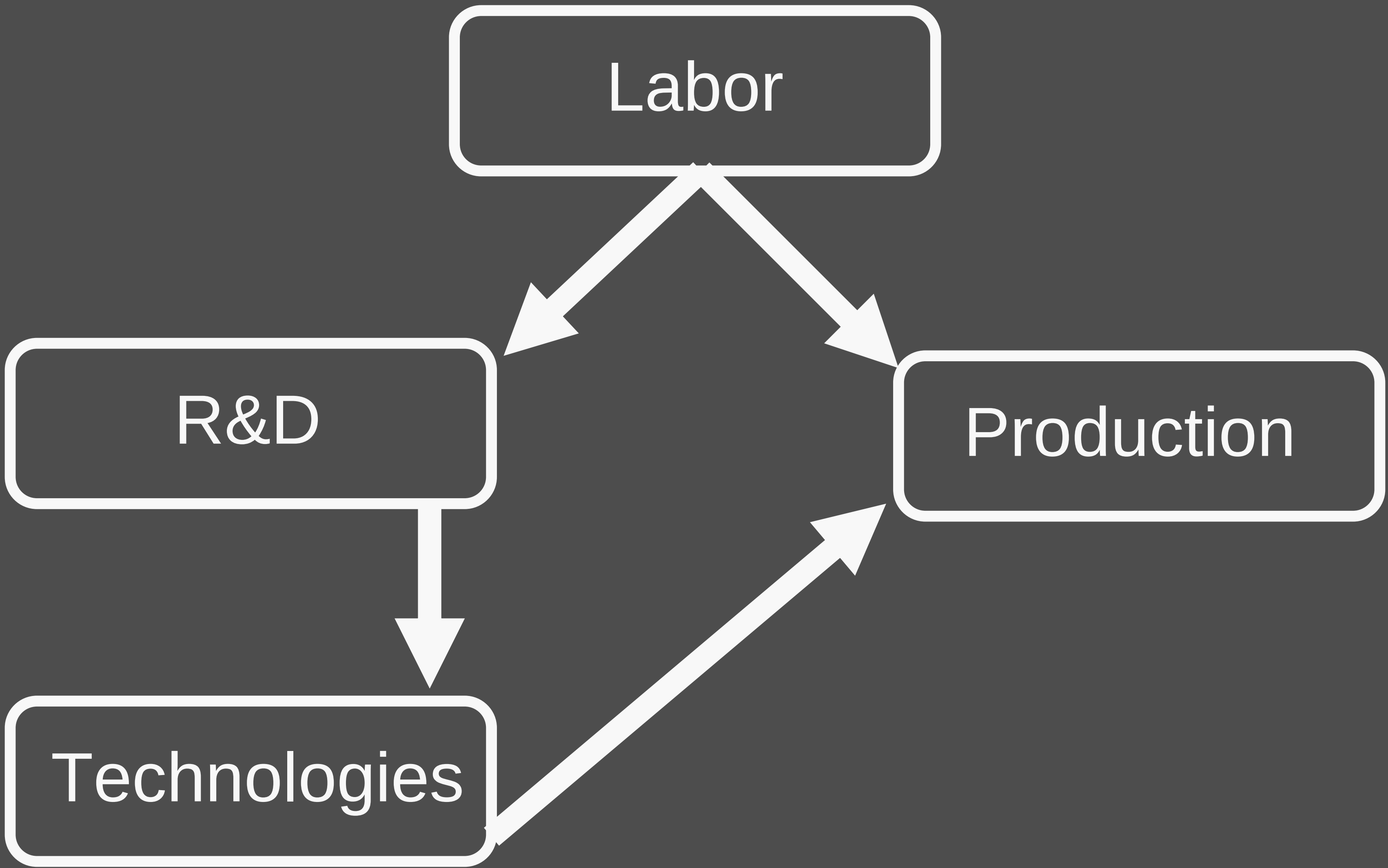
Labor

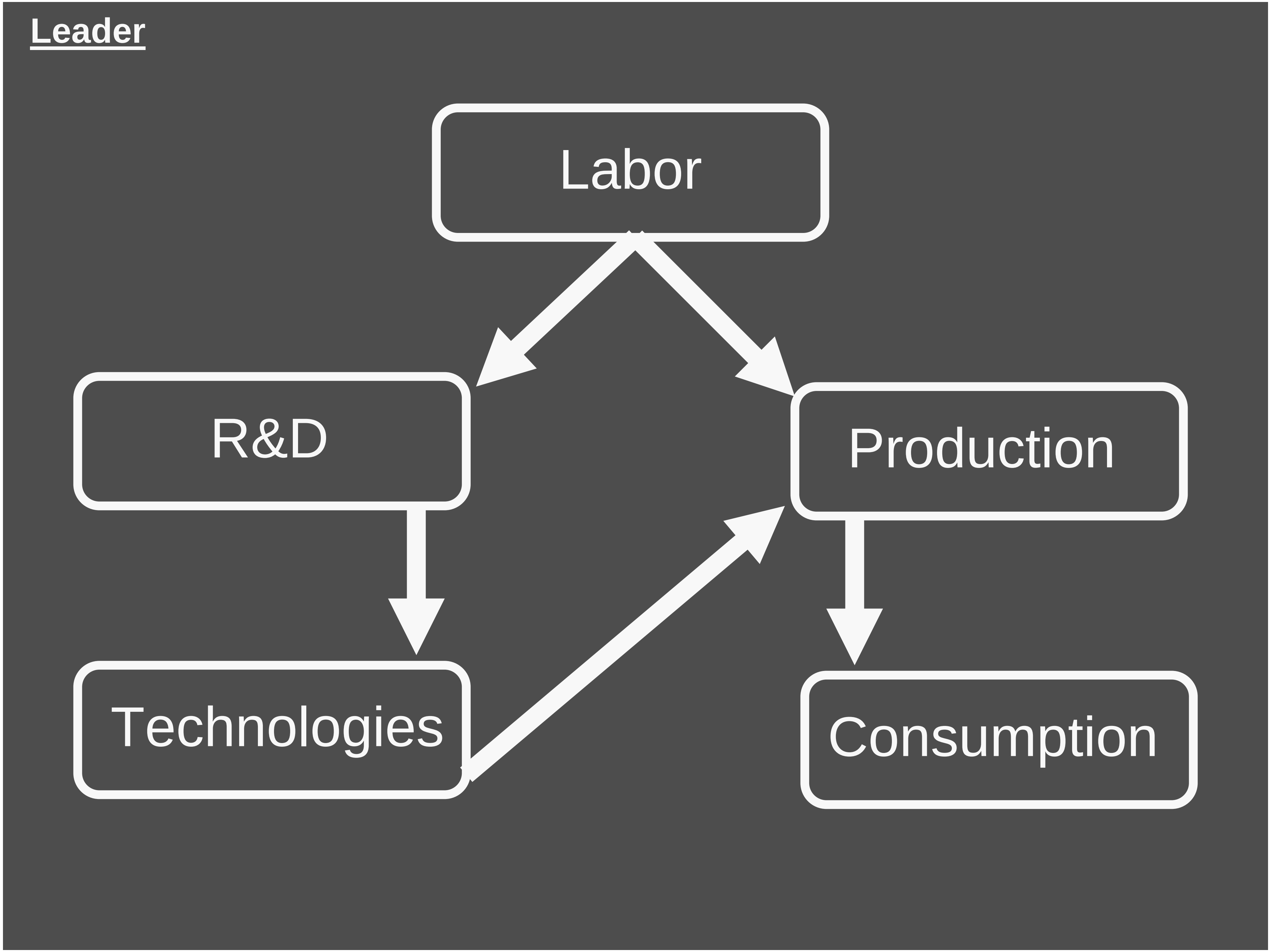
Leader




Leader
Labor
R&D
Production
Technologies

Leader




Leader
Labor
R&D
Production
Technologies
Consumption

Leader

Annual growth in technology $\approx \dfrac{\text{Labor in R\&D}}{\text{Technology stock}}$

$$T_{i+1} - T_i = L_i^{R\&D} T_i / a$$

Leader

$$\text{Annual growth in technology} \approx \frac{\text{Labor in R\&D}}{\text{Technology stock}}$$

$$T_{i+1} - T_i = L_i^{R\&D} T_i / a$$

$$\text{Annual consumption} \approx \frac{\text{Labor in production}}{\text{Technology stock}}$$

$$C_i = (\overline{L} - L_i^{R\&D}) T_i^{1/\alpha - 1}$$

Leader

Annual growth in technology $\approx$ Labor in R&D Technology stock

$$T_{i+1} - T_i = L_i^{R\&D} T_i / a$$

Annual consumption $\approx$ Labor in production Technology stock

$$C_i = (\overline{L} - L_i^{R\&D}) T_i^{1/\alpha - 1}$$

Utility

$$U = \sum_i (1-\rho)^i [(1/\alpha - 1)\log T_i + \log(\overline{L} - L_i^{R\&D})]$$

Leader

Annual growth in technology ≈ Labor in R&D × Technology stock

$$T_{i+1} - T_i = L_i^{R\&D} T_i / a$$

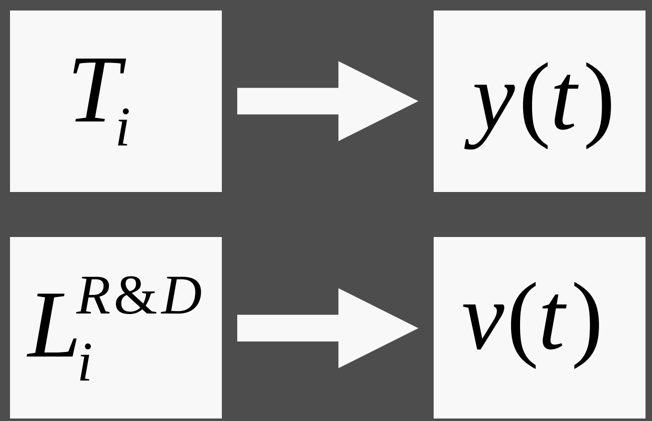
Annual consumption ≈ Labor in production × Technology stock

$$C_i = (\bar{L} - L_i^{R\&D}) T_i^{1/\alpha - 1}$$

Utility

$$U = \sum_i (1-\rho)^i [(1/\alpha - 1)\log T_i + \log(\bar{L} - L_i^{R\&D})]$$

Leader



Leader

maximize

$$\bar{J} = \int_0^\infty e^{-\rho t}[\kappa \log y(t) + \log(\bar{L} - v(t))]dt$$

$$\dot{y}(t) = v(t)y(t)/a$$

$$y(0) = y^0$$

$$v(t) \in [0, \bar{L}]$$

T_i → $y(t)$

$L_i^{R\&D}$ → $v(t)$

Leader

maximize

$$\bar{J} = \int_0^\infty e^{-\rho t}[\kappa \log y(t) + \log(\bar{L} - v(t))]dt$$

$$\dot{y}(t) = v(t)y(t)/a$$

$$y(0) = y^0$$

$$v(t) \in [0, \bar{L}]$$

$$v(t) = \bar{v} = \max[0, \bar{L} - a\alpha\rho/(1-\alpha)]$$

Leader

maximize

$$\bar{J} = \int_0^\infty e^{-\rho t}[\kappa \log y(t) + \log(\bar{L} - v(t))]dt$$

$$\dot{y}(t) = v(t)y(t)/a$$

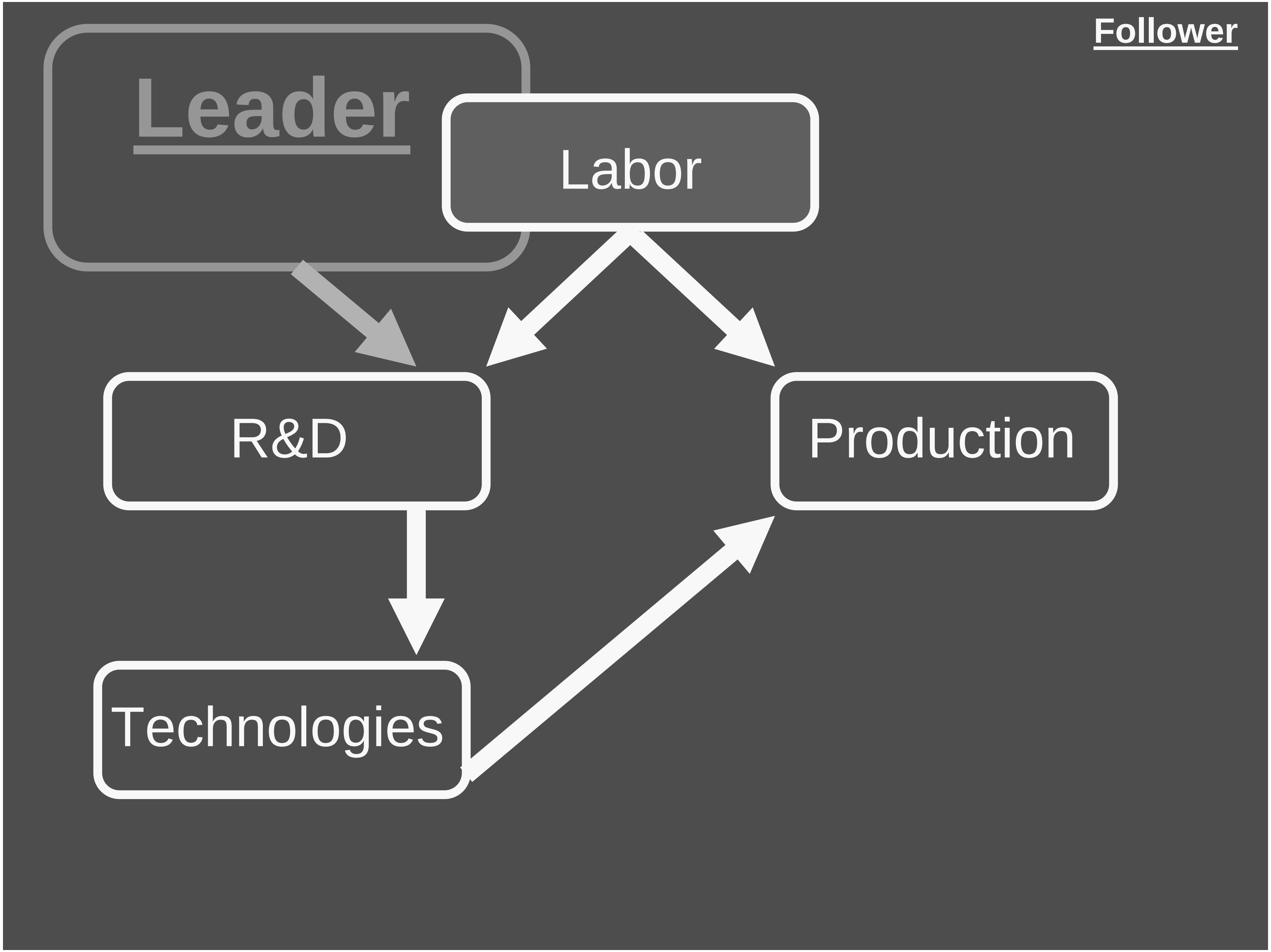
$$y(0) = y^0$$

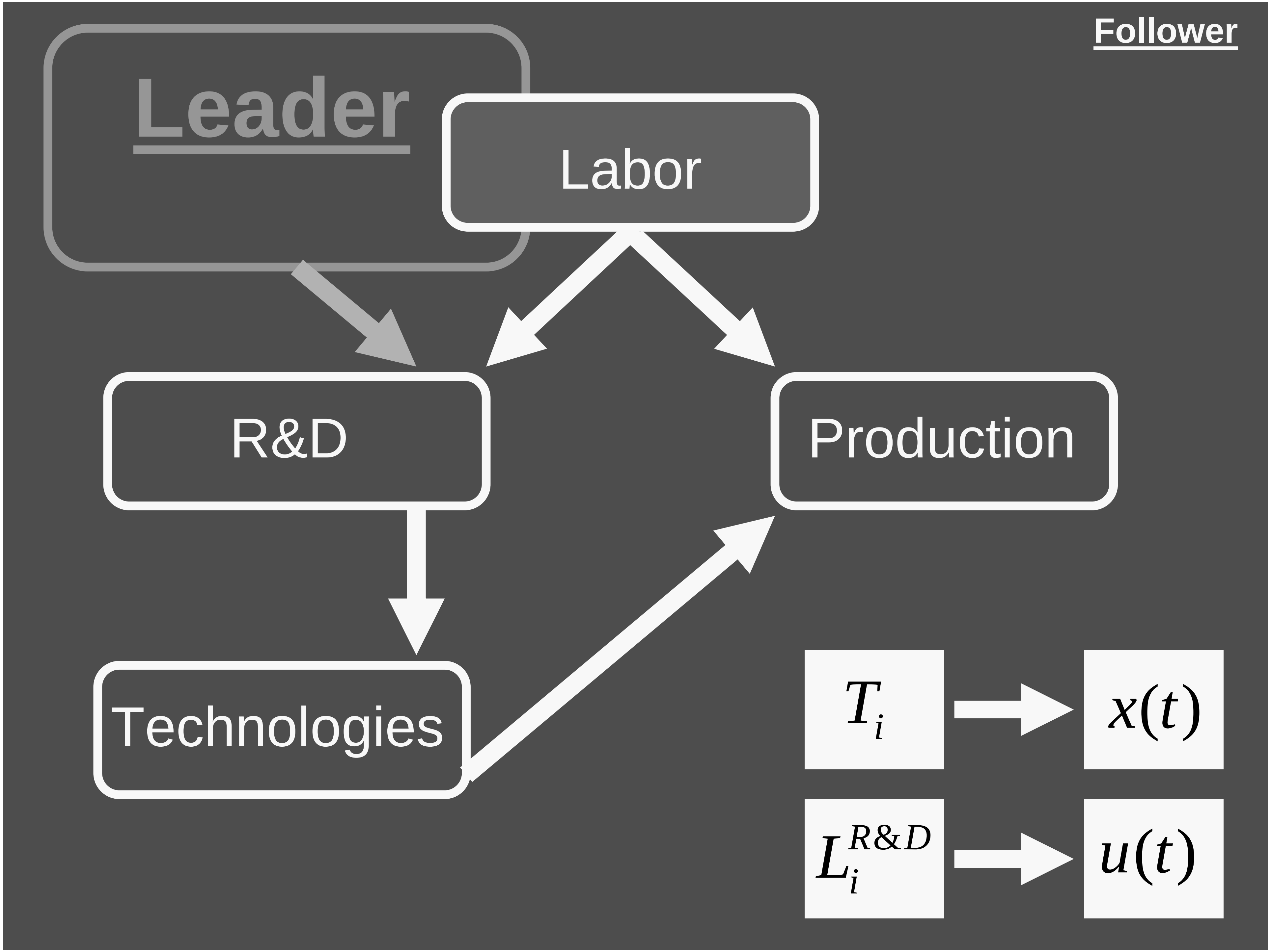
$$v(t) \in [0, \bar{L}]$$

$$v(t) = \bar{v} = \max[0, \bar{L} - a\alpha\rho/(1-\alpha)]$$

$$\dot{y}(t) = \nu y(t)$$

$$\nu = \bar{v}$$


Follower
Leader
Labor
R&D
Production
Technologies



Follower
Leader
Labor
R&D
Production
Technologies
T_i
$x(t)$
$L_i^{R\&D}$
$u(t)$

maximize

$$J = \int_0^\infty e^{-\rho t}[\kappa \log y(t) + \log(b - u(t))]dt$$

$$\dot{x}(t) = u(t)[x(t) + \gamma y(t)]$$

$$x(0) = x^0$$

$$u(t) \in [0, b)$$

maximize

$$J = \int_0^\infty e^{-\rho t}[\kappa \log y(t) + \log(b - u(t))]dt$$

$$\dot{x}(t) = u(t)[x(t) + \gamma y(t)]$$

$$x(0) = x^0$$

$$u(t) \in [0, b)$$

absorption capacity

Follower

maximize

$$J = \int_0^\infty e^{-\rho t}[\kappa \log y(t) + \log(b - u(t))]dt$$

$$\dot{x}(t) = u(t)[x(t) + \gamma y(t)]$$

$$x(0) = x^0$$

$$u(t) \in [0, b]$$

absorption capacity

$$\dot{y}(t) = \nu y(t)$$

$$y(0) = y^0$$

Follower

maximize

$$J = \int_0^\infty e^{-\rho t}[\kappa \log y(t) + \log(b - u(t))]dt$$

$$\dot{x}(t) = u(t)[x(t) + \gamma y(t)]$$

$$x(0) = x^0$$

$$u(t) \in [0, b)$$

$$\dot{y}(t) = \nu y(t)$$

$$y(0) = y^0$$

Follower

maximize

$$J = \int_0^\infty e^{-\rho t}[\kappa \log y(t) + \log(b - u(t))]dt$$

$$\dot{x}(t) = u(t)[x(t) + \gamma y(t)]$$

$$x(0) = x^0$$

$$u(t) \in [0, b)$$

$$\dot{y}(t) = \nu y(t)$$

$$y(0) = y^0$$

$$z(t) = x(t) / y(t)$$

$$z(0) = x^0 / y^0$$

Follower

maximize

$$J = \int_0^\infty e^{-\rho t}[\kappa \log z(t) + \log(b - u(t))]dt$$

$$\dot{z}(t) = u(t)[z(t) + \gamma] - \nu z(t)$$

$$z(0) = z^0$$

$$u(t) \in [0, b)$$

maximize

$$J = \int_0^\infty e^{-\rho t}[\kappa \log z(t) + \log(b - u(t))]dt$$

$$\dot{z}(t) = u(t)[z(t) + \gamma] - \nu z(t)$$

$$z(0) = z^0$$

$$u(t) \in [0, b)$$

maximize

$$J = \int_0^\infty e^{-\rho t}[\kappa \log z(t) + \log(b - u(t))]dt$$

$$\dot{z}(t) = u(t)[z(t) + \gamma] - \nu z(t)$$

$$z(0) = z^0$$

$$u(t) \in [0, b)$$

$$\dot{p}(t) = -[u(t) - \nu - \rho]p(t) - \kappa / z(t)$$

Follower

maximize

$$J = \int_0^\infty e^{-\rho t}[\kappa \log z(t) + \log(b - u(t))]dt$$

$$\dot{z}(t) = u(t)[z(t) + \gamma] - \nu z(t)$$

$$z(0) = z^0$$

$$u(t) \in [0, b)$$

$$\dot{p}(t) = -[u(t) - \nu - \rho]p(t) - \kappa / z(t)$$

$$u(t) = \arg\max\{\tilde{M}(z(t), p(t), u) : u \in [0, b)\}$$

maximize

$$J = \int_0^\infty e^{-\rho t}[\kappa \log z(t) + \log(b - u(t))]dt$$

$$\dot{z}(t) = u(t)[z(t) + \gamma] - \nu z(t)$$

$$z(0) = z^0$$

$$u(t) \in [0, b)$$

$$\dot{p}(t) = -[u(t) - \nu - \rho]p(t) - \kappa / z(t)$$

$$u(t) = \arg\max\{\tilde{M}(z(t), p(t), u) : u \in [0, b)\}$$

$\tilde{M}(z(t), p(t), u)$ current Hamilton-Pontryagin function

Follower

maximize

$$J = \int_0^{\infty} e^{-\rho t}[\kappa \log z(t) + \log(b - u(t))]dt$$

$$\dot{z}(t) = u(t)[z(t) + \gamma] - \nu z(t)$$

$$z(0) = z^0$$

$$u(t) \in [0, b)$$

$$\dot{p}(t) = -[u(t) - \nu - \rho]p(t) - \kappa / z(t)$$

$$u(t) = \arg\max\{\tilde{M}(z(t), p(t), u) : u \in [0, b)\}$$

$\tilde{M}(z(t), p(t), u)$ current Hamilton-Pontryagin function

Follower

maximize

$$J = \int_0^{\infty} e^{-\rho t}[\kappa \log z(t) + \log(b - u(t))]dt$$

$$\dot{z}(t) = u(t)[z(t) + \gamma] - \nu z(t)$$

$$z(0) = z^0$$

$$u(t) \in [0, b)$$

$$\dot{p}(t) = -[u(t) - \nu - \rho]p(t) - \kappa / z(t)$$

$$u(t) = \arg\max\{\tilde{M}(z(t), p(t), u) : u \in [0, b)\}$$

$\tilde{M}(z(t), p(t), u)$ current Hamilton-Pontryagin function

Follower

maximize

$$J = \int_0^\infty e^{-\rho t}[\kappa \log z(t) + \log(b - u(t))]dt$$

$$\dot{z}(t) = u(t)[z(t) + \gamma] - \nu z(t)$$

$$z(0) = z^0$$

$$u(t) \in [0, b)$$

$$\dot{p}(t) = -[u(t) - \nu - \rho]p(t) - \kappa / z(t)$$

$$u(t) = \arg\max\{\tilde{M}(z(t), p(t), u) : u \in [0, b)\}$$

$\tilde{M}(z(t), p(t), u)$ current Hamilton-Pontryagin function

Follower

$$\dot{z}(t) = \varphi_1(z(t), p(t))$$
$$\dot{p}(t) = \varphi_2(z(t), p(t))$$

Hamiltonian system

$$\dot{z}(t) = u(t)[z(t) + \gamma] - \nu z(t)$$
$$z(0) = z^0$$
$$u(t) \in [0, b)$$

$$\dot{p}(t) = -[u(t) - \nu - \rho]p(t) - \kappa / z(t)$$
$$u(t) = \arg\max\{\tilde{M}(z(t), p(t), u) : u \in [0, b)\}$$

$\tilde{M}(z(t), p(t), u)$ current Hamilton-Pontryagin function

Follower

$$\dot{z}(t) = \varphi_1(z(t), p(t))$$
$$\dot{p}(t) = \varphi_2(z(t), p(t))$$

Hamiltonian system

$$\dot{z}(t) = u(t)[z(t) + \gamma] - \nu z(t)$$
$$z(0) = z^0$$
$$u(t) \in [0, b)$$

$$\dot{p}(t) = -[u(t) - \nu - \rho]p(t) - \kappa / z(t)$$
$$u(t) = \arg\max\{\tilde{M}(z(t), p(t), u) : u \in [0, b)\}$$

$\tilde{M}(z(t), p(t), u)$ current Hamilton-Pontryagin function

Follower

$$\dot{z}(t) = \varphi_1(z(t), p(t))$$
$$\dot{p}(t) = \varphi_2(z(t), p(t))$$

Hamiltonian system

$$\dot{z}(t) = \varphi_1(z(t), p(t))$$
$$\dot{p}(t) = \varphi_2(z(t), p(t))$$

Hamiltonian system

Pontryagin maximum principle

$$\dot{z}(t) = \varphi_1(z(t), p(t))$$
$$\dot{p}(t) = \varphi_2(z(t), p(t))$$

Hamiltonian system

Pontryagin maximum principle

If $z(t)$ is optimal, then there is a positive $p(t)$ such that $(z(t), p(t))$ solves the Hamiltonian system and

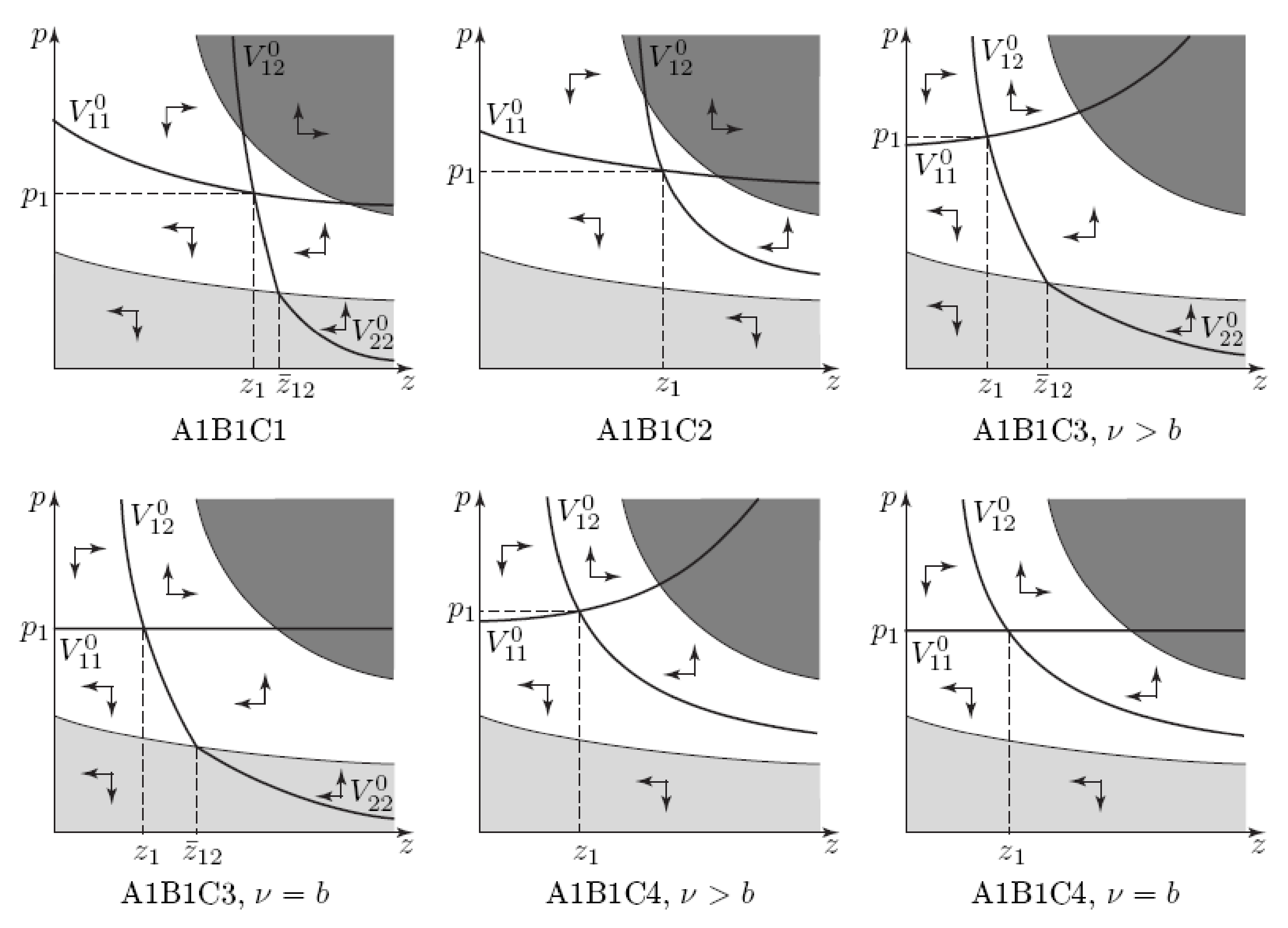
$$z(t)\,p(t) \leq \frac{\kappa}{\rho}$$

Vector field of the Hamiltonian system

<u>**Follower**</u>

Follower

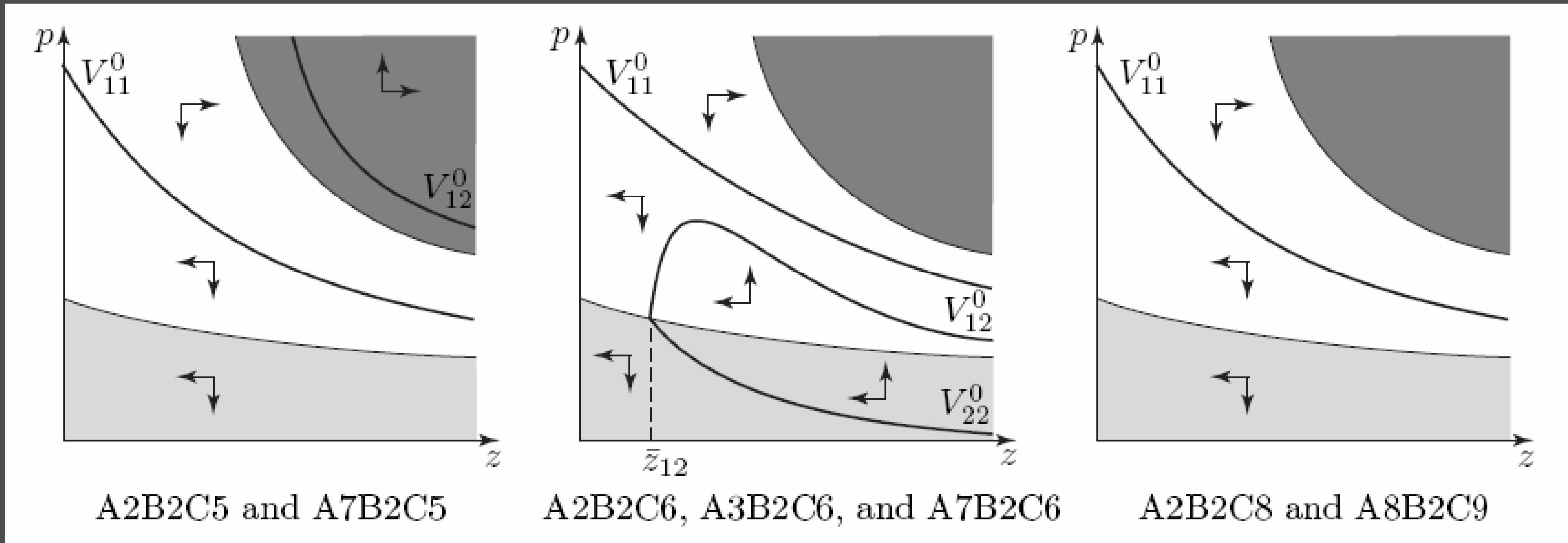
Vector field of the Hamiltonian system



Regular non-degenerate cases

Vector field of the Hamiltonian system

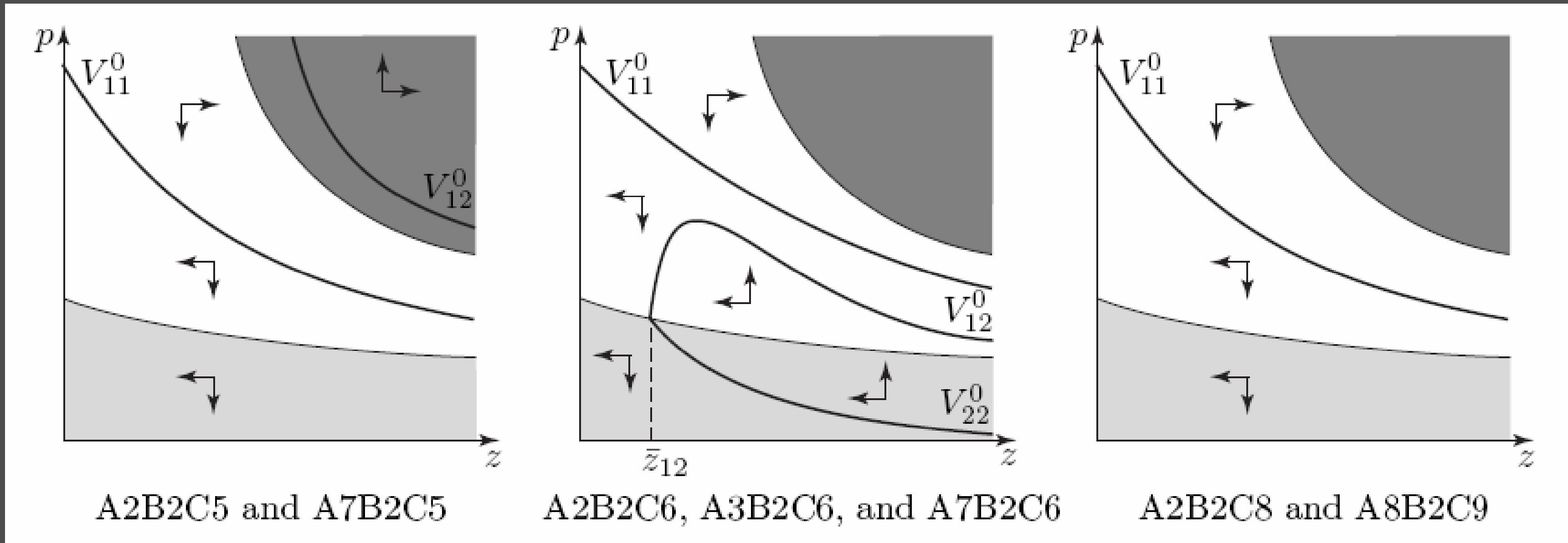
Follower



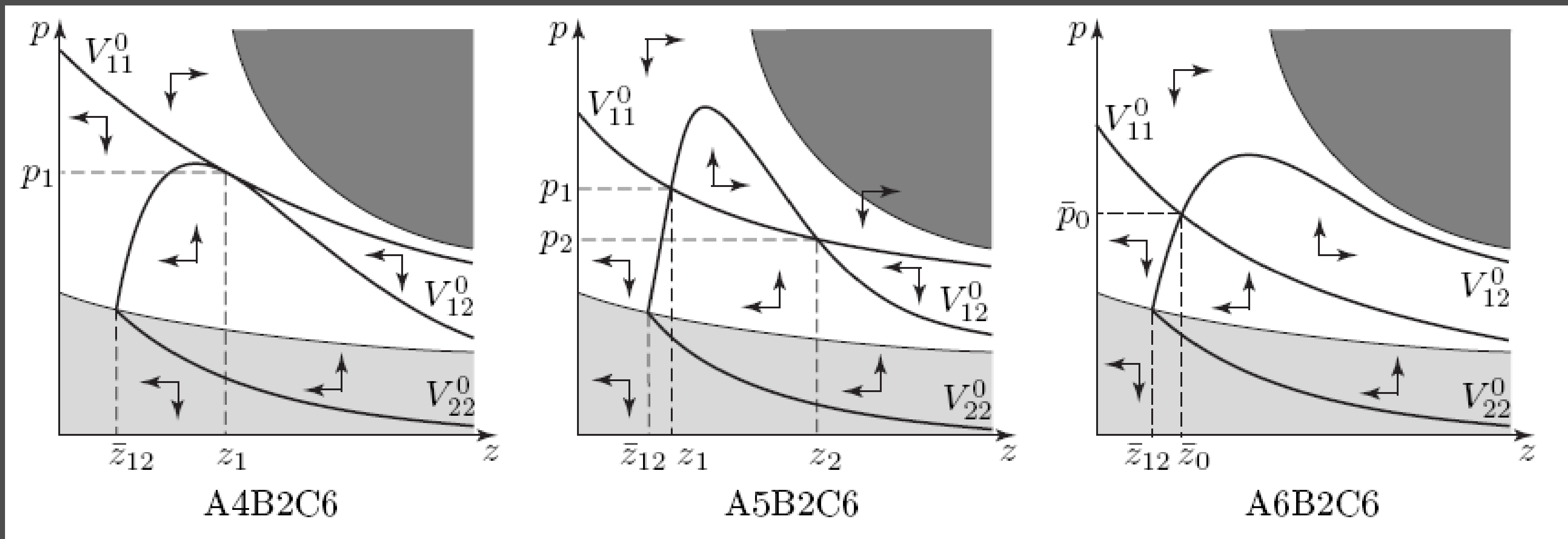
Degenerate cases

Vector field of the Hamiltonian system

Follower



Degenerate cases



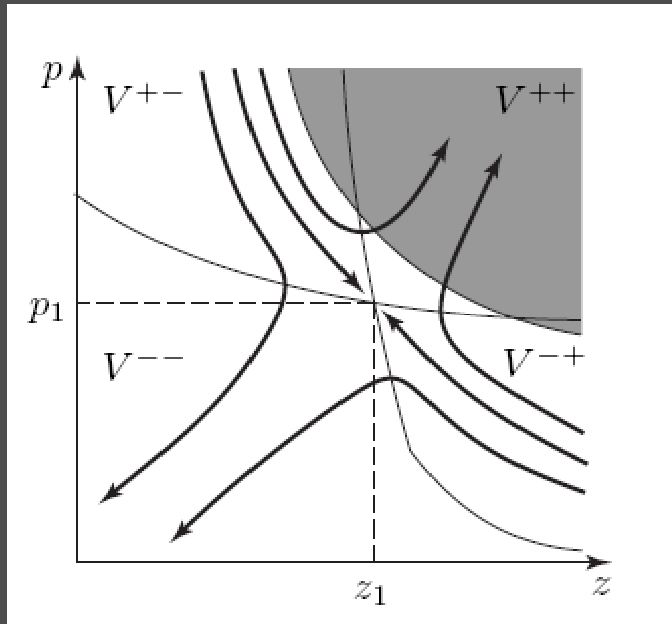
Singular non-degenerate cases

Solutions in regular non-degenerate cases

Follower

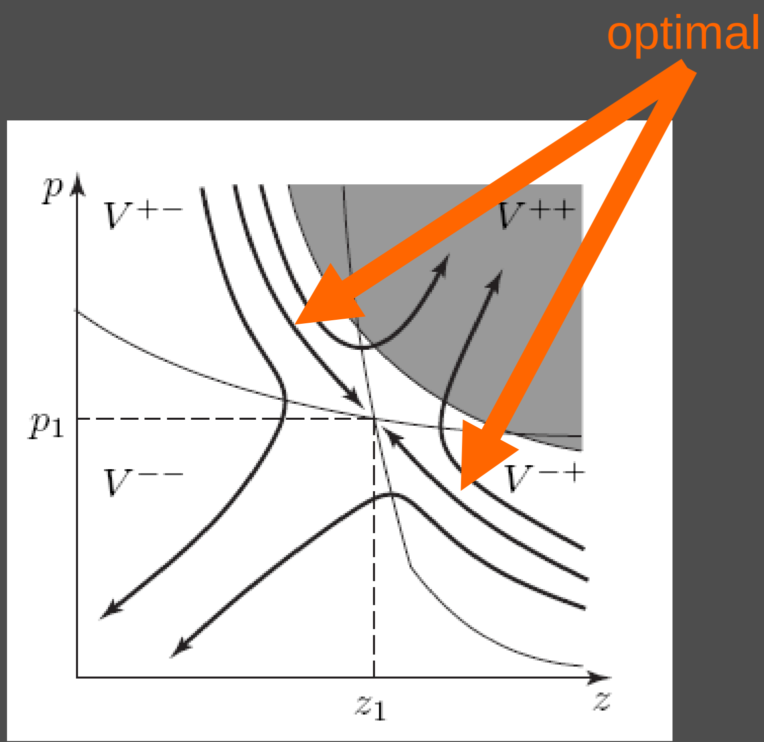
Follower

Solutions in regular non-degenerate cases



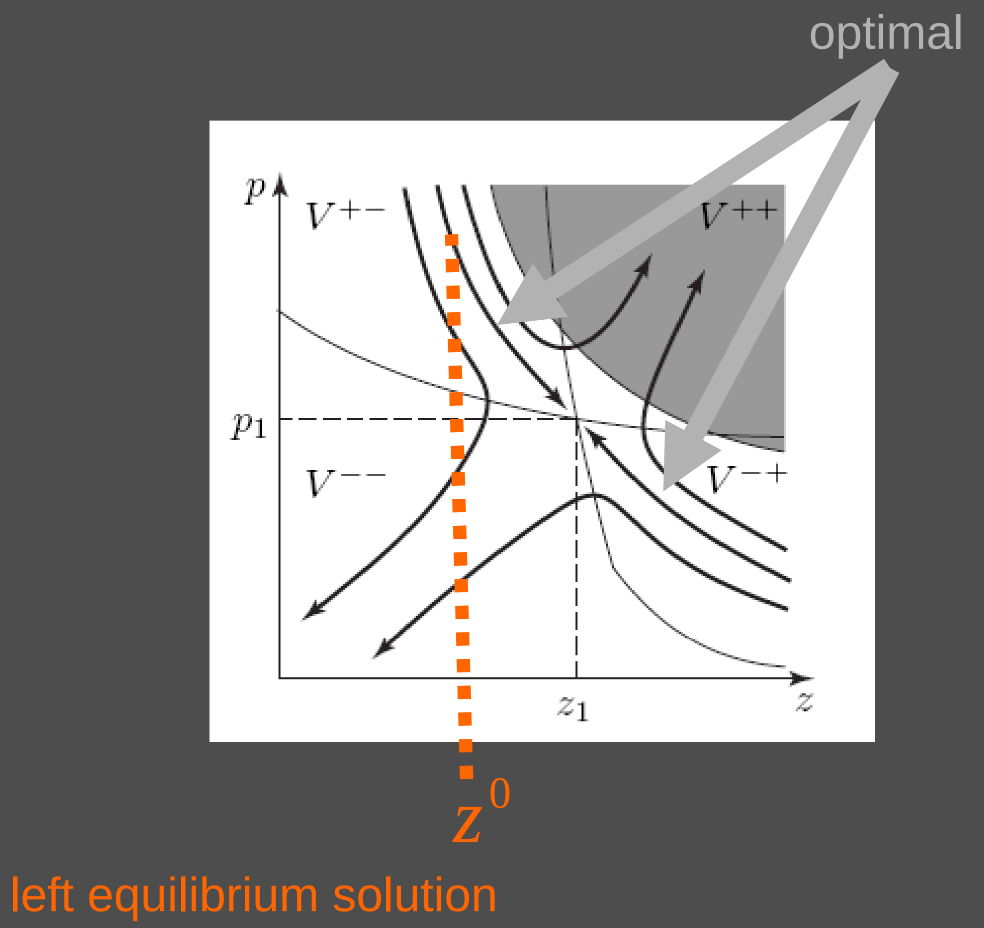
Solutions in regular non-degenerate cases

Follower

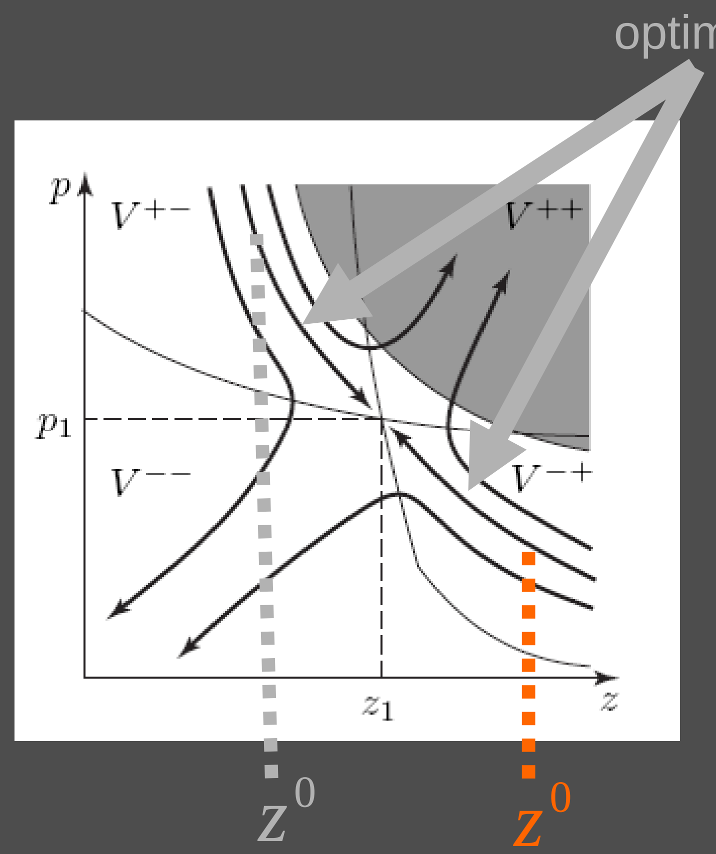


Solutions in regular non-degenerate cases

Follower



Solutions in regular non-degenerate cases



left equilibrium solution

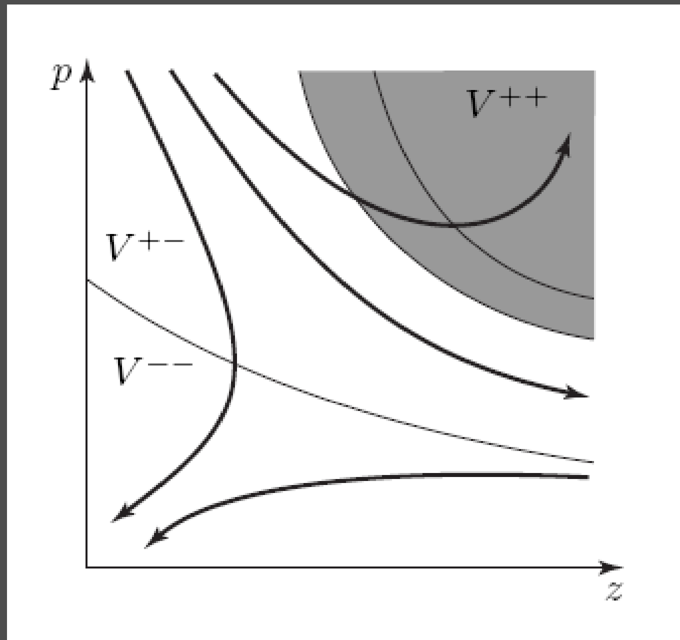
right equilibrium solution

Follower

Solutions in degenerate cases

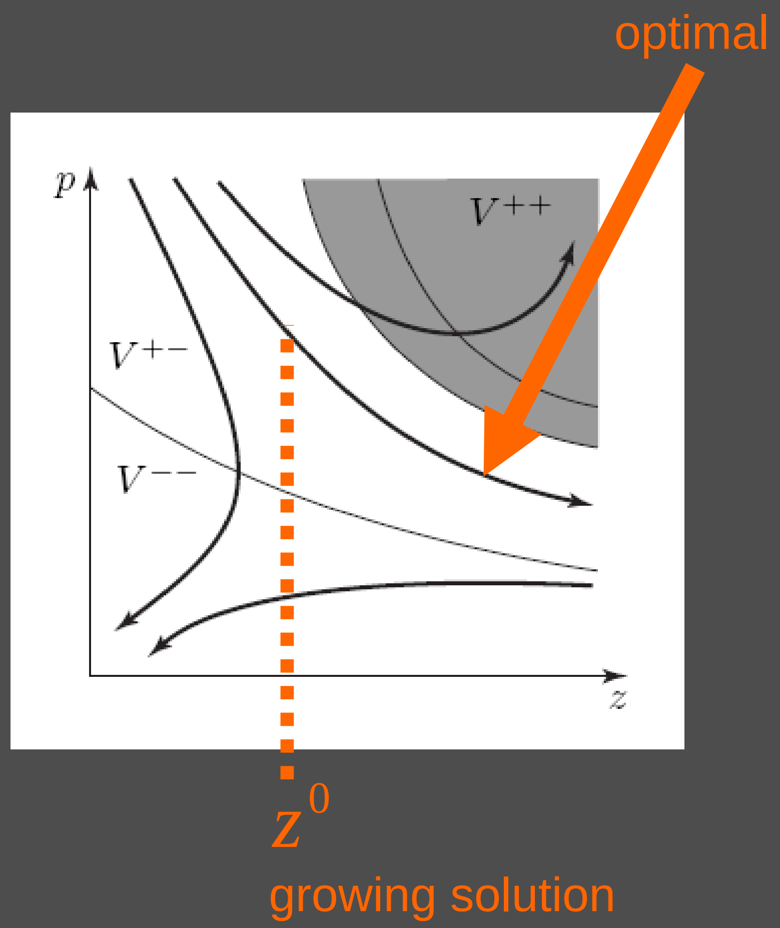
Follower

Solutions in degenerate cases



Solutions in degenerate cases

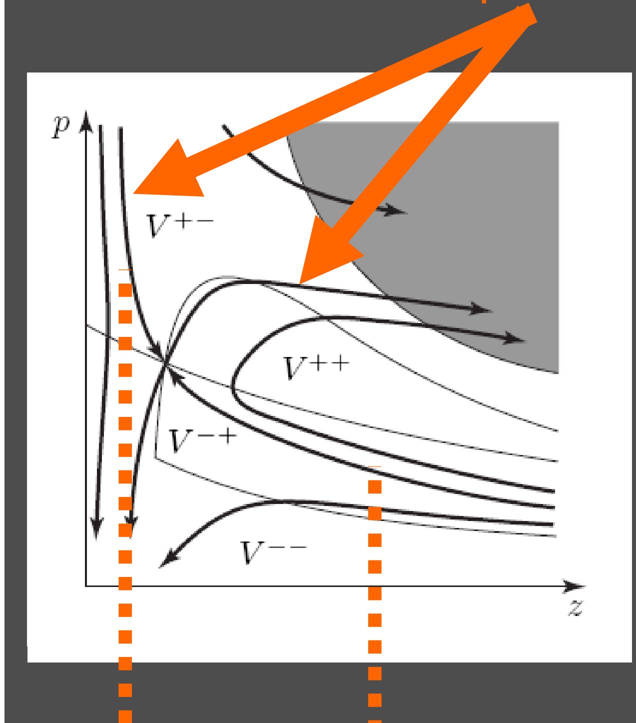
Follower



Solutions in singular non-degenerate cases

Solutions in singular non-degenerate cases

optimal



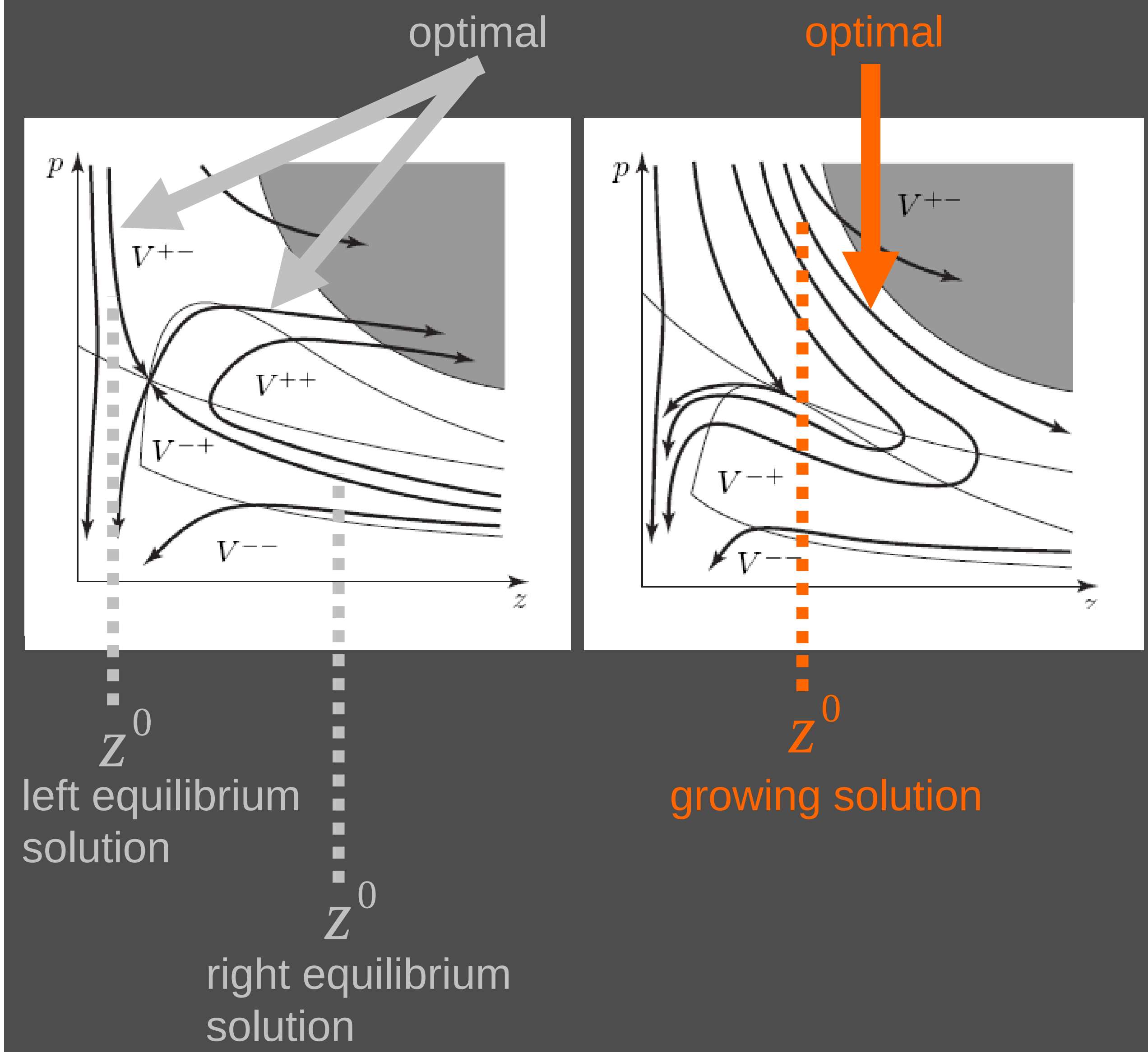
z^0

left equilibrium solution

z^0

right equilibrium solution

Solutions in singular non-degenerate cases



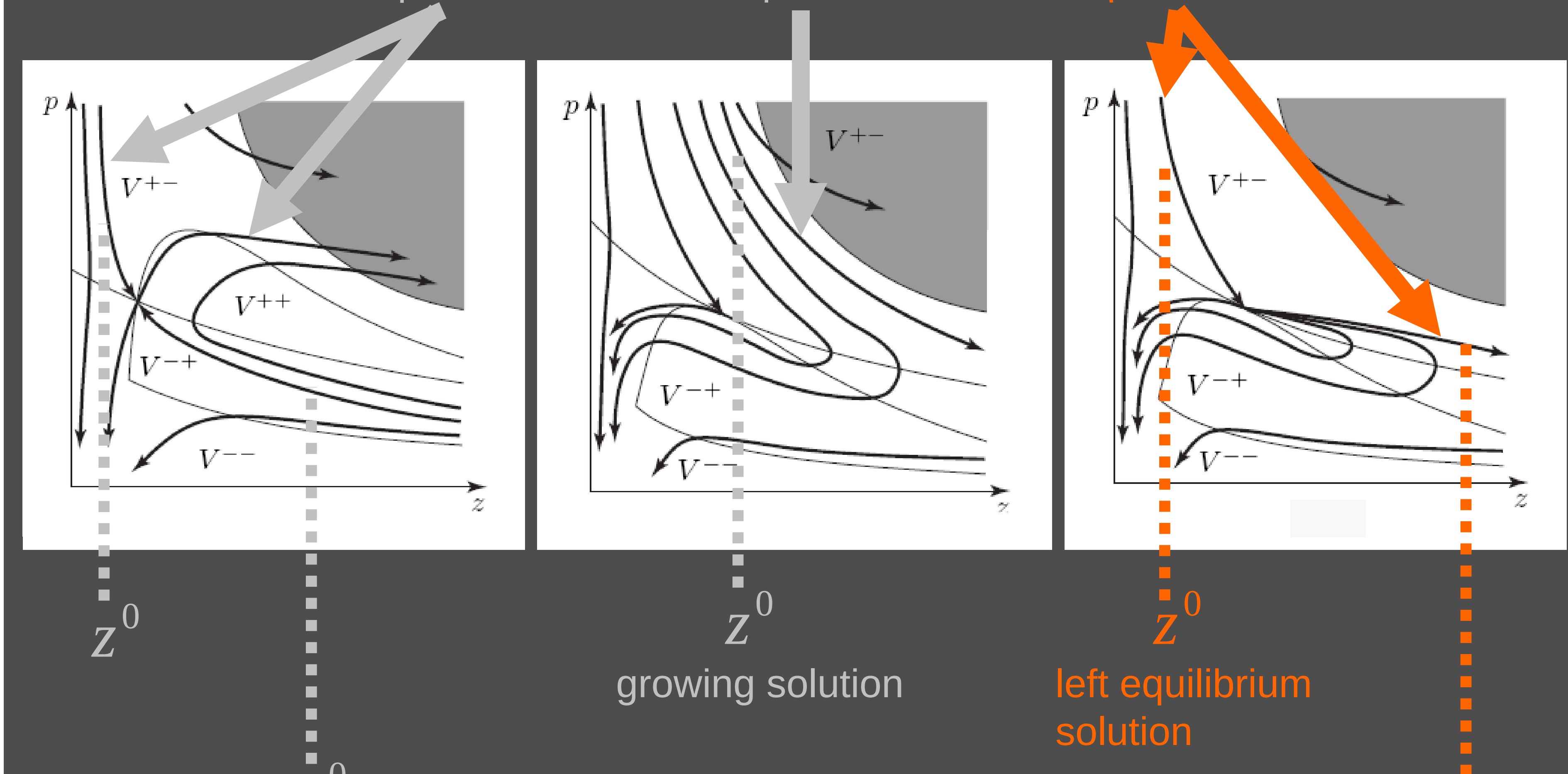
Follower

Solutions in singular non-degenerate cases

optimal

optimal

optimal



z^0

z^0

right equilibrium solution

z^0

growing solution

z^0

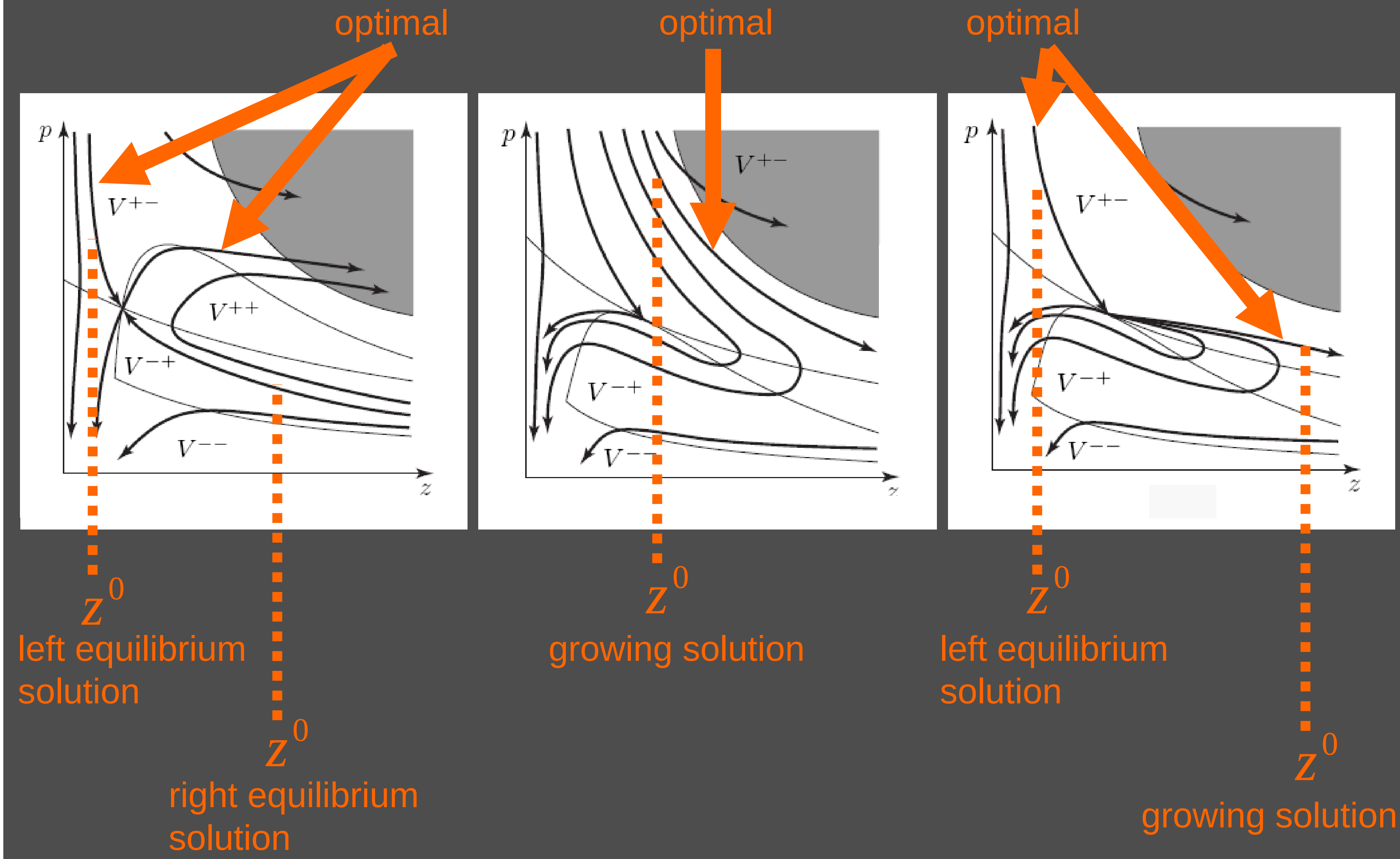
left equilibrium solution

z^0

growing solution

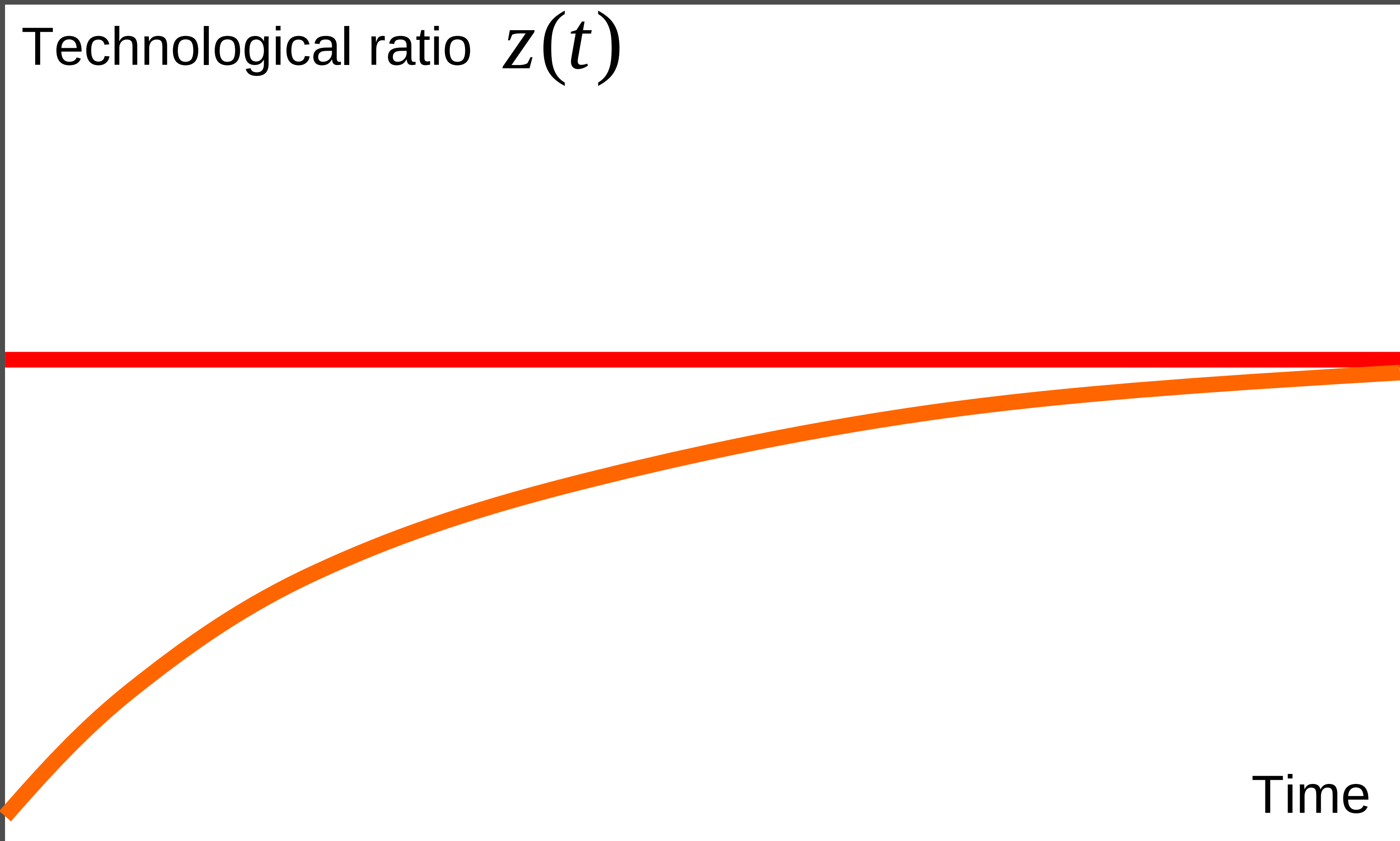
Solutions in singular non-degenerate cases

Follower

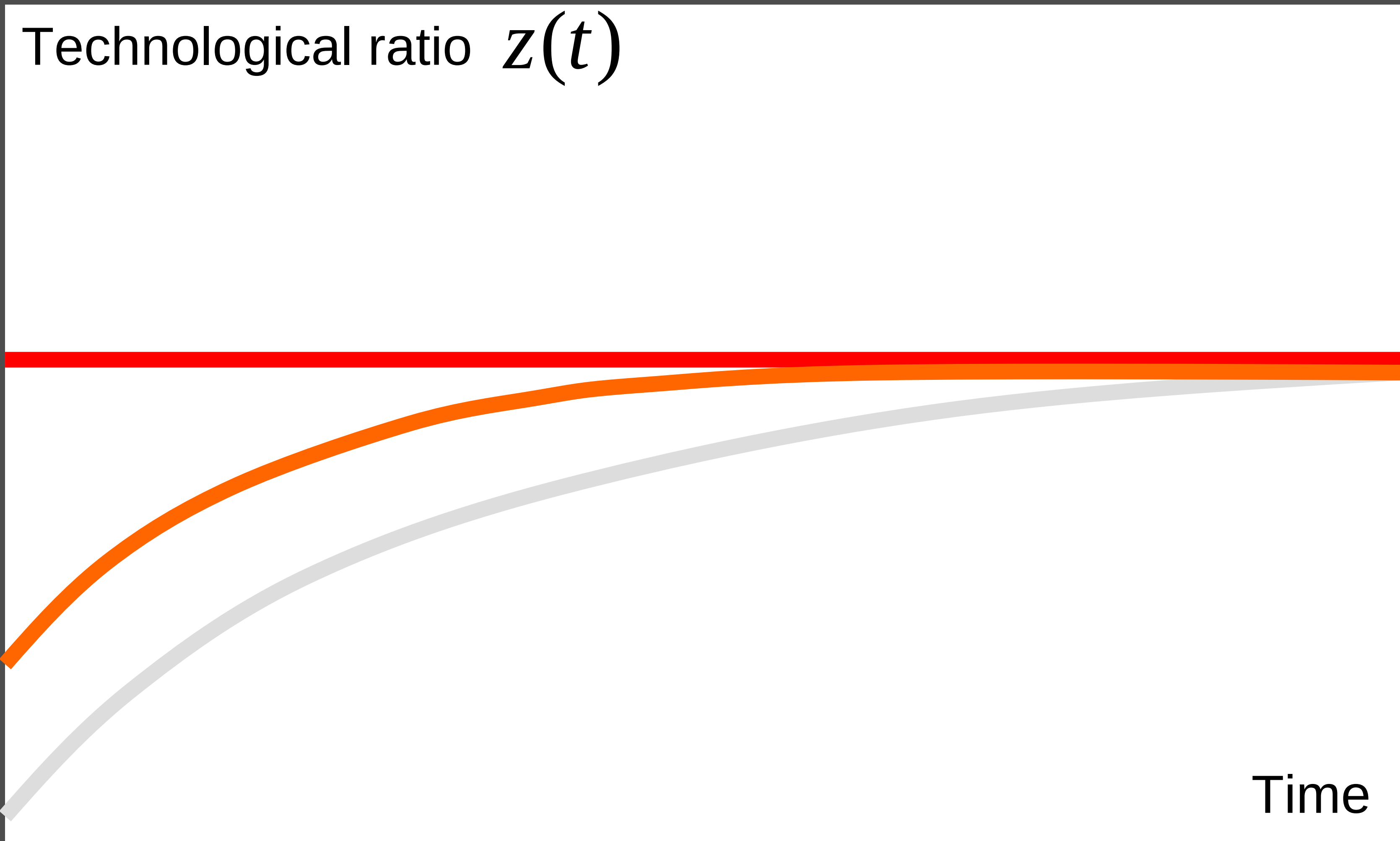


Equilibrium solution: catching up

Equilibrium solution: catching up

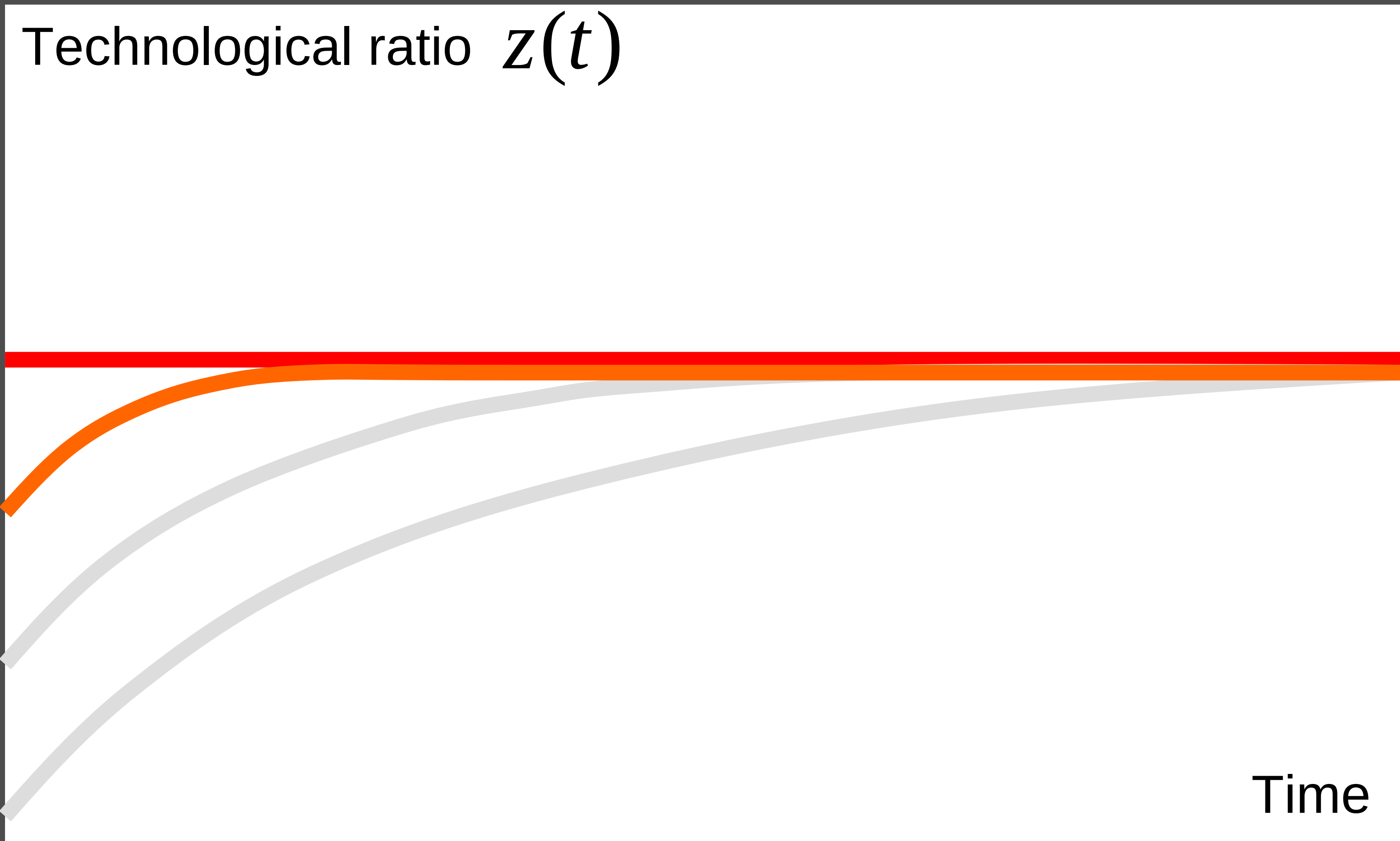


Equilibrium solution: catching up

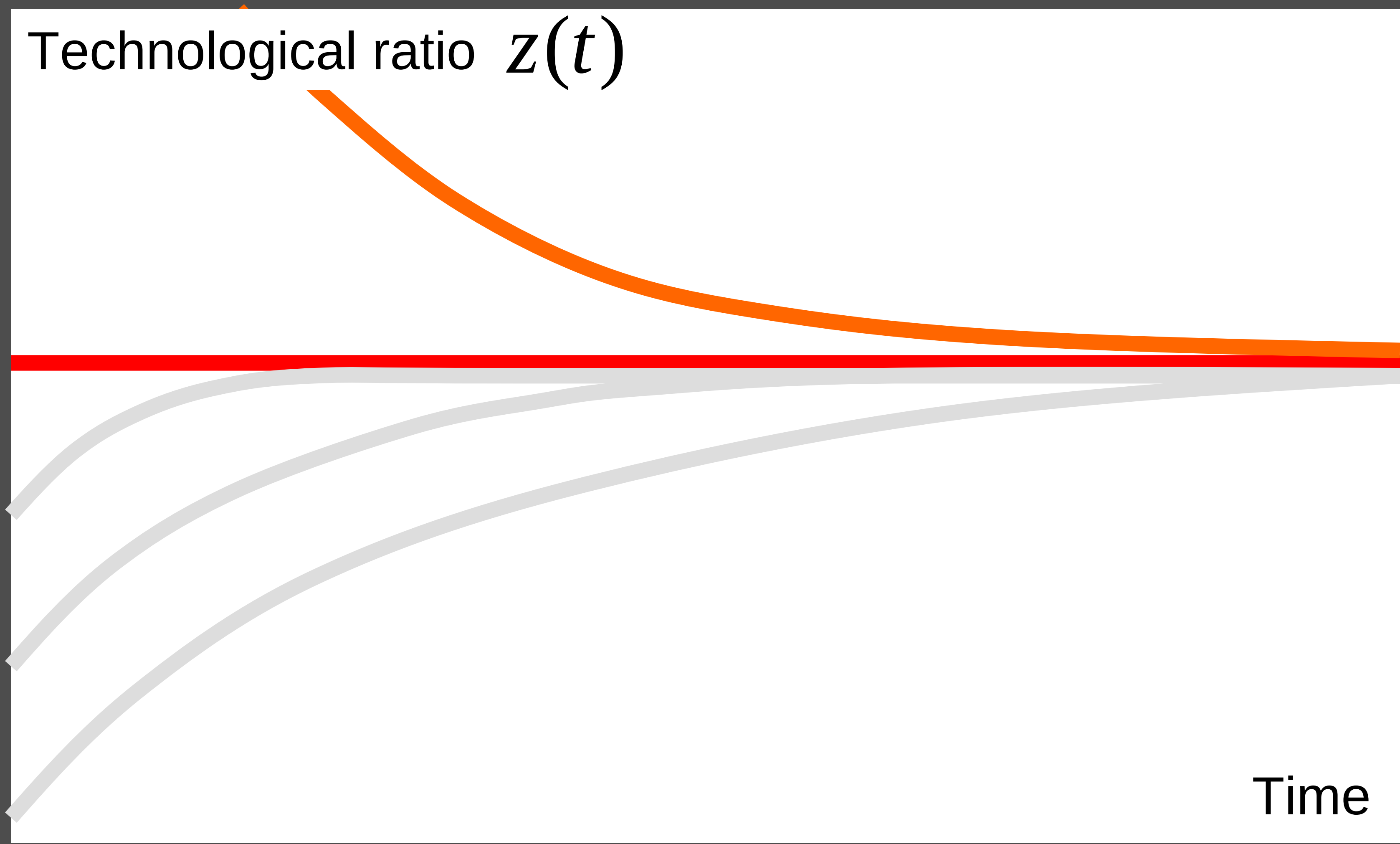


Equilibrium solution: catching up

Follower

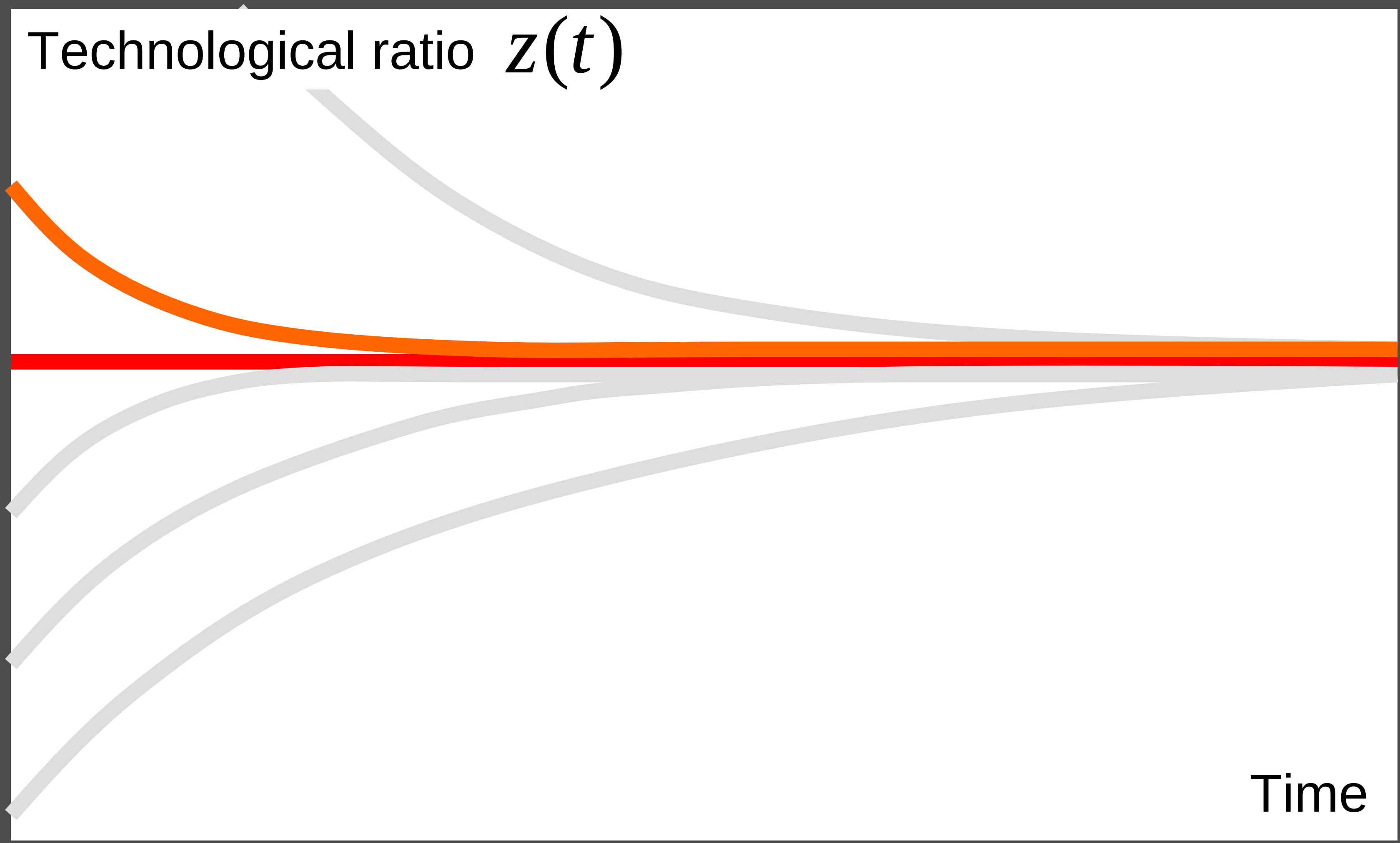


Equilibrium solution: catching up



Equilibrium solution: catching up

Follower

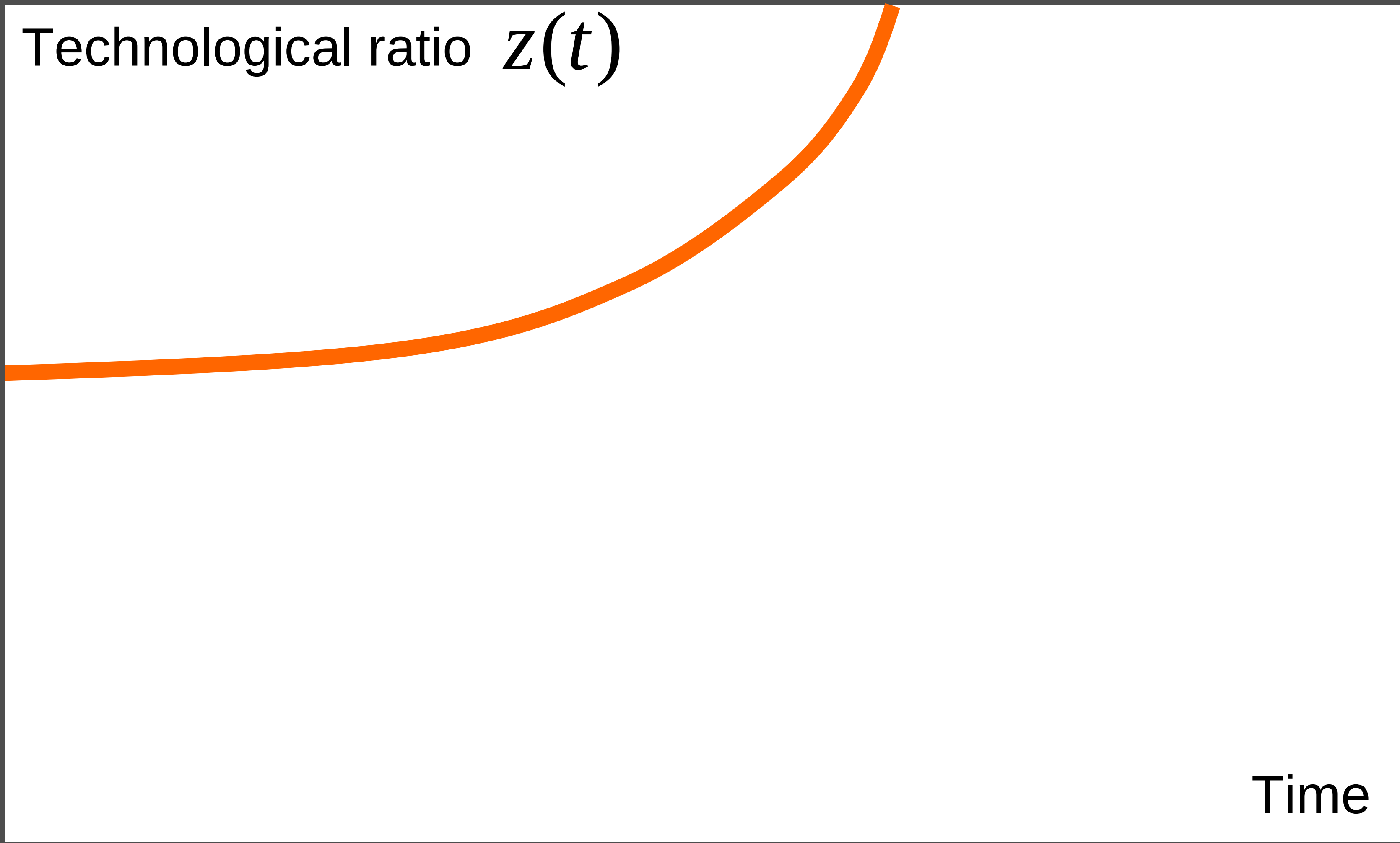


Growing solution: overtaking

Follower

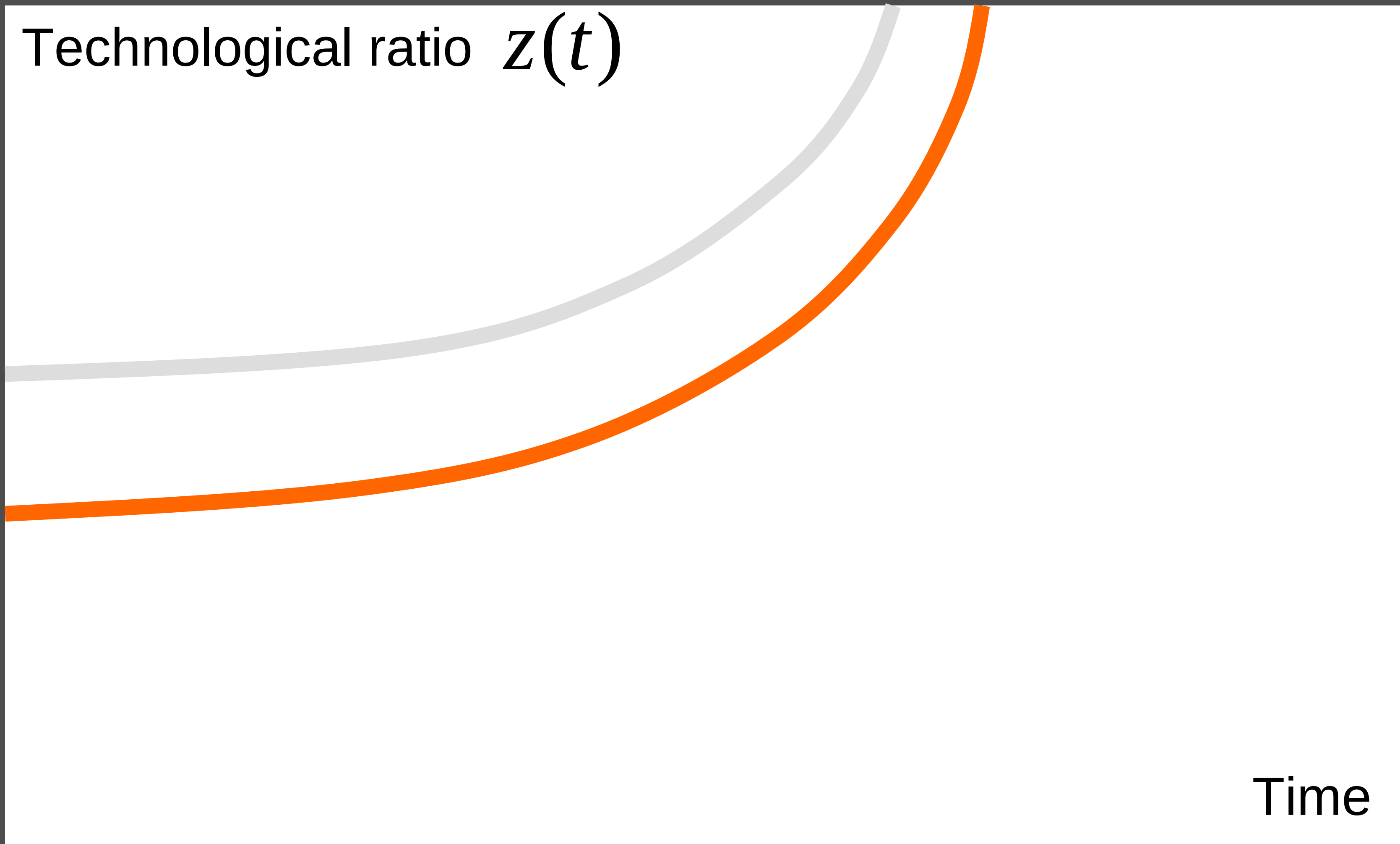
Follower

Growing solution: overtaking



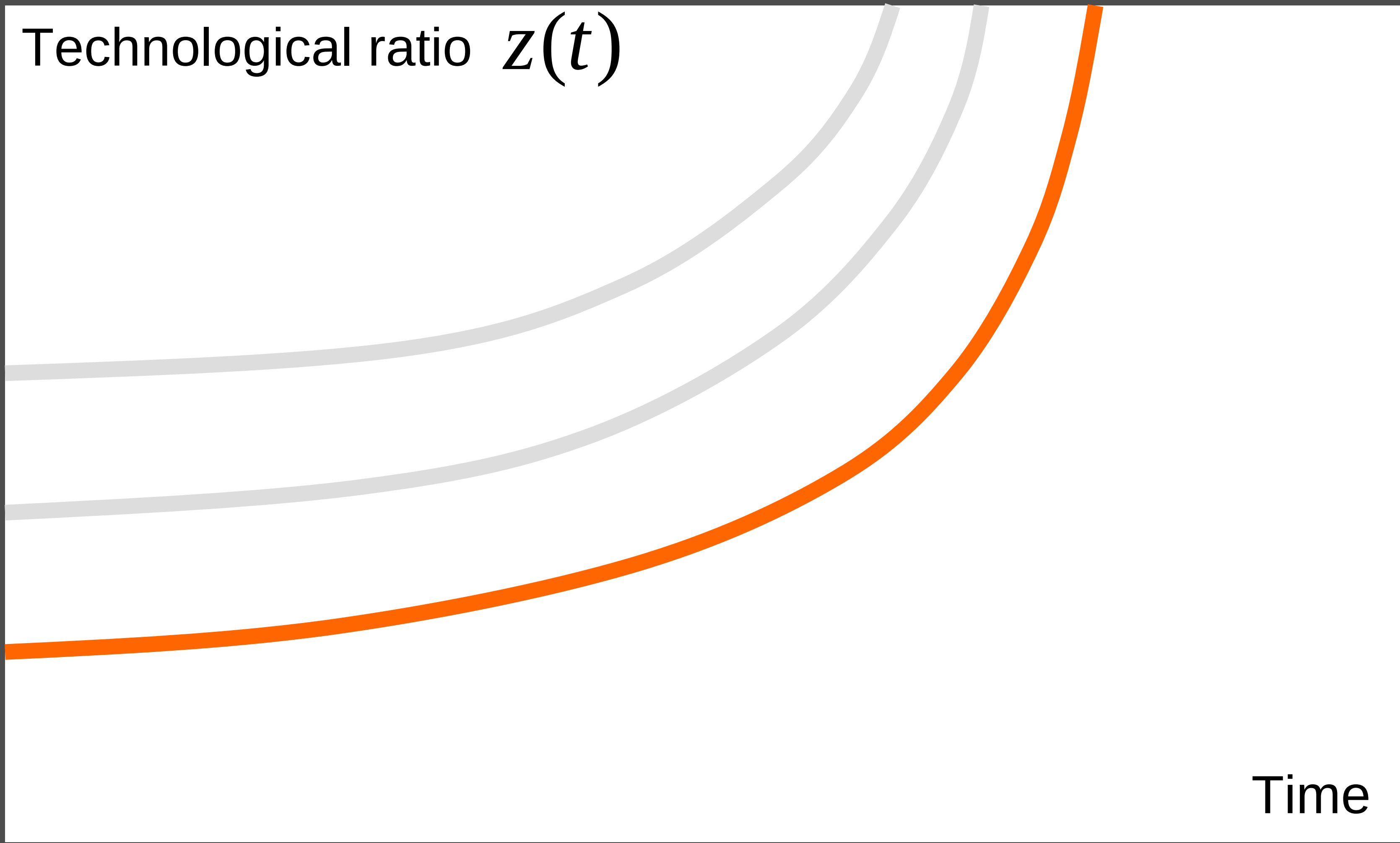
Growing solution: overtaking

Follower



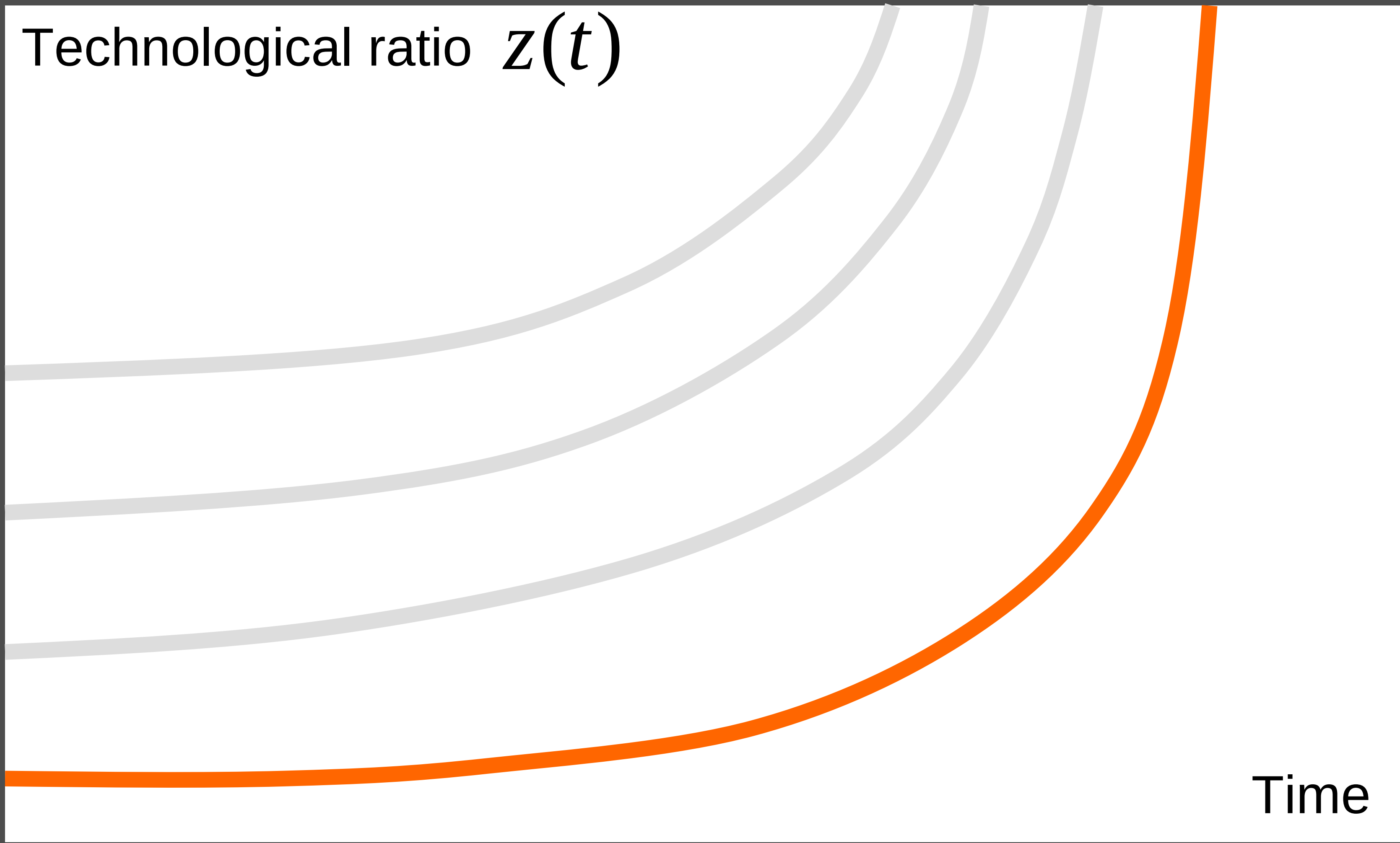
Growing solution: overtaking

Follower



Follower

Growing solution: overtaking

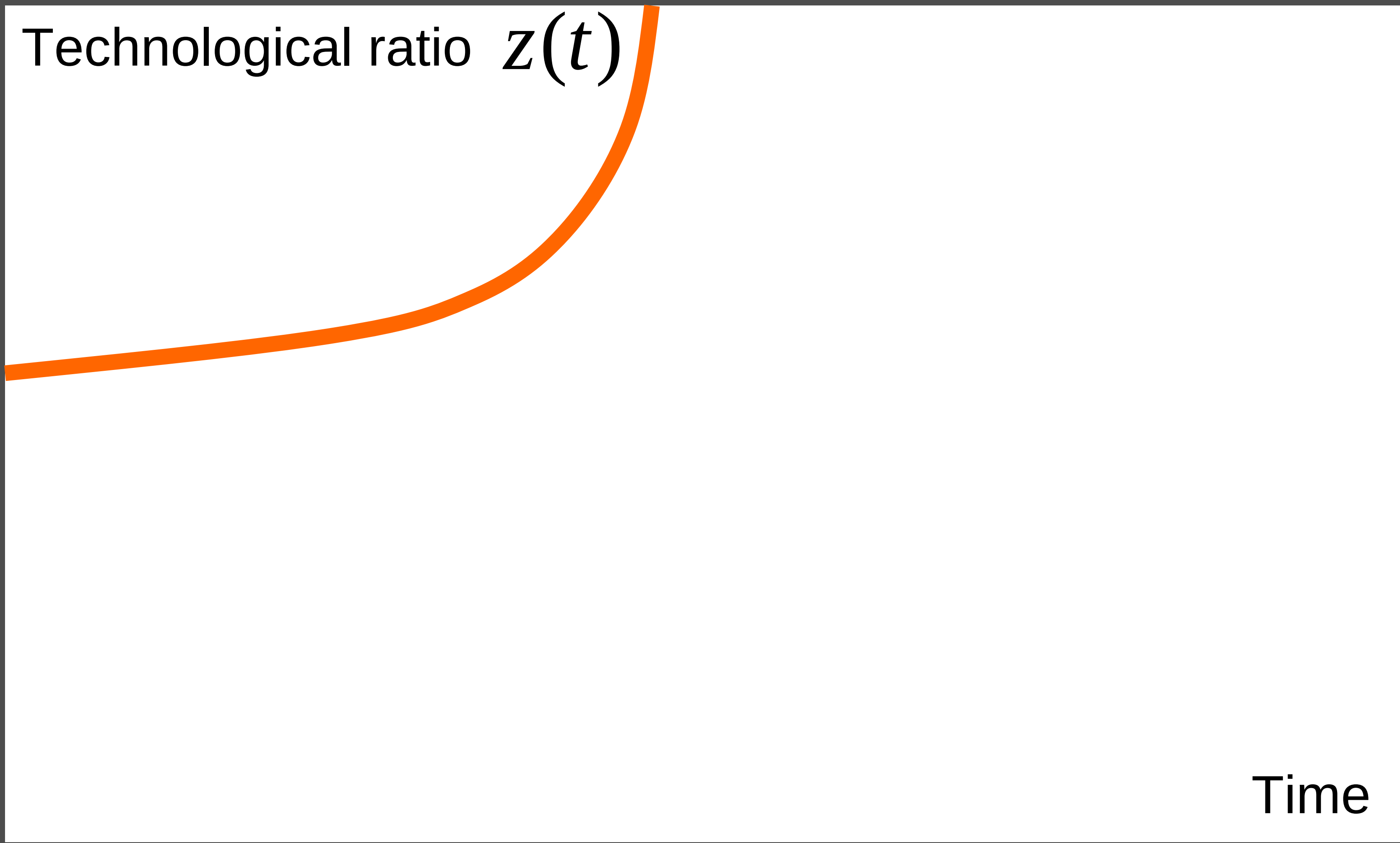


Sensitivity in ρ

$$J = \int_0^{\infty} e^{-\rho t}[\kappa \log y(t) + \log(b - u(t))]dt$$

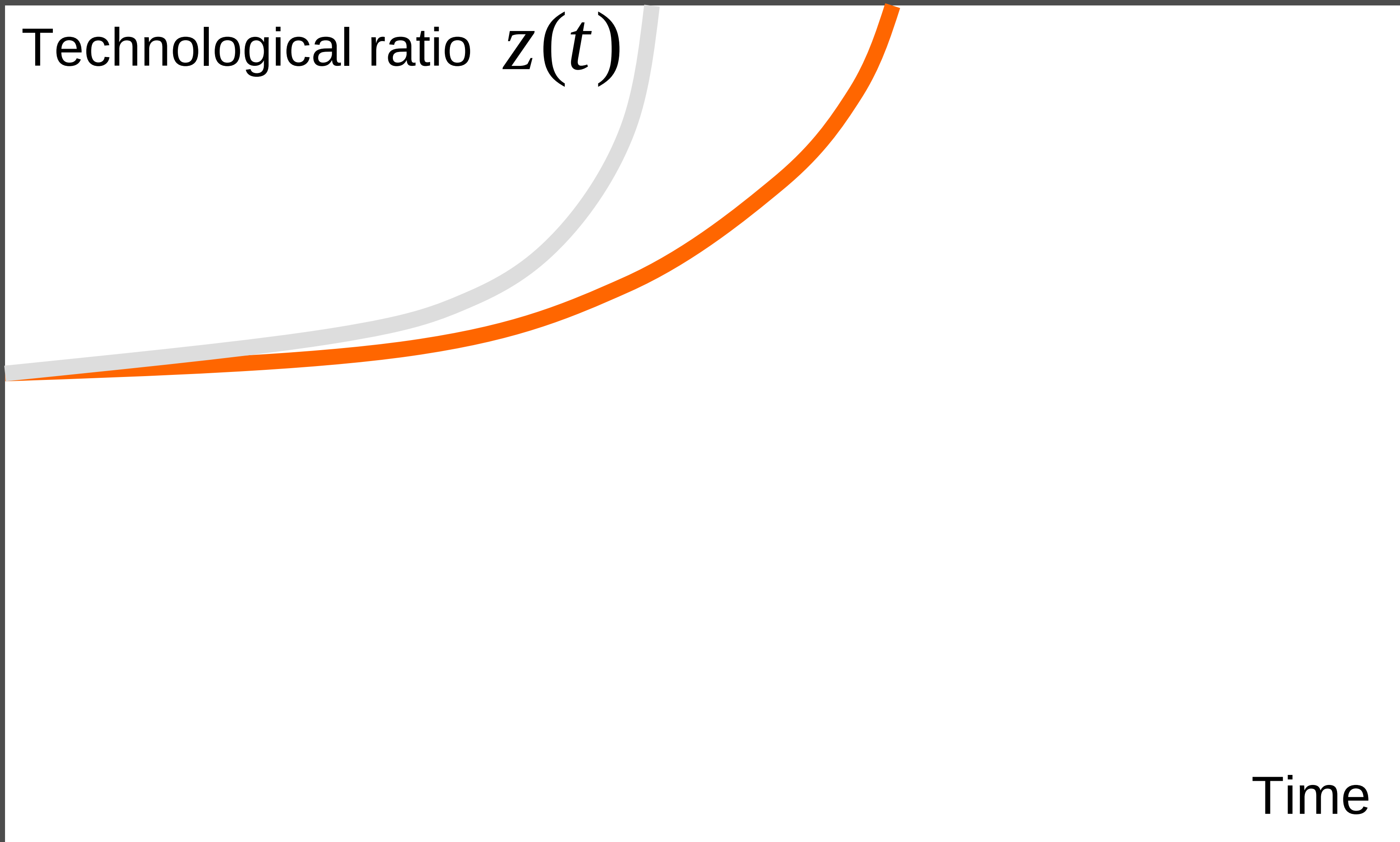
Follower

Sensitivity in ρ



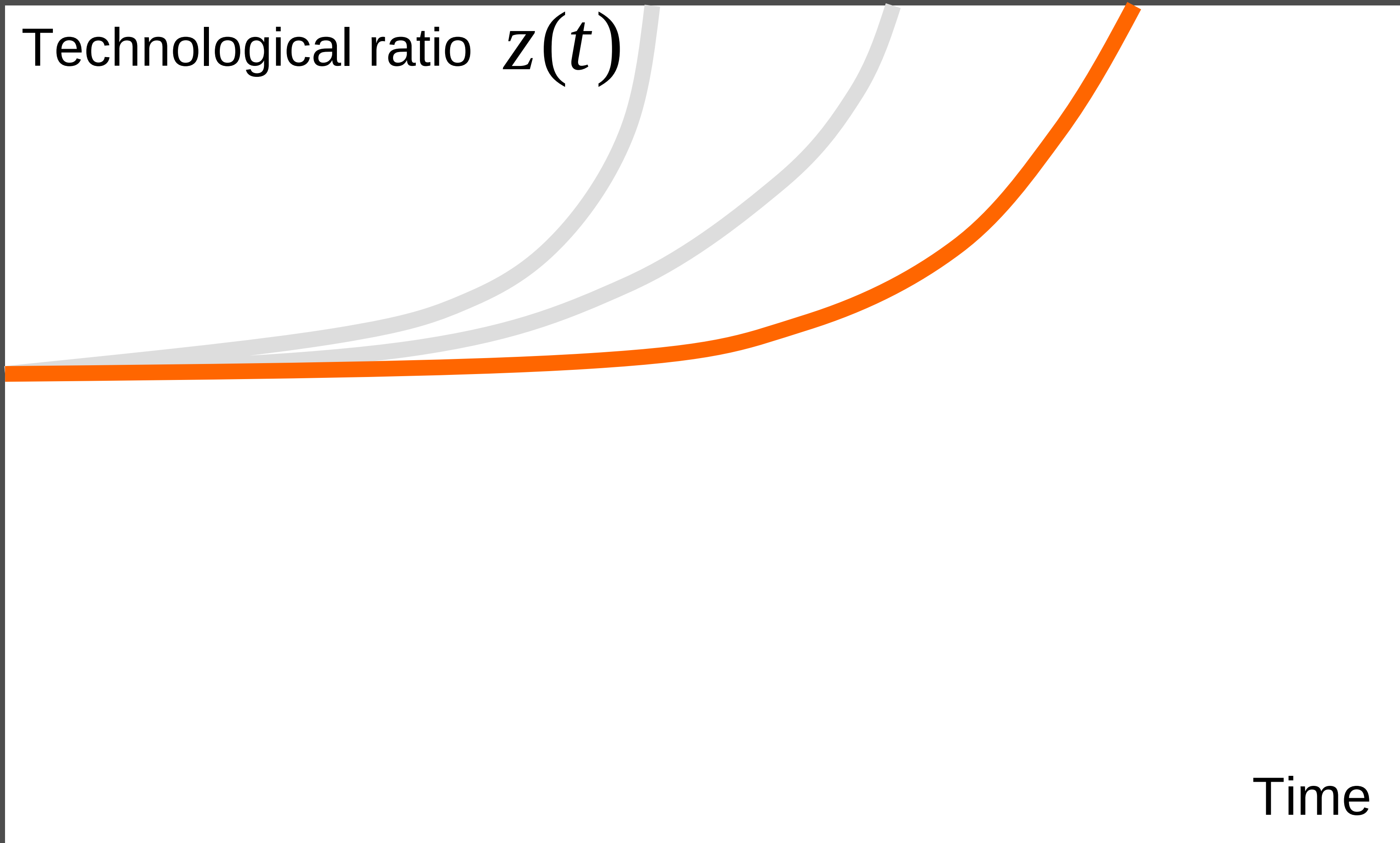
Follower

Sensitivity in ρ



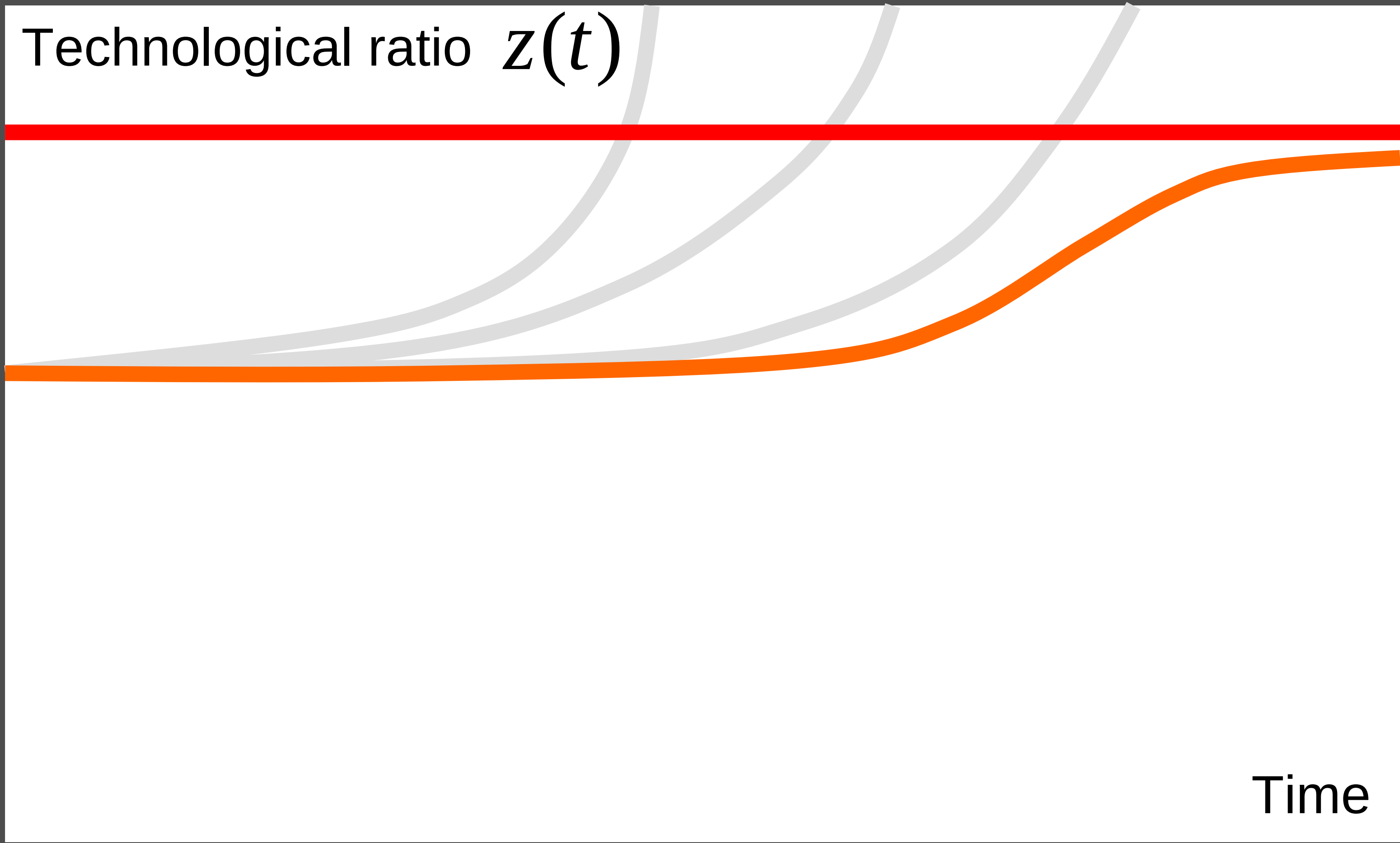
Follower

Sensitivity in ρ

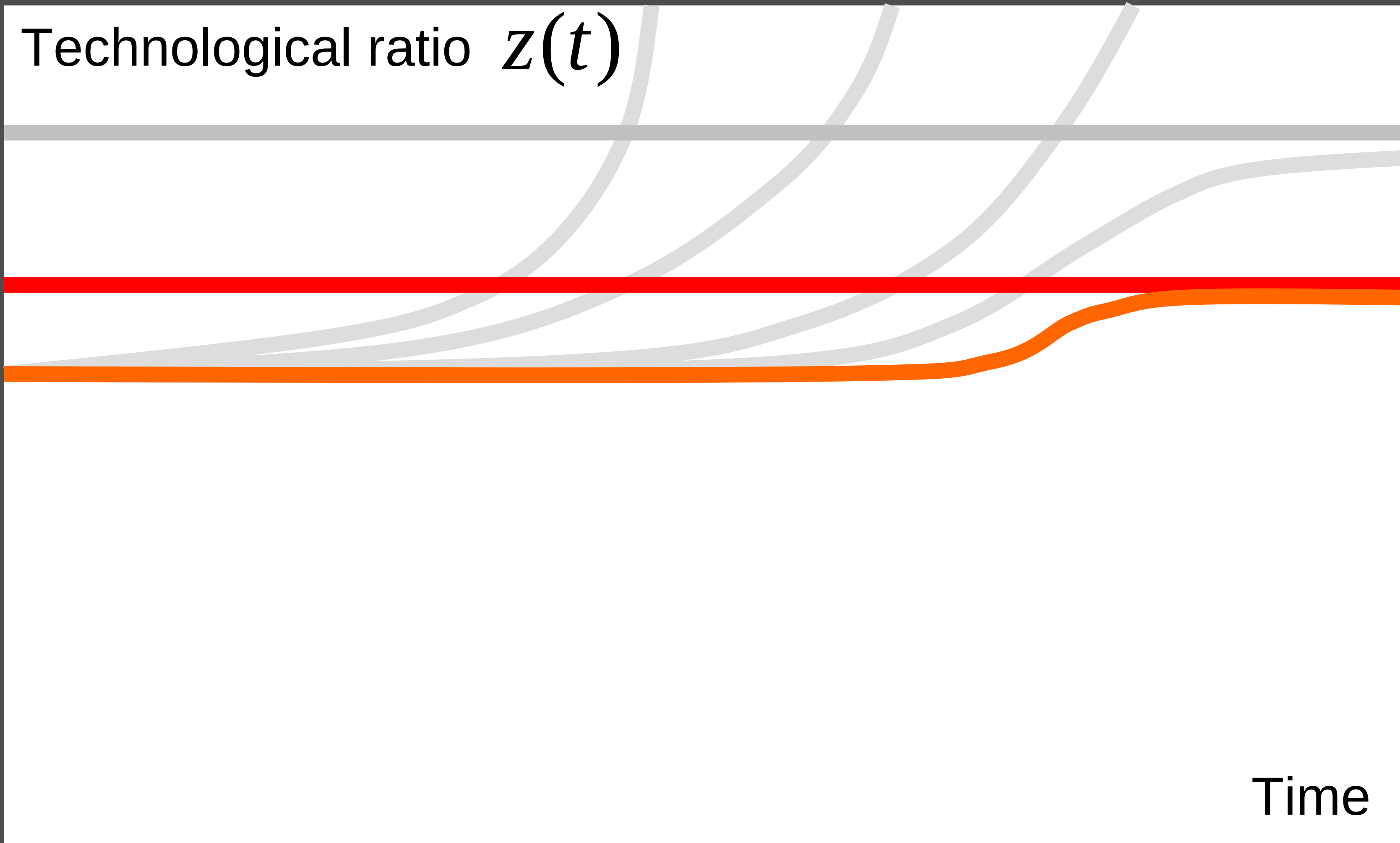


Follower

Sensitivity in ρ

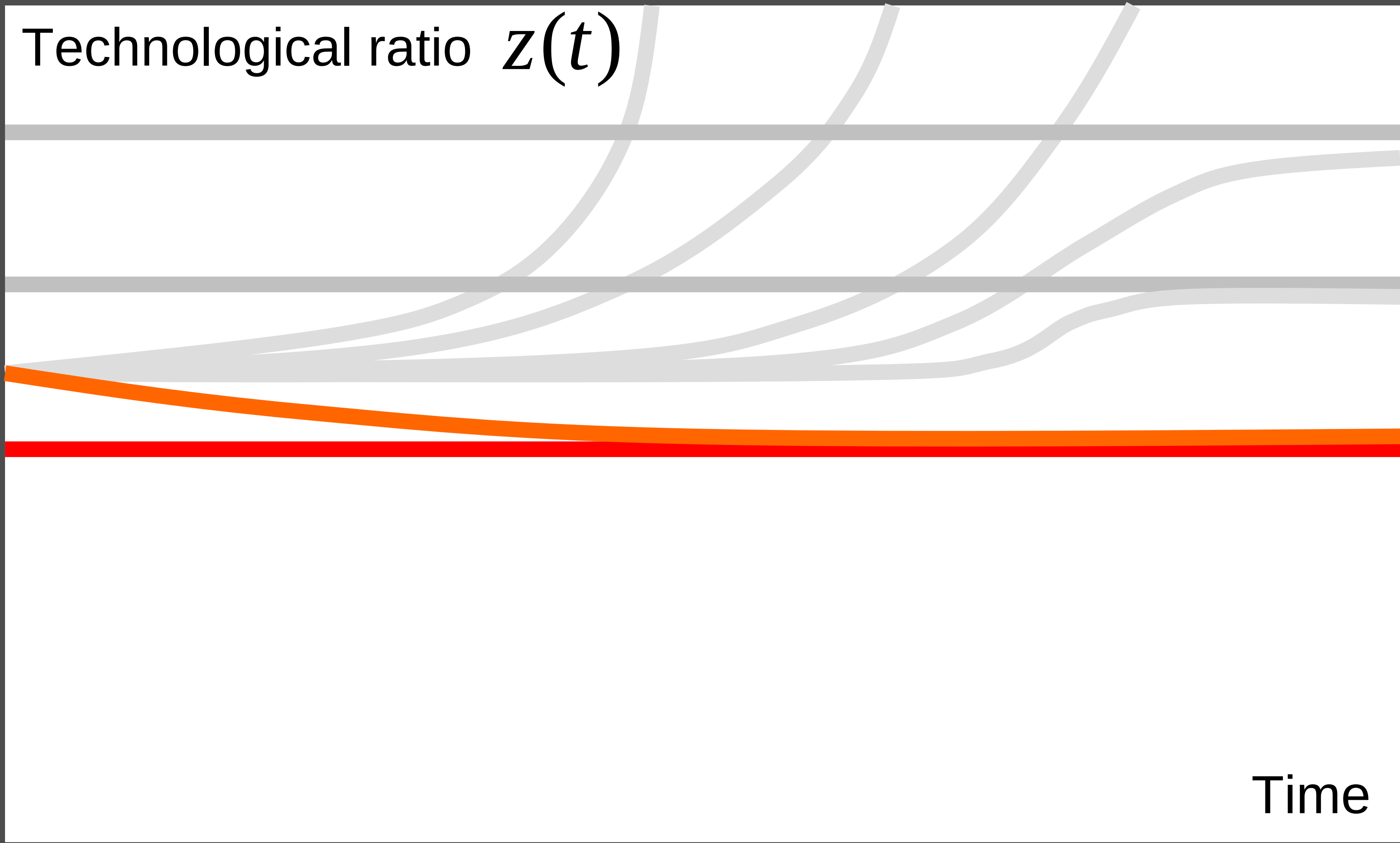


Sensitivity in ρ



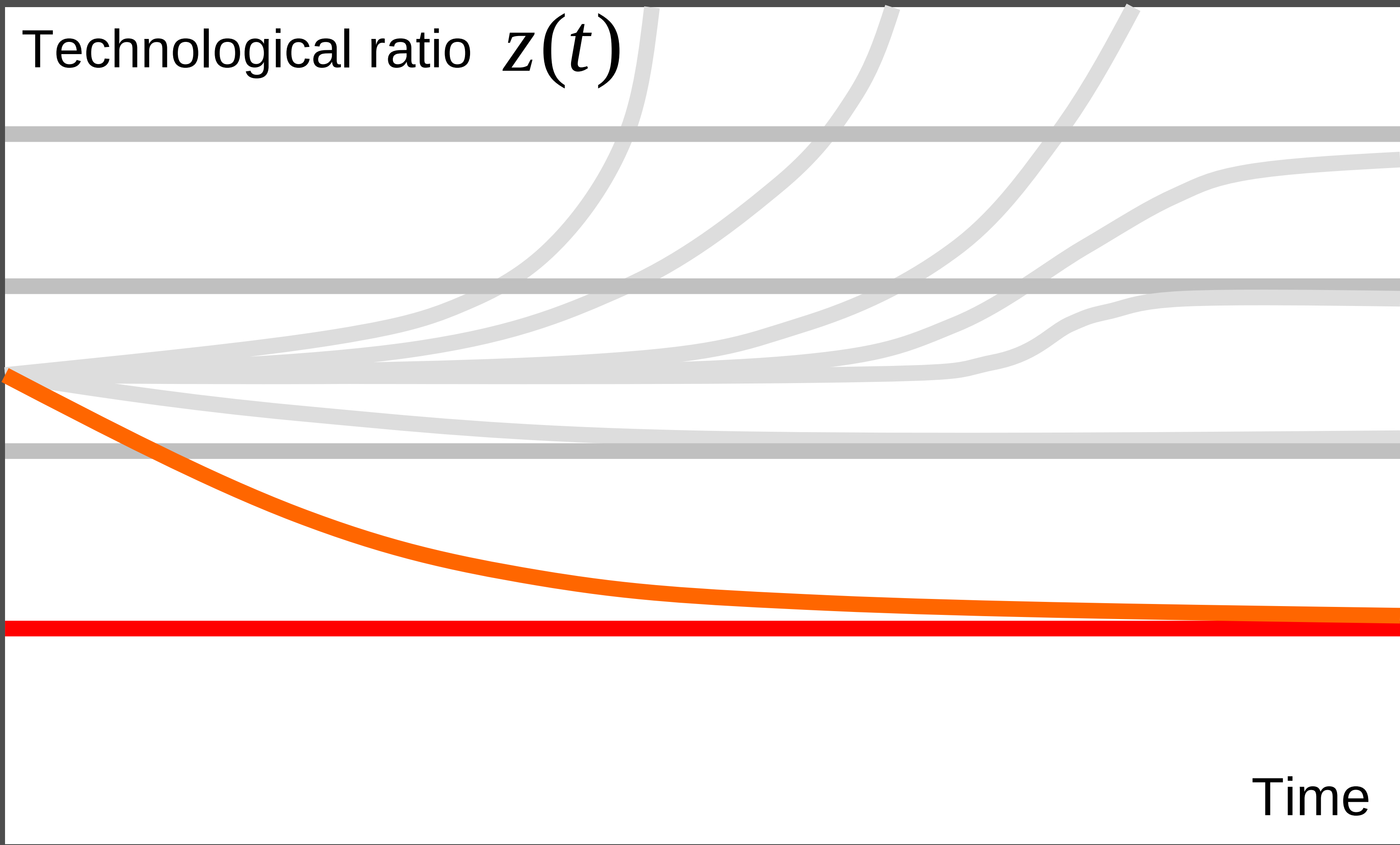
Follower

Sensitivity in ρ



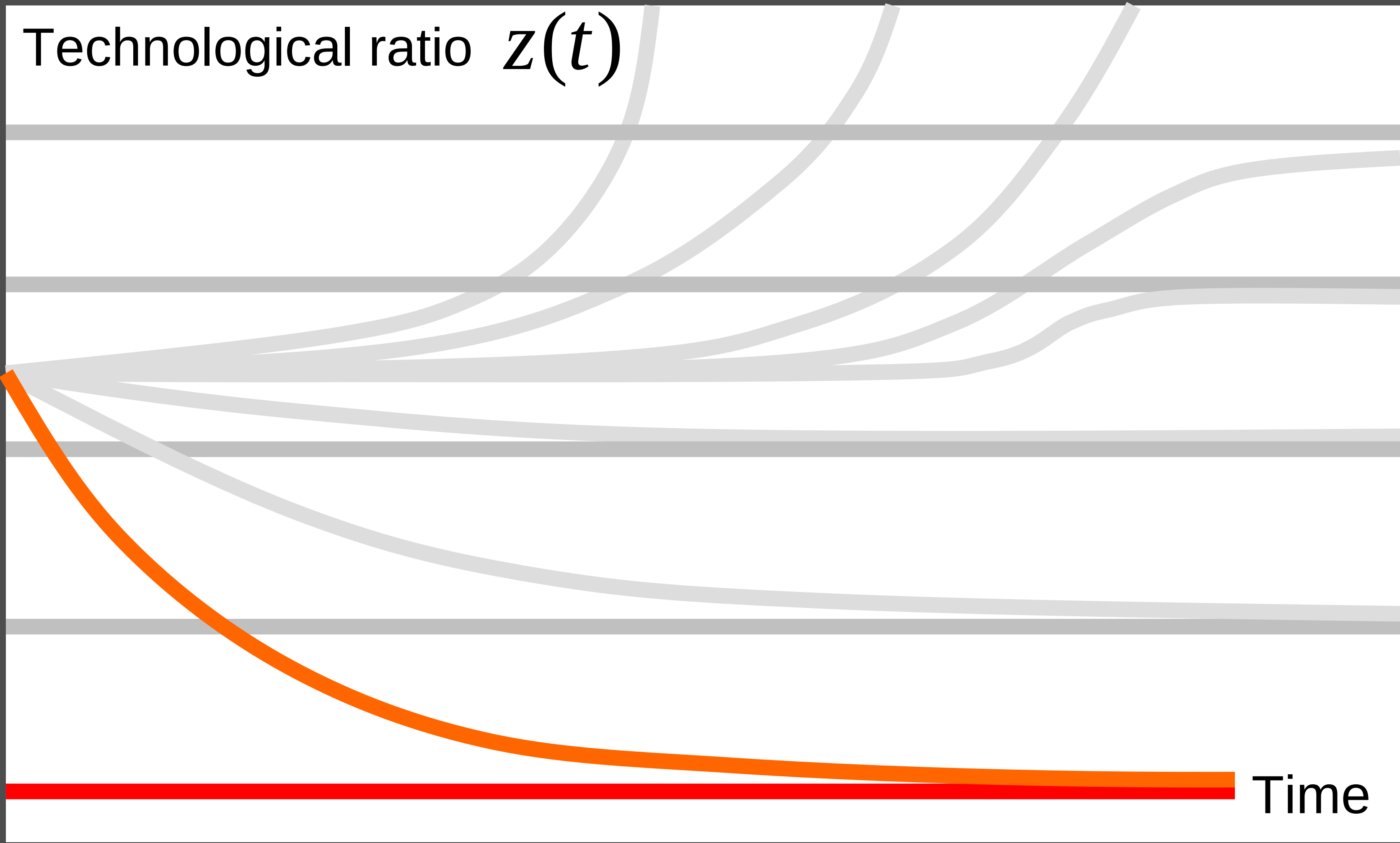
Follower

Sensitivity in ρ



Follower

Sensitivity in ρ



Acknowledgements

Gernot Hutschenreiter, *Austrian Institute for Economic Research*
Chihiro Watanabe, *Tokyo Institute of Technology*
Masakazu Katsumoto, *Kyoto Institute of Technology*
Tapio Palokangas, *Helsinki University*