ON PERMUTATIONS AND PERMUTATION POLYTOPES

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PREFACE

The development of better methods to compute optimal solutions to models is a basic task in the System and Decision Sciences Area. Of particular interest are linear models with linear constraints. An important problem here is to be able to characterize when one solution is "adjacent" to another, since most practical algorithms for linear models operate by moving from one solution to an adjacent "better" one.

In this paper a convenient criterion is formulated to characterize adjacency for constraint systems that arise from orderings. The results have application to models in which the object is to find a best *ordering* of alternatives using some linear criterion, for example. In particular, it was used as a technique for studying collective decision criteria (H.P. Young and A. Levenglick, *A Consistent Extension of Condorcet's Election Principle*, RR-77-15, International Institute for Applied Systems Analysis, Laxenburg, Austria).

SUMMARY

Permutation polytopes arise in a class of problems in which the objective is to find an optimal complete ordering of some given alternatives, subject to a linear objective criterion. In this paper an easy characterization is given of neighbors on permutation polytopes. Using this characterization it is shown that the graph of any such polytope is Hamiltonian, and that the diameter is two.

The methods used are combinatorial in nature.

<center>On Permutations and Permutation Polytopes</center>

1. INTRODUCTION

A *permutation*, or *linear ordering*, on the n symbols $1, 2, \ldots, n$
will be denoted by the n-tuple $\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(n))$. We also
write h σ k if h precedes k in the order σ. One useful algebraic
representation of a permutation is by a permutation matrix. Here
we shall consider another way of representing permutations by $(0,1)$
matrices that has application to a variety of problems including
computer design [2] and collective decision making [5].

For any linear ordering $\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(n))$ let E^{σ} be the
$n \times n$ matrix with 1 in position (i,j) if i σ j and 0 otherwise. The
permutation polytope P_n is defined to be the set of all convex com-
binations of the matrices E^{σ}, σ a permutation of order n. It is
an unsolved problem to determine the complete set of inequalities
defining a general P_n (those proposed in [1] being insufficient).
On the other hand various characterizations of neighbors on these
polytopes are known. Here we shall give a computationally easy way
of recognizing neighbors in terms of the structure of the permuta-
tions. This approach also reveals some facts about the structure
of a class of graphs (implicitly introduced in [3,4] called *trans-
position graphs* that are intimately connected with adjacency ques-
tions on the graph of P_n.

2. BLOCKS

An *interval* of a permutation $\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(n))$ is a
subsequence of form $(\sigma(i), \sigma(i+1), \ldots, \sigma(j))$. We also allow the
empty subsequence. Let $e = (1, 2, \ldots, n)$ be the *identity* permutation.
A *block* of σ is any interval of σ that can be rearranged to be an
interval of e. A *common interval* of σ and e is an interval (without
rearrangement) of both σ and e.

Example 1. Let $e = (1, 2, 3, \ldots, 10, 11)$ and let $\sigma = (2, 10, 4, 11,$
$6, 5, 8, 7, 9, 3, 1)$. The blocks of σ, aside from ϕ and the singletons,
are:

$$\beta_1 = (2,10,4,11,6,5,8,7,9,3)$$
$$\beta_2 = \sigma$$
$$\beta_3 = (10,4,11,6,5,8,7,9)$$
$$\beta_4 = (10,4,11,6,5,8,7,9,3)$$
$$\beta_5 = (6,5)$$
$$\beta_6 = (6,5,8,7)$$
$$\beta_7 = (6,5,8,7,9)$$
$$\beta_8 = (8,7)$$
$$\beta_9 = (8,7,9)$$

The transposition set of σ, $S(\sigma)$, is the set of all unordered pairs $\{i,j\}$ $i \neq j$ inverted by $\sigma : S(\sigma) = \{\{i,j\} : i < j \text{ and } j\sigma i\}$. The transposition set of a block β of σ, $S(\beta) = \{\{i,j\} : i,j \in \beta, i < j \text{ and } j\sigma i\}$.

Notice that for any two blocks β_1, β_2 of σ, $\beta_1 \cap \beta_2$ is also a block and

$$S(\beta_1 \cap \beta_2) = S(\beta_1) \cap S(\beta_2) \quad .$$

Let $L_\sigma = \{S : S = S(\beta_i) \text{ for some block } \beta_i \text{ of } \sigma\}$. Then the elements of L_σ, ordered by inclusion, form a *lattice*, called a *permutation lattice*. In general, l.u.b.$\{S_i\}_{S_i \in L_\sigma} \supseteq \bigcup_i S_i$, but equality may not hold.

For the σ of Example 1, we obtain the following lattice.
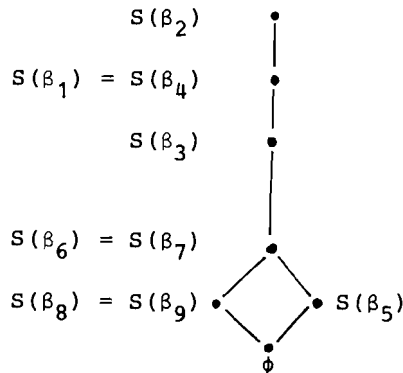
3. CHARACTERIZATION OF NEIGHBORS BY BLOCKS

Two permutations σ and τ are said to be *neighbors* written $\sigma \sim \tau$ if E^σ and E^τ are neighboring extreme points on P_n. We may now give the following easy characterization of neighbors on permutation polytopes.

(1) $\sigma \neq e$ is a neighbor of e on P_n if and only if L_σ has a unique nonempty element.

This will be established below as a consequence of Theorem 1.

For the moment let us notice that this criterion leads to a highly efficient method for checking whether a given σ is a neighbor of e.

First, find the least p^0, $1 \le p^0 \le n$ such that $\sigma(p^0) \neq p^0$ (if none exists, $\sigma = e$). Next, order the pairs (p,q), $p^0 \le p < q \le n$, lexicographically. For each successive pair we check to see whether the interval $(\sigma(p),\ldots,\sigma(q))$ is a block: this is true iff $q - p = M - m$, where $M = \max\limits_{p \le i \le q} \sigma(i)$, $m = \min\limits_{p \le i \le q} \sigma(i)$. Moreover, this block is nontrivial (that is, it is not an interval of e) if $M \neq \sigma(q)$. If (p',q') is the first pair for which a nontrivial block is found, we check that $\sigma(i) = i$ for $q' < i < n$: if this is false, σ is *not* a neighbor of e. If true, repeat the above process for successive pairs (p,q), $(p'+1) \le p < q \le q'$: if any nontrivial block is found, σ is *not* a neighbor of e; otherwise σ *is* a neighbor of e.

Notice that the computation of the new M and m at each stage is easy, since the pairs (p,q) are taken in lexicographic order. The number of pairs (p,q) that have to be considered is at most $\binom{n}{2}$.

4. CHARACTERIZATION OF NEIGHBORS BY GRAPHS

A second way of characterizing neighbors on permutation poly-topes is based on a certain class of comparability graphs. An un-directed graph G is said to be a *comparability graph* if its edges can be directed so as to form a strict partial order Θ: i.e.,

(i)   exactly one of (i,j), (j,i) ε Θ iff $\{i,j\}$ is an edge of G,

(ii) (i,j), (j,k) ε Θ implies (i,k) ε Θ.

Given a permutation $\sigma = (\sigma(1),\ldots,\sigma(n))$, the *permutation graph* [3] of σ, G_σ, is defined to be the undirected graph with vertex set $\{1,2,\ldots,n\}$ and edge set $S(\sigma)$. Notice that G_σ is a comparability graph by the partial order $\Theta = \{(j,i) : i < j$ and $\{i,j\} \varepsilon S(\sigma)\}$ (or the reverse order $\overline{\Theta}$). Similarly, G_σ^C is a comparability graph for the order $\Theta' = \{(i,j) : i < j$ and $\{i,j\} \not\varepsilon S(\sigma)\}$. Conversely, if G is some graph such that both G and G^C are comparability graphs then $G = G_\sigma$ for some σ [3].

Given permutation σ, a graph related to G_σ is the transposition *set* F_σ defined as follows: Γ_σ has vertex set $S(\sigma)$, and $\{i,j\} \varepsilon S(\sigma)$ is adjacent to $\{j,k\} \varepsilon S(\sigma)$ (written $\{i,j\}\Gamma\{j,k\}$) if and only if $i \neq k$ and $\{i,k\} \not\varepsilon S(\sigma)$; these are all the adjacencies in Γ_σ.

The following connection between Γ_σ and neighbors on the permutation polytope is due to Alan Hoffman.

(2) $\sigma \neq e$ is a neighbor of e if and only if Γ_σ is connected

To prove sufficiency, suppose Γ_σ is connected and $\sigma \neq e$ is not a neighbor of e. Then $S(\sigma) = \bigcup_{r=1}^{s} S(\pi_r)$ for distinct permutations $\pi_r \neq \sigma$, $r \geq 2$.

Since Γ_σ is connected, there is $\pi_r \neq \pi_{r'}$ and $\{i,j\} \varepsilon S(\pi_r) - S(\pi_{r'})$, $\{j,k\} \varepsilon S(\pi_{r'})$, such that $\{i,j\}\Gamma\{j,k\}$, i.e. $\{i,k\} \not\varepsilon S(\sigma)$. Hence $\{i,k\} \not\varepsilon S(\pi_r) \cup S(\pi_{r'})$. Hence, of i,j,k, $\pi_{r'}$ reverses only j and k, and π_r reverses j and i but not i and k. Therefore, either $i < j < k$ or $k < j < i$. But σ reverses $\{i,j\}$ *and* $\{j,k\}$ and not $\{i,k\}$, an absurdity. The converse will be established as a consequence of the following connection between Γ_σ and the permutation lattice L_σ.

Theorem 1. For any σ, the components of Γ_σ are precisely the nonempty sets of form $S - \bigcup_{\substack{S_i \subsetneq S \\ S_i \varepsilon L_\sigma}} S_i$ where $S \varepsilon L_\sigma$.

We proceed by several lemmas.

Lemma 1.1. If C is any component of Γ_σ and $C \cap S \neq \phi$ for some $S \in L_\sigma$, then $C \subsetneq S$.

Proof: If $C \not\subseteq S$ then $\{i,j\} \Gamma \{j,k\}$ for some $\{i,j\} \in C \cap S$ and some $\{j,k\} \in C - S$. Let $S = S(\beta)$, $\beta = (\sigma(p),...,\sigma(q))$; then $i,j \in \beta$, $k \notin \beta$. Since σ reverses j and k it must also reverse i and k, whence $\{i,k\} \in S$, a contradiction.

Recall that G_σ has vertex set $V = \{1,2,...,n\}$ and edge set $S(\sigma)$.

Following [3], we say that a subset $V' \subseteq V$ is *uniformly hinged* (u.h.) in G_σ if for every $k \notin V'$, either $\{k,i\} \in S(\sigma)$ for *all* $i \in V'$ or $\{k,i\} \in S(\sigma)$ for *no* $i \in V'$. Clearly any block of σ regarded as a set of vertices of G_σ is u.h., but not every u.h. set constitutes a block. For any $V' \subseteq V$ and any permutation π define $cl_\pi(V')$ as the smallest interval of π containing V'.

Lemma 1.2. If V' is u.h. in G_σ then, *as sets*, $cl_e(V')$ equals $cl_\sigma(V')$, in which case $cl_\sigma(V')$ is a block of σ.

Proof: Suppose V' is u.h., and let $cl_\sigma(V') = (\sigma(p),...,\sigma(q))$. Then $\sigma(p), \sigma(q) \in V'$ by definition. If $i \in cl_\sigma(V') - V'$, and $\sigma(p) > \sigma(q)$, then u.h. implies $\sigma(p) > i > \sigma(q)$; similarly, if $\sigma(p) < \sigma(q)$ then $\sigma(p) < i < \sigma(q)$. Hence $cl_\sigma(V') \subseteq cl_e(V')$. On the other hand, if $i \in cl_e(V') - V'$ then for some $h,k \in V'$ we have $h < i < k$. Thus i cannot precede (or follow) *both* h and k in σ without contradicting u.h., hence $i \in cl_\sigma(V')$ and $cl_\sigma(V') = cl_e(V')$.

For any subset of edges $E \subseteq S(\sigma)$, let $V(E) \subseteq V$ be the set of endpoints of edges in E. The next two results are immediate consequences of Lemmas 2 and 4, respectively in [3].

Lemma 1.3. If C is a component of Γ_σ then $V(C)$ is u.h. in G_σ.

Lemma 1.4. If C and C' are distinct components of Γ_σ then $V(C) \neq V(C')$.

Proof of Theorem 1: By Lemma 1.1, any component that meets $\overline{S} = S - \bigcup_{S_i \subsetneq S, S_i \in L_\sigma} S_i \neq \phi$ must be contained in $\overline{S}$, hence it will suffice to show that $\overline{S}$ is connected in Γ_σ.

Let C be *any* component of Γ_σ contained in $\overline{S}$, and let $W = V(C)$. Since W is u.h., Lemma 1.2 implies $cl_\sigma(W) = \beta$ is a block of σ, say $\beta = (\sigma(p),\ldots,\sigma(q))$.

Suppose $\beta - W \neq \phi$. (Here we think of β as a set as well as a sequence.)

Let $h \in \beta - W$ and

$$W_h^+ = \{k \in W : k\sigma h\} \quad,$$

$$W_h^- = \{k \in W : h\sigma k\} \quad.$$

Notice that since $\sigma(p)$, $\sigma(q) \in W$, W_h^+, $W_h^- \neq \phi$; moreover W_h^+ and W_h^- are each u.h. Therefore $\beta_h^+ = cl_\sigma(W_h^+)$ and $\beta_h^- = cl_\sigma(W_h^-)$ are non-empty disjoint blocks.

In general, for any *disjoint* $V', V'' \subseteq V$, let (V', V'') denote the set of edges of G_σ with one end in V' and the other end in V''.

By Lemma 1.1, $C \cap S(\beta_h^+) = \phi$ and $C \cap S(\beta_h^-) = \phi$, hence $C \subseteq (W_h^+, W_h^-)$.

Suppose now that some $k \in \beta_h^+ - W_h^+ \neq \phi$. Then by a similar argument we have $C \subseteq (W_k^+, W_k^-)$. By choice of k, $W_h^+ - W_k^+ \neq \phi$, say $\ell \in W_h^+ - W_k^+$. Then $\ell \in W_k^-$ and ℓ are the endpoints of some edge in C. The other endpoint, ℓ', is therefore in $W_k^+ \subseteq W_h^+$, but then both ends are in W_h^+, a contradiction. Therefore $\beta_h^+ = W_h^+$ is a block and similarly $\beta_h^- = W_h^-$ is a block; say $\beta_h^+ = (\sigma(p),\ldots,\sigma(p'))$, $\beta_h^- = (\sigma(q'),\ldots,\sigma(q))$, and let $\alpha = (\sigma(p'+1),\ldots,\sigma(q'-1)) \neq \phi$. Since $C \neq \phi$ and $C \subseteq (\beta_h^+, \beta_h^-)$ we must have $\beta_h^+ > \beta_h^-$ (i.e., $i > j$ for all $i \in \beta_h^+$, $j \in \beta_h^-$). Further, since $W = \beta_h^+ \cup \beta_h^-$ is u.h. we have $p_h^+ > \alpha > \beta_h^-$, so α is also a nonempty block of σ.

In summary, for any component $C \subsetneq \overline{S}$, if $V(C) = W$ and $\beta = cl_\sigma(W)$, then either $W = \beta$ *is* a block or β has form $\beta = (W^+, \alpha, W^-)$ where $W^+ > \alpha > W^-$ are each nonempty blocks.

If $\overline{S}$ is *not* connected then there is another component $\tilde{C}$ of Γ_σ such that $C \neq \tilde{C} \subsetneq \overline{S}$ and corresponding $\tilde{W}$, $\tilde{\beta}$. By definition of $\overline{S}$, $S(\tilde{\beta}) = S(\beta) = \overline{S}$, so $\beta = \tilde{\beta}$.

Now $W = \beta = \tilde{\beta} = \tilde{W}$ is impossible by Lemma 1.4. Hence say, without loss of generality, that either

(i) $\beta = (W^+, \alpha, W^-) = \tilde{W}$,

   or

(ii) $\beta = (W^+, \alpha, W^-) = (\tilde{W}^+, \tilde{\alpha}, \tilde{W}^-) = \tilde{\beta}$,

where $W^+ > \alpha > W^-$ and $\tilde{W}^+ > \tilde{\alpha} > \tilde{W}^-$ and each is a nonempty block.

In the first case notice that both (W^+, α) and (α, W^-) are blocks. Choose any $k \in \alpha$; then $\{j, k\} \in \tilde{C}$ for some j and either $\{j, k\} \in S(W^+, \alpha)$ or $\{j, k\} \in S(\alpha, W^-)$. But neither $S(W^+, \alpha)$ nor $S(\alpha, W^-)$ equals S, contradicting the definition of $\bar{S}$.

In the second case suppose $W^+ \neq \tilde{W}^+$; without loss of generality $W^+ \subsetneqq \tilde{W}^+$. Then $(\tilde{W}^+ - W^+, \alpha, \tilde{W}^-) = \delta$ is a block. For any $k \in \tilde{W}^+ - W^+, \{k, j\} \in \tilde{C}$ for some j and either $\{k, j\} \in S(W^+)$ or $\{k, j\} \in S(\delta)$. Since neither $S(\delta)$ nor $S(W^+)$ equals S, this contradicts the definition of $\bar{S}$. Thus $W^+ = \tilde{W}^+$, and likewise $W^- = \tilde{W}^-$. But then $W = \tilde{W}$, contradicting Lemma 1.4.□

The proof of statement (1) is now as follows.

If L_σ has one nonempty element S, then $S = \bar{S}$ is the unique component of Γ_σ; hence by the argument following (2) σ is a neighbor of e. If on the other hand L_σ has distinct, nonempty elements, let $S_0 = S(\beta)$ be a *minimal* nonempty element of L_σ. Then within β we may rearrange the elements to be an interval of e, and the resulting permutation σ' satisfies $S(\sigma') = S(\sigma) - S_0$. Likewise we may rearrange σ to agree with e everywhere *except within* β, and the resulting σ'' has $S(\sigma'') = S_0$. But then $E^{\sigma'} + E^{\sigma''} = E^\sigma + E^e$, showing that σ is *not* a neighbor of e. Statement (2) now follows at once also, since if σ is a neighbor of e, then L_σ has a unique nonempty element; hence by Theorem 1 Γ_σ is connected.

Example 2. For $n = 3$, the neighbors of $e = (1,2,3)$ are seen by inspection to be: $(2,1,3)$, $(2,3,1)$, $(1,3,2)$, $(3,1,2)$.

For n = 4, the neighbors of e = (1,2,3,4) are seen by in-spection to be the following twelve permutations:

(1,2,4,3)	(2,1,3,4)	(3,1,2,4)	(4,1,2,3)
(1,3,2,4)	(2,3,1,4)	(3,1,4,2)	
(1,3,4,2)	(2,3,4,1)	(3,4,1,2)	
(1,4,2,3)	(2,4,1,3)		

5. THE CIRCUMFERENCE OF P_n

The *graph* of P_n, $H(P_n)$, is the undirected graph whose vertices are the permutations $\sigma = (\sigma(1),\ldots,\sigma(n))$, where σ and τ are adjacent if and only if $\sigma \neq \tau$ and σ is a neighbor of τ.

Theorem 2. The graph $H(P_n)$ is Hamiltonian for any $n \geq 3$.

Proof: For $n = 3$ a Hamiltonian circuit is

$$
\begin{array}{ccccccccccccc}
1 & & 2 & & 2 & & 3 & & 3 & & 1 & & 1 \\
2 & \rightarrow & 1 & \rightarrow & 3 & \rightarrow & 2 & \rightarrow & 1 & \rightarrow & 3 & \rightarrow & 2 \\
3 & & 3 & & 1 & & 1 & & 2 & & 2 & & 3
\end{array} \ .
$$

Suppose the result has been proved for $n - 1$, and let $\tau_1,\ldots,$ $\tau_{(n-1)!},\tau_1$ be a Hamiltonian circuit on the permutations τ_i of $1,2,\ldots,n-1$. For each *odd* i, $1 \leq i \leq (n-1)!$, let $\tau_i = (\tau_i(1),\ldots,$ $\tau_i(n-1))$, and for each k, $1 \leq k \leq n$ define the permutations σ_{ik} of $1,2,\ldots,n$ as follows:

$$\sigma_{i1} = (\tau_i(1),\ldots,\tau_i(n-1),n)$$

$$\sigma_{i2} = (\tau_i(1),\ldots,n,\tau_i(n-1))$$

$$\vdots$$

$$\sigma_{in} = (n,\tau_i(1),\ldots,\tau_i(n-1)) \ .$$

For i *even*, let

$$\sigma_{i1} = (n, \tau_i(1), \ldots, \tau_i(n-1))$$

$$\sigma_{i2} = (\tau_i(1), n, \ldots, \tau_i(n-1))$$

$$\vdots$$

$$\sigma_{in} = (\tau_i(1), \ldots, \tau_i(n-1), n) \quad .$$

Since $(n-1)!$ is even, (1) implies that $\sigma_{11}, \sigma_{12}, \ldots, \sigma_{1n}, \sigma_{21} \ldots, \sigma_{2n}, \sigma_{(n-1)!n}, \sigma_{11}$ is a Hamiltonian circuit in $H(P_n)$. $\Box$

It will be noticed that each successive pair of permutations on the Hamiltonian circuit are "close" neighbors in the sense that they differ by just one adjacent transposition. This construction is known in combinatorics, see for example [6,7].

6. AN INDUCTIVE LEMMA

Pick m, $1 \le m \le n$, and let $m_1 = m - 1$, $m_2 = m + 1$. (If $m = 1$, only m_2 is defined, and if $m = n$ only m_1 is defined.)

Given any permutation π, let $\hat{\pi}$ be the permutation on $n - 1$ symbols obtained from π by removing m from the sequence. Thus, if $\pi = (2,3,5,1,4)$ and $m = 3$ then $\hat{e} = (1,2,4,5)$ and $\hat{\pi} = (2,5,1,4)$.

The $n - 1$ elements of $\hat{\pi}$ we call *dots* and the n positions defined between the dots or at either end of the dots we call *slots*. Given any *dot sequence* β (i.e., an interval of $\hat{\pi}$,) where $|\beta| = k$, let (β) denote the sequence of k dots and $k - 1$ slots beginning with the first dot in β and ending with the last dot in β; let $(\beta]$ denote the sequence with k dots and k slots consisting of (β) and the end slot to the *right* of β; let $[\beta)$ be (β) and the end slot to the *left* of β, and let $[\beta]$ be (β) together with *both* end slots. We call these *open, half-open* and *closed slot-dot sequences*, respectively. We adopt the convention that if β is the empty sequence then $[\beta]$ consists of one slot, and the others are empty. We also write $m \in (\beta)$, $m \in [\beta]$ etc., when we mean that m occupies a slot in that sequence.

Let γ_1 be the longest *initial* interval of $\hat{\pi}$ that is also an initial interval of $\hat{e}$, γ_2 the longest *terminal* interval of $\hat{\pi}$ that is also a terminal interval of $\hat{e}$ (either may be empty). $\hat{\pi} = \gamma_1 \overset{\wedge}{\hat{\pi}} \gamma_2$ where $\overset{\wedge}{\hat{\pi}}$ is a block of $\hat{\pi}$ (relative to $\hat{e}$) and $S(\overset{\wedge}{\hat{\pi}}) = S(\hat{\pi})$. Define δ_1 to be the *longest common interval* of $\hat{\pi}$ and $\hat{e}$ *ending with* m_1, δ_2 to be the *longest common interval* of $\hat{\pi}$ and $\hat{e}$ *beginning with* m_2. (If $m = 1$ or $m = n$, one of these sequences is vacuous.)

Lemma 3.1. Let $\hat{\pi}$ be a neighbor of $\hat{e}$. Then π is a neighbor of e if and only if

(i) $\quad m \notin [\delta_1) \, , m \notin (\delta_2] \quad$,

and

(ii) $[\delta_1) \subseteq [\gamma_1)$ implies $m \notin [\gamma_2]$; $(\delta_2] \subseteq (\gamma_2]$ implies $m \notin [\gamma_1]$.

Proof: Let the conditions (i) and (ii) be satisfied and suppose β, β' are any two blocks of π. We must show that

(3) $\qquad S(\beta) \neq \phi \quad , \quad S(\beta') \neq \phi$ implies $S(\beta) = S(\beta) \quad$.

For any block β of π, let $\hat{\beta}$ denote β with m removed. Note that $\hat{\beta}$ is then a block of $\hat{\pi}$ (with respect to the "identity", $\hat{e}$).

If $m \notin \beta$, $S(\beta) \neq \phi$, then $S(\beta) = S(\hat{\beta}) \neq \phi$ so $S(\hat{\beta}) = S(\hat{\pi}) = S(\overset{\wedge}{\hat{\pi}})$, since $\hat{\pi}$ is a neighbor of $\hat{e}$. If $m \in \beta$, $S(\beta) \neq \phi$, then since $\hat{\pi}$ is a neighbor of $\hat{e}$, either $S(\hat{\beta}) = \phi$, or $S(\hat{\beta}) = S(\hat{\pi}) = S(\overset{\wedge}{\hat{\pi}})$. In the former case $\hat{\beta}$ is a common interval of $\hat{e}$ and $\hat{\pi}$ and since β is a block with $S(\beta) \neq \phi$, either $m_1 \in \hat{\beta}$ or $m_2 \in \hat{\beta}$. It follows that either β has a subinterval of form (m, δ_1') where δ_1' is a common interval of $\hat{\pi}$ and $\hat{e}$ ending with m_1, or β has a subinterval of form (δ_2', m), where δ_2' is a common interval of $\hat{\pi}$ and $\hat{e}$ beginning with m_2. Both situations violate condition (i). Hence

(4) $\qquad S(\beta) \neq \phi$ implies $S(\hat{\beta}) = S(\overset{\wedge}{\hat{\pi}})$ and $\overset{\wedge}{\hat{\pi}} \subseteq \hat{\beta}$.

Suppose both m_1, m_2 are defined, i.e. $m \neq 1,n$.

If $m_1, m_2 \in \hat{\hat{\pi}}$, then for any block β of π, $S(\beta) \neq \phi$ implies $\hat{\hat{\pi}} \subseteq \hat{\beta}$, hence $m_1, m_2 \in \beta$, so $m \in \beta$. Then the sequence $\mu = c\ell_\pi(\hat{\hat{\pi}},m)$ is a block of π, $\mu \subseteq \beta$, and $S(\mu) = S(\beta)$, so (3) holds.

If $m_1 \in \hat{\hat{\pi}}$, $m_2 \notin \hat{\hat{\pi}}$, then m_1 is maximal in $\hat{\hat{\pi}}$, so $m_2 \in \gamma_2$ and in fact m_2 is the initial "dot" of γ_2, so $\delta_2 = \gamma_2$. Then conditions (i) and (ii) imply $m \in (\hat{\hat{\pi}}]$. Hence the sequence ω obtained from $\hat{\hat{\pi}}$ by re-inserting m in $(\hat{\hat{\pi}}]$ is a *block* of π and m is its maximal element. If m is not the last element of the sequence ω, i.e. if $m \in (\hat{\hat{\pi}})$, then for any block β of π, $S(\beta) \neq \phi$ implies $\hat{\hat{\pi}} \subseteq \hat{\beta}$, so $\omega \subseteq \beta$ and $S(\omega) = S(\beta)$, so (3) holds. If m is the last element of ω, then m maximal in ω implies $\hat{\hat{\pi}}$ is also a block of π and $S(\omega) = S(\hat{\hat{\pi}})$. By choice of $\hat{\hat{\pi}}$ we then have $\hat{\hat{\pi}} \subseteq \beta$ and $S(\hat{\hat{\pi}}) = S(\beta)$ for any block β of π such that $S(\beta) \neq \phi$, and again (3) holds. A similar argument applies if $m_1 \notin \hat{\hat{\pi}}$, $m_2 \in \hat{\hat{\pi}}$.

Finally, if $m_1 \notin \hat{\hat{\pi}}$, $m_2 \notin \hat{\hat{\pi}}$, then by choice of $\hat{\hat{\pi}}$, m_1 is adjacent to m_2 in $\hat{\pi}$ and both are in γ_1 or in γ_2; say without loss of generality $m_1, m_2 \in \gamma_1$. Then $[\gamma_1] = [\delta_1) \circ (\delta_2]$, and conditions (i) and (ii) imply that in π, either m is between m_1 and m_2, or $m \in (\hat{\hat{\pi}})$. In the former case $\hat{\pi}$ is also a block of π and it is clearly the unique nontrivial block. In the latter case, let β be any block of π such that $S(\beta) \neq \phi$. Then $\hat{\hat{\pi}} \subseteq \beta$ hence $m \in \beta$; hence $m_2 \in \beta$ (since β is a block and the elements of $\hat{\hat{\pi}}$ are larger than m). Therefore $\nu = c\ell_\pi(m_2, \hat{\hat{\pi}})$ is a block of π, $\beta \supseteq \nu$, and $S(\beta) = S(\nu)$, so (3) holds.

In case $m = n$, δ_2 is empty and $m \notin [\gamma_1]$ by condition (ii). Moreover we must have $m_1 \in \hat{\hat{\pi}}$, since if not then $\gamma_1 = \hat{\pi} = \hat{e}$ contrary to the assumption that $\hat{\pi}$ is a *neighbor* of $\hat{e}$ (and hence $\hat{\pi} \neq \hat{e}$). The argument of the second paragraph preceding then implies (3). Similar remarks hold if $m = 1$. The converse is left to the reader. $\square$

Given $\hat{\pi}$, we represent slots by the symbol $\circ$; we also (where relevant) mark a bracket with an arrow "$\rightarrow$" if it coincides with the left end of $\hat{\pi}$, and with an arrow "$\leftarrow$" if it coincides with the right end of $\hat{\pi}$. The proof of Lemma 3.1 then actually shows that

the "forbidden" slot-dot sequences for m must have one of the following forms relative to $\hat{\pi}$:

(A1) $\ldots [o \ldots m_1) o \overset{\mp\phi}{\overbrace{\ldots}} o (m_2 \ldots o] \ldots$

(A2) $\overset{\mp\phi}{\overbrace{\ldots}} o (m_2 \ldots o] \ldots [o \ldots m_1) o \overset{\mp\phi}{\overbrace{\ldots}}$

(B1) $\to [o \overset{\gamma_1 \mp\phi}{\overbrace{\ldots}} o] \bullet \overset{\mp\phi}{\overbrace{\ldots}} \bullet [o \ldots m_1) o \overset{\mp\phi}{\overbrace{\ldots}} o (m_2 \ldots o] \leftarrow$

or

(B2) $\to [o \ldots m_1) o \overset{\mp\phi}{\overbrace{\ldots}} o (m_2 \ldots o] \bullet \overset{\mp\phi}{\overbrace{\ldots}} \bullet [o \overset{\gamma_2 \mp\phi}{\overbrace{\ldots}} o] \leftarrow$.

The indicated sequences in every case are *dot* disjoint, and, as *dot* sequences, are subintervals of e. Of course, if some m_i' does not exist, the corresponding half open sequence is empty. Since in any case some slot is *not* forbidden, we have the following result.

(5) <u>Corollary.</u> For any m, $1 \le m \le n$, let $\hat{e}$ be as above and let η be a permutation on the $n-1$ symbols $\{1,\ldots,n\} - \{m\}$ that is a neighbor of $\hat{e}$. Then there is some slot in η in which m can be inserted such that the resulting permutation on n symbols is a neighbor of e.

7. <u>THE DIAMETER OF P_n</u>

Theorem 3. For $n \ge 2$ the diameter of P_n is two.

Proof. The proof is constructive, and proceeds by induction on n. For $n = 2,3$ it is obvious.

(6) Suppose then that the result is proved for $2 \le n' < n$, $n \ge 4$, and let σ be a permutation on n symbols such that $\sigma \ne e$ and σ is not a neighbor of e.

We say that the permutation σ is *degenerate* if

(i) σ and e have a common interval of length ≥ 2,

or

    (ii)   σ and e have the same first symbol or the same last
           symbol.

    If σ is degenerate as in case (i) then by treating γ as a
new symbol we obtain permutations σ̃ and ẽ on n − |γ| + 1 symbols
such that σ̃ ≠ ẽ and σ̃ is not a neighbor of ẽ. Hence by induc-
tion σ̃ ~ π̃ ~ ẽ for some π̃ and it follows that σ ~ π ~ e for the
corresponding π on {1,2,...,n}. Induction also yields the
theorem if σ is degenerate as in case (ii).

    Hence we may assume in the sequel that (6) holds and σ is
nondegenerate.

(7) We say that a symbol m is *inadmissible* in σ if

    (i)   m is the predecessor* in σ of m−1 or the successor in
           σ of m+1;
or
    (ii)   m = 1 and 1 is last in σ or m = n and n is first in σ;
or
    (iii) m is the predecessor in σ of some block β', where β'
           is a rearrangement of e' = (m−j,...,m−1) for some j ≥ 3
           and β' is a neighbor of e';
or
    (iv)   m is the successor in σ of some block β", where β" is
           a rearrangement of e" = (m+1,...,m+k) for some k ≥ 3
           and β" is a neighbor of e".

    For any given symbol m, 1 ≤ m ≤ n, let π̂, ê, etc. be de-
fined as in Section 6.

    Lemma 3.2. Suppose n ≥ 4, σ is nondegenerate, and m is
admissible. If σ̂ ~ η ~ ê for some permutation η on symbols
{1,2,...,n} − {m}, then there is a slot of η into which m can
be inserted, such that the resulting η' satisfies σ ~ η' ~ e.

    The proof is given later in this Report.

_____

*i.e., the immediate predecessor.

Any permutation σ that can be partitioned into subintervals
σ = δ₁δ₂ ... δ_r such that

(8) $\delta_1 = \phi$ or $\delta_1 = (n)$,

           $\delta_r = \phi$ or $\delta_r = (1)$,

and, for 1 < p < r,

           δ_p is an interval of $\bar{e} = (n, n-1, \ldots, 2, 1)$, $|\delta_p| \geq 2$,

will be called a *quasi-inverse* of e*. ($\bar{e}$ is the *inverse* of e.)

   Lemma 3.3.   If $n \geq 4$, σ is nondegenerate, and every m is
inadmissible in σ, then σ is a quasi-inverse of e.   (The proof
is given below.)

   Theorem 3 now follows from the following explicit construc-
tion.

   Lemma 3.4.   Let σ be a quasi-inverse of e, where $e = (1, 2, \ldots, n)$
and $n \geq 4$:

   (i)   if n is even, and 1 is not the predecessor of
            n in σ, then $\pi = (2, 4, \ldots, n, 1, 3, \ldots, n-1)$ is a
            neighbor of both σ and e;

   (ii)   if n is even, and 1 is the predecessor of n in
            σ, then $\pi = (1, 3, \ldots, n-1, 2, 4, \ldots, n)$ is a neighbor
            of both σ and e;

   (iii)   if n is odd, and 1 is the predecessor of n - 1
            in σ, then $\pi = (1, 3, \ldots, n, 2, 4, \ldots, n-1)$ is a
            neighbor of both σ and e;

   (iv)   if n is odd, 1 is not the predecessor of n-1 in
            σ, and n is not first in σ, then $\pi = (2, 4, \ldots, n-1,$
            $1, 3, \ldots, n)$ is a neighbor of both σ and e;

---

*For example, (8,7,6,3,2,5,4,1) is a quasi-inverse of (1,2,3,4,
 5,6,7,8) and it can be partitioned as in (8) by (8)(76)(32)(54)(1)
 as well as by ()(876)(32)(54)(1).

(v)   if n is odd, 1 is not the predecessor of n-1 in
        σ, and n is first in σ, then $\pi = (2,4,\ldots,n-1,$
        $1,n,3,\ldots,n-2)$ is a neighbor of both σ and e.

Proof of Lemma 3.4. We illustrate the method of proof
for case (i); the others are similar.

Let β be any interval of e, $|\beta| \geq 2$, that can be properly
rearranged to be an interval $\bar{\beta}$ of π. By *properly* we mean $\bar{\beta} \neq$
β. Then $\bar{\beta}$ contains an odd number and an even number, so
$1,n \in \bar{\beta}$, hence $1,n \in \beta$ and $\beta = $ e. Thus, π and e are neighbors.

Let β be any interval of σ, $|\beta| \geq 2$, that can be properly
rearranged to be an interval $\bar{\beta}$ of π. Suppose $|\beta| = 2$, say $\beta = $
(j,j'). We cannot have $|j - j'| = 1$, because then $\bar{\beta}$ would not
be an interval of π. Hence by (8), j is the last element of
some δ_p, j' the first in δ_{p+1}. Since $\bar{\beta}$ is an interval of π,
either $|j - j'| = 2$ or $(j,j') = (n,1)$. In the former case σ
could not be a quasi-inverse of e. In the latter case n pre-
cedes 1 in σ (by case (i) hypothesis), so $\bar{\beta}$ is not a proper
rearrangement of β.

We may therefore assume that $|\beta| \geq 3$. Since σ is a quasi-
inverse, $n \geq 4$, β contains some interval of form $(k,k-1)$. But
then $c\ell_\pi(k,k-1) \subseteq \beta$, and by construction of π,

(9)   for any consecutive integers j,j' in e at least one of j,j'$\in\beta$.

Moreover, $1,n \in \beta$. Suppose $2 \notin \beta$. If 2 succeeds β then 1
must also succeed 2, whence $1 \notin \beta$, contradicting (9), hence 2
precedes β and (since $n \in \beta$) we must have $\sigma = (2,\overbrace{1\quad\quad}^{\beta}\ldots)$.
In particular, $4 \in \beta$, for otherwise there is no position for 3
consistent with σ being a quasi-inverse. Similarly, if $n - 1 \notin \beta$,
then $n - 1$ is last in σ, and $n - 3 \in \beta$. Therefore $\sigma = (2,\beta,n-1)$
or $\sigma = (2,\beta)$ or $\sigma = (\beta,n-1)$, or $\sigma = \beta$ so π is a neighbor of σ. $\square$

Proof of Lemma 3.3. Let (i), (ii), (iii) or (iv) hold in
(7) for every m, $1 \leq m \leq n$. Suppose that *neither* (i) *nor* (ii)
holds for some particular m. Say β'' is as in (iv) for m (a
similar argument holds if m satisfies (iii) with β').

Let $m+j$ be the predecessor of m in σ, $m+j \neq m+1$ by hypothesis. Since $m+j$ is inadmissible, is neither first nor last in σ, and cannot satisfy (iii) we have that either

(a) $m+j$ is inadmissible by (iv),

or

(b) $m+j$ is the successor of $m+j+1$.

In case (a) $m+j$ must be the successor of a block γ which is a rearrangement of $(m+j+1,\ldots,m+j+\ell)$, $\ell \geq 3$. Clearly $\gamma \subsetneq \beta''$, since $m+1 \notin \gamma$ but precedes $m+j$ in β''. Since β'' is a *neighbor* of e", and $(\gamma,m+j)$ is a nontrivial subblock of β'', γ must be an *interval* of e, that is $\gamma = (m+j+1,\ldots,m+j+\ell)$. But then σ is degenerate, a contradiction.

Therefore (b) holds. Again, since β'' is a neighbor of e", and $(m+j+1 , m+j)$ is a nontrivial subblock of β'', we conclude that the predecessors of $m+j+1$ in b" form an interval β''' of e. Then σ is degenerate unless $|\beta'''| = 1$. Thus $\beta'' = (m+1 , m+3 , m+2 , m)$. But then $m + 1$ is admissible, a contradiction.

Thus 7(i) or 7(ii) holds for every m, $1 \leq m \leq n$, from which it follows that σ has a decomposition as in (8), i.e. σ is a quasi-inverse of e. This completes the proof of Lemma 3.3. $\square$

Proof of Lemma 3.2. Let $n \geq 4$, σ nondegenerate, m admissible, and suppose $\hat{\sigma} \sim \eta \sim \hat{e}$. We cannot have $\hat{\sigma} = \hat{e}$ (since σ is nondegenerate). If $\eta = \hat{e}$ or $\eta = \hat{\sigma}$, then the Corollary to Lemma 3.1 implies there is an η' such that $\sigma \sim \eta' \sim e$. Hence we may assume that

(10) $\hat{\sigma}, \eta, \hat{e}$ are *distinct*.

By Lemma 3.1, the collection $\mathcal{C}_e$ of forbidden slot-dot sequences in η are of form (A) or (B). That is, if m is put in a slot which is in no sequence $s \in \mathcal{C}_e$ then the resulting η' is a neighbor of e. Symmetrically, there is a collection $\mathcal{C}_\sigma$ of slot-dot sequences in η having form (A) or (B) (with m_i replaced by m_i', $i = 1,2$) such that if m is put in a slot which is in no

$s \varepsilon \mathscr{C}_\sigma$ then η' is a neighbor of σ. To show that there is some η' that is a neighbor of both e and σ, it suffices to show that

(11) $\bigcup_{s \varepsilon \tilde{\mathscr{B}}_e \cup \tilde{\mathscr{B}}_\sigma} s$ does not contain every slot of η.

Suppose both $\tilde{\mathscr{B}}_e$ and $\tilde{\mathscr{B}}_\sigma$ are of type (A). The sequences (there are at most two of them) in $\tilde{\mathscr{B}}_e$ are dot-disjoint, as are the sequences in $\mathscr{C}_\sigma$. Moreover, each has equally many slots and dots. But η has n slots and $n-1$ dots, hence if (11) is *false*, there is some $s \varepsilon \tilde{\mathscr{B}}_e$, $s' \varepsilon \tilde{\mathscr{B}}_\sigma$ such that $s \cap s'$ contains more dots than slots. But if $s \cap s'$ contains ≥ 2 dots, then the dots in $s \cap s'$ constitute a common interval of e and σ, contrary to the nondegeneracy of σ. Hence $s \cap s'$ consists of one dot, but then this dot must be $m_i = m_j'$ for some i and j, contradicting the hypotheses that σ is nondegenerate and m is admissible.

Therefore we may assume that at least one of $\mathscr{C}_e$, $\mathscr{C}_\sigma$ is of type B, say without loss of generality that $\mathscr{C}_\sigma$ has form B1:

(12) $\rightarrow [\circ \ldots \circ] \overbrace{\bullet \ldots \bullet}^{u} \underbrace{[\circ \ldots m_1') \overbrace{\circ \ldots \circ}^{v}}_{\beta} (m_2' \ldots \circ] \leftarrow$, $u \neq \phi$, $v \neq \phi$.

(13) Suppose, by way of contradiction, that every slot is forbidden by sequences in either $\mathscr{C}_\sigma$ or $\mathscr{C}_e$. Suppose also that $[\delta_1) = [\circ \ldots m_1) \varepsilon \mathscr{C}_e$ *meets* the slot-dot sequence v. If $[\delta_1) \cap (m_2' \ldots \circ] \neq \phi$, then either $m_1 = m_2'$, or they overlap in ≥ 2 dots; the first is impossible because m is admissible, the second violates the assumption that σ is nondegenerate. Therefore the slot immediately to the *right* of $[\delta_1)$ is in v and not forebidden by $\mathscr{C}_e$, a contradiction.

If $\mathscr{C}_e$ is of type (A), this means that $v \subseteq (\delta_2] = (m_2 \ldots \circ]$. Since $m_2 \notin [\circ \ldots m_1')$, it then follows that the slot immediately to the *left* of $(\delta_2]$ is not forbidden by $\mathscr{C}_e$ or $\mathscr{C}_\sigma$ (a contradiction) *unless* $\beta \subseteq (\delta_2]$. Now $\hat{\sigma}$ is obtained from η by rearranging only the elements within β. Let e" be the subinterval of δ_2 beginning

with $m_2 = m + 1$ and ending with the last element of β. e'' is by definition a subinterval of e, and its rearrangement, β'', in $\hat{\sigma}$ is a *neighbor* of e'', because $\hat{\sigma}$ is a neighbor of η. Moreover, m_1' is the *last* element in the rearrangement β'', and by definition m is the successor in σ of β''. If $|\beta''| = 2$ then $\beta'' = e'' = (m_2, m_1')$ and $\eta = \hat{\sigma}$, contrary to assumption. Hence $|\beta''| \geq 3$ and (β'', m) is a subsequence of σ satisfying (iv), whence m is inadmissible, a contradiction.

If $\mathscr{C}_e$ is of type (B1), we have by paragraph (13) that $v \subseteq [\gamma_1] \cup (\delta_2]$. If $v \subseteq (\delta_2]$, then, as above, we find that $\beta \subseteq (\delta_2]$, which results in the same sort of contradiction. The case $v \subseteq [\gamma_1]$ is seen to be symmetric by interchanging the roles of σ and e.

Suppose $\mathscr{C}_e$ is of type (B2). If m_2' exists and $m_2' \in [\gamma_2]$, then n is last in both σ and e, contradicting nondegeneracy. Therefore if m_2' exists, $v \subseteq (m_2 \ldots o]$ by paragraph (13), hence $\beta \subseteq (m_2 \ldots o]$ and a contradiction is obtained as before. We may therefore assume that m_2' does not exist, and, symmetrically, that m_1 does not exist. But then $m = 1$ was last in σ, so m was inadmissible, a contradiction. $\square$

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