International Institute for
Applied Systems Analysis
Schlossplatz 1
A-2361 Laxenburg, Austria

Tel: +43 2236 807 342
Fax: +43 2236 71313
E-mail: publications@iiasa.ac.at
Web: www.iiasa.ac.at

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Polyhedral coherent risk measures, portfolio optimization and investment allocation problems

Vladimir Kirilyuk (kirilyuk@d130.icyb.kiev.ua)

Approved by

Marek Makowski (marek@iiasa.ac.at)
Leader, Integrated Modeling Environtment Project
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Foreword

The main focus of the research reported here is the development of appropriate concept of coherent risk measures which can be used in flood risk management. The proposed concept of polyhedral risk measures practically includes all well known coherent risk measures. The attractive feature of polyhedral measures is their simplicity from computational point of view: similar to Conditional Value-at-Risk, polyhedral risk measures require only linear programming methods. In this report important properties of polyhedral risk measures are discussed, the connections of these measures with well known coherent risk measures are analyzed and an example of investment allocation under catastrophic floods is considered.

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Abstract

The class of polyhedral coherent risk measures that could be used in decision- making under uncertainty is studied. Properties of these measures and invariant operations are considered. Portfolio optimization problems on the return–risk ratio using these risk measures are analyzed.

The developed mathematical technique allows to solve large-scale portfolio problems by standard linear programming methods as an example of applications, investment allocation problems under risk of catastrophic floods are considered.

Keywords: *coherent risk measure, polyhedral coherent risk measure, conditional value-at-risk, second order stochastic domination, portfolio optimization, linear programming, catastrophic flood.*

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About the Author

Vladimir Kirilyuk is the leading scientific researcher of Glushkov Institute of Cybernetics of the National Academy of Sciences of Ukraine, and a participant of Ukrainian System Analysis Committee team collaborating with IIASA projects. He received the Master degrees and PhD from Kiev National University, and the degree of Doctor of Sciences in mathematics from Glushkov Institute of Cybernetics. He was a participant of the YSSP at IIASA in 1993.

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Polyhedral Coherent Risk Measures, Portfolio Optimization and Investment Allocation Problems

Vladimir Kirilyuk (*kirilyuk@d130.icyb.kiev.ua*)

1. Introduction

Making effective decisions in risky situations is often irrational in practice. Impacts of incorrect decisions can be huge, involving catastrophic losses and consequences. How should resources for protection against accidents and for mitigation of their consequences be distributed? How should economic activity be performed in regions which are under threat of natural disasters and cataclysms? How should effective financial decisions be made? Quite often such decisions have a long-term character, and their consequences crucially depend on future uncertainties.

Let us note that in situations where results have no critical consequences and decisions are made frequently, it is possible to use traditional statements of optimization problems, e.g., to optimize some average values (return, efficiency, etc). Otherwise, it is necessary to consider some risk measures as well. For example, it is necessary to design an investment portfolio in which not only total future returns, but also corresponding risks should be taken into account, in particular, catastrophic flood risks.

Often, financial decisions are made by two criteria: the average return value and a risk measure. The second criterion allows a decision-maker to choose decisions which are more reliable (robust) with regard to their potential consequences. The main methodological question is concerned with the choice of an appropriate risk measure. Inadequate choice of the measure can lead to wrong decisions.

Recently, various functions were used for risk measure, for instance, deviation, semi-deviation, Value-at-Risk (VaR), and others [1–4]. An important step was made in [5], where four axioms were proposed for risk functions. Functions which satisfy the axioms were called coherent risk measures. In particular, such a measure is CVaR (Conditional Value-at-Risk) [6, 7] or shortfall [8] that can be considered as competitor of VaR in numerous financial applications. The concept of a spectral coherent risk measure (SCRM), which was proposed in [9], generalizes CVaR; it is, in essence, a convex combination of CVaR_α measures with various values of parameter α. As a result, SCRM preserves all attractive properties of CVaR.

A class of polyhedral coherent risk measures (PCRM), which is an important subset of the class of coherent measures, was introduced in [10]. It contains well-known coherent risk measures and guarantees a possibility of reducing portfolio optimization problems to linear programming (LP) problems.

In this paper, we consider properties of this class of risk measures, invariant operations on the class, interpretation of such risk measures, their consistency with second order stochastic domination and the reduction of portfolio optimization to appropriate LP problems. Also some generalizations of PCRM are proposed. We also analyze situations with unknown probability distributions.

As an example of application, investment allocation problems under risk of catastrophic floods are considered.

Our presentation is restricted by a consideration of discretely distributed random variables (on a finite set of elementary scenarios-events) since this allows us to simplify the presentation and is sufficient for modeling financial applications. Results can be extended to more general probabilistic spaces after refinement of appropriate technical details.

2. Definitions and some examples of risk measures

Consider only discrete random variables defined by n scenarios and a vector of scenario probabilities

$$p_0 = (p_1^0,..., p_n^0),\ p_i^0 > 0, i = 1,..., n, \sum_1^n p_i^0 = 1.$$

Let us introduce the following notations: **1** = (1, …, 1) and **0** = (0,…, 0) are n-dimensional vectors that consist of unities and zeros, respectively, $S^n = \{x= (x_1,…,x_n): \Sigma x_i \le 1, x_i \ge 0, i=1,…, n\}$ is a unit simplex, co $M = \{\Sigma\lambda_i x_i: \lambda_i \ge 0, \Sigma\lambda_i=1, x_i \in M, i=1,2,…\}$ is the convex hull of a set M, and ri M and cl M are, respectively, the relative interior and closure of the set M. By the relation $x_1 \ge x_2$ for $x_1, x_2 \in R^n$ we understand the corresponding component wise inequality.

We recall [5] that a function ρ: $R^n \to R$ is called a coherent risk measure if the following axioms are fulfilled:

$A1$) $\rho(x+c\mathbf{1}) = \rho(x) - c$ for $c \in R$ (translation equivariance);

$A2$) $\rho(\mathbf{0}) = 0$, $\rho(\lambda x) = \lambda\, \rho(x)$ (positive homogeneity);

$A3$) $\rho(x_1+ x_2) \le \rho(x_1) + \rho(x_2)$ (subadditivity);

$A4$) $\rho(x_1) \le \rho(x_2)$ if $x_1 \ge x_2$ (monotonicity).

In this case [5], the function is of the form

$$\rho(x) = \sup\{E_p[-X] \,/\, p \in P\}, \qquad (1)$$

where P is some closed convex set of probability measures (for discrete finite distributed random values it is a set of scenario probabilities), i.e., we have $P \subseteq S^n$. Since there exists a one-to-one correspondence between the function $\rho(\cdot)$ and the set P, a specification of the set P by relation (1) actually specifies a coherent measure $\rho(\cdot)$. This fact was used to define the following class of risk measures in [10].

Definition 1. Functions of the form (1) for which the set P is representable as a convex hull consisting of a finite number of points are called by *polyhedral coherent risk measures* (PCRM).

More precisely, if the set P is specified in the form

$$P = \text{co}\{p_i: i=1,…, k\},$$

or, in the equivalent form

$$P = \{p: B\, p \le c, p \ge 0\}, \qquad (2)$$

where B and c are a matrix and a vector (of corresponding dimensions), then relations (1) and (2) uniquely specify a polyhedral coherent risk measure.

We first consider some examples of PCRM that are well known in financial applications. Let a distribution $x = (x_1,\ldots, x_n)$ describe a return obtained as a result of realization of scenarios with the corresponding probabilities $p_0 = (p_1^0,..., p_n^0)$.

Example 1. Worst-case risk (WCR) is the case of worst losses [11]. Then we have $WCR(x) = \max\ \{-x_i\colon i=1,\ldots, n\}$, and the set P is of the form

$$P_{WCR} = \{p=(p_1,..., p_n)\colon p_i \geq 0,\ i=1,\ldots, n, \sum_1^n p_i = 1\}. \quad (3)$$

Example 2. Conditional value-at-risk (CVaR_α) is the conditional mean of losses on α-tail of the distribution [6]. To avoid technical details, we consider the following interpretation of this notion for discrete distributions:

$$CVaR_\alpha(x) = \max\left\{\frac{1}{\alpha}\left(\left(\sum_{k_\alpha+1}^{n} p_{i_j}^0 - 1 + \alpha\right)(-x_{i_{k_\alpha+1}}) + \sum_1^{k_\alpha} p_{i_j}^0(-x_{i_j})\right) : \sum_1^{k_\alpha} p_{i_j}^0 < \alpha \leq \sum_1^{k_\alpha+1} p_{i_j}^0\right\}.$$

In the case, appropriate set P from the unit simplex is described as

$$P_{CVaR_\alpha} = \{p = (p_1,\ldots, p_n)\colon p_i \leq p_i^0/\alpha\ ,\ p_i \geq 0,\ i=1,\ldots, n, \sum_1^n p_i = 1\}, \quad (4)$$

where $p_0 = \left(p_1^0,..., p_n^0\right)$ is the vector of initial scenario probabilities.

Note that, when $\alpha \leq \min\{\ p_i^0\ ,\ i=1,\ldots, n\}$, CVaR_α coincides with WCR since, as it is easily seen, the sets P described by relations (3) and (4) are identical in this case.

Example 3. Worst conditional expectation (WCE_α) from [5] is described as

$$WCE_\alpha(x) = \max\left\{\frac{1}{\sum_1^k p_{i_j}^0}\sum_1^k p_{i_j}^0(-x_{i_j}) : \sum_1^k p_{i_j}^0 > \alpha\right\}.$$

Therefore, we have

$$P_{WCE_\alpha} = \text{co}\{(p_1,\ldots, p_n)/ \text{ for} \sum_1^m p_{i_j}^0 > \alpha,\ p_{i_j} = p_{i_j}^0 / \sum_1^m p_{i_j}^0, j \leq m; p_{i_j} = 0, j > m\}. \quad (5)$$

Note that the coherent risk measures from Examples 1–3 are polyhedral. However, in Example 3, in contrast to Examples 1-2, it is rather difficult to write the set P in the form of explicit relation (2). The reason is that the implicitly described set from the right side of relation (5) should be presented in the form of extreme supports.

3. Interpretation of the PCRM

In paper [5], an appropriate set P used in definition (1) was called as a set of generalized probabilities of scenarios, or testing probabilities. However, risk measure (1) requires some interpretation in order to justify the construction and use of adequate risk measures in modeling.

It was assumed [12] that in decision-making, investors, estimating actual probabilities p_0 of scenarios perform monotone transformation $\pi(\cdot)$ of them. Therefore actual decisions are made on the basis of $\pi(p_0)$ rather than on actual probabilities p_0.

Various attempts have been made to explain numerous cases where different investors who observe the same distributions act as if these distributions are different. Based on similar considerations, we can propose the following interpretation of construction of the measure by relation (1).

There is a vector of actual probabilities (scenarios) p_0 estimated by a decision-maker as the set of admissible scenario probabilities P depending on initial p_0. Then a risk measure described by (1) estimates the risk of random value x as the maximal (worst) average loss for x over the set probabilities $P(p_0)$. Therefore,

$$\rho(x) = \max\{E_p[-x] \,/\, p \in P(p_0)\}, \qquad (6)$$

where $P(p_0)$ is a set of estimations of p_0 by a decision-maker.

This interpretation of P in constructing a coherent risk measure as a set of estimates of the vector of actual probabilities p_0 makes construction (1) better understandable. This approach can also be used irrespective of motives for choice of such $P(p_0)$, i.e., irrespective of the fact whether a decision-maker really estimates p_0 by $P(p_0)$ or it guarantees against some unlikely but possible case.

In particular, in Example 1 we have $P(p_0) = S^n$, i.e., any p from S^n is possible (the complete uncertainty of p_0) and, in Example 2 the set $P(p_0)$ contains all the conditional probabilities of the α-tail of worst losses of a distribution for some fixed α. Hence, in Example 1 a person guarantees against the worst-case scenario, whereas, in Example 2 the one guarantees against the case of realizing conditional probabilities of the α-tail of worst losses.

Note that if a coherent risk measure is polyhedral, then the corresponding $P(p_0)$ describes the set of scenario probabilities as a convex polyhedron, i.e. the set is estimated by some extreme points of scenario probabilities and their convex combinations. It seems likely that, in practice, this is a way to construct these sets that take into account some numerous but finite set of extreme points of these probabilities and all their possible convex combinations. Henceforth, we assume that

$$P(p_0) = \{p\colon B(p_0)\, p \le c(p_0),\, p \ge 0\}. \qquad (7)$$

4. Some invariant operations of PCRM

Since coherent risk measures are convex functions of the form (1), it is natural to introduce some elementary operations that save this type of functions. We assume that there are m coherent risk measures represented in the form (1), i.e., we have

$\rho_i(x) = \max\{<-x, p> / p \in P_i\}, i=1,\ldots, m,$

and consider the following operations defined on these measures,

(1) the operation of convex combination,

(2) the maximum function,

(3) the infimal convolution $\cdot$,

where the operation denoted by $\cdot$ is defined for convex functions f1(x), f2(x) as:

$\mathrm{f1(x)} \cdot \mathrm{f2(x)} = \inf\{\mathrm{f1(x1)} + \mathrm{f2(x2)} / \mathrm{x1} + \mathrm{x2} = \mathrm{x}\}.$

The following theorems can be easily obtained by use of a standard technique of convex analysis, for instance from [13].

Theorem 1. A convex combination, the maximum function, or infimal convolution of coherent risk measures is also a coherent risk measure and, besides, the following relations are also true:

$$\sum_1^m \lambda_i \rho_i(x) = \max\{< -x, p >: p \in \sum_1^m \lambda_i P_i\} \text{ for } \lambda_i \ge 0, \sum_1^m \lambda_i = 1; \tag{8}$$

$$\max\{\rho_i(x),\ i=1,\dots,m\}=\max\{<-x, p>: p\in \mathrm{co}(P_i,\ i=1,\dots,m)\}; \tag{9}$$

$$\mathrm{cl}(\rho_1(x),\dots,\rho_m(x))=\max\{<-x, p>: p\in \bigcap_1^m P_i\} \text{ for } \bigcap_1^m P_i \ne \emptyset, \tag{10}$$

the closure sign "cl" in the last equality can be eliminated if $\bigcap_1^m riP_i \ne \emptyset$.

These operations can be refined if they are performed on polyhedral measures, i.e. $\rho_i(x) = \max\{<-x, p> / B_i\, p \le c_i, p \ge 0\}$, $i=1,\dots,m$.

Theorem 2. A convex combination, the maximum function, or infimal convolution of PCRM are risk measures. Besides, $\lambda_i P_i = \{p: B_i\, p \le \lambda_i c_i, p\ge 0\}$, $i=1,\dots,m$ in (8), and the closure sign "cl" in equality (10) can be eliminated.

Corollary 1. If polyhedral coherent risk measures are described by identical matrices, i.e., we have $B_i = B$, $i=1,\dots, m$, then the following relation is true:

$$\sum_1^m \lambda_i \rho_i(x) = \max\{< -x, p >: Bp \le \sum_1^m \lambda_i c_i, p \ge 0\} \text{ for } \lambda_i \ge 0, \sum_1^m \lambda_i = 1. \tag{11}$$

Let us consider one more example of a coherent risk measure.

Example 4. A spectral coherent risk measure (SCRM) was introduced in [9] as follows:

$$M_\varphi(X) = -\int_0^1 \varphi(p) F_X^{\leftarrow}(p)dp. \tag{12}$$

In this case, we have

$F_X^{\leftarrow}(\alpha) \equiv \inf\{y / F_X(y) \ge \alpha\} = -VaR_{1-\alpha}(X)$, $\varphi(p) = c\delta(p) + \tilde{\varphi}(p)$,

where $\delta(.)$ is the Dirac delta-function, $c\in[0,1]$, and a function $\tilde{\varphi}: [0,1]\to R$ satisfies the following conditions:

$$1)\ \tilde{\varphi}(p) \ge 0 \forall p;\ 2)\ p_1 < p_2 \Rightarrow \tilde{\varphi}(p_1) \ge \tilde{\varphi}(p_2);\ 3) \int_0^1 \tilde{\varphi}(p)dp = 1 - c.$$

As it is well known from [9], the measure $M_\varphi(.)$ can be represented for discrete distributions as a convex combination of CVaR_α measures (for different parameters α), i.e., for some $\lambda_j \ge 0, \mathrm{j} = 1,\dots,m, \sum_1^m \lambda_j = 1$, we have

$$M_{\varphi}(x)=\sum_{1}^{m}\lambda_j CVaR_{\alpha_j}(x)\,. \qquad (13)$$

Taking into account Example 2 and Theorem 2, it is easy to see that a spectral measure is a PCRM, and its representation in form (1)-(2) can easily be obtained from (4) and (13) by relation (11). We formulate this fact in the form of the following corollary.

Corollary 2. For discrete distributions, SCRM can be presented in form (1), (2) and, in this case, the corresponding set of scenario probabilities P from (2) has the following form:

$$P_{SCRM}=\{p=(p_1,\ldots,p_n):\ p_i \le \left(\sum_{1}^{m}(\lambda_j/\alpha_j)\right)p_i^0,\ p_i\ge 0,\ i=1,\ldots,n,\ \sum_{1}^{n}p_i=1\}, \qquad (14)$$

where $p_0=\left(p_1^0,\ldots,p_n^0\right)$ is the vector of initial scenario probabilities.

Thus, the class of PCRM is wide enough since it contains these well-known coherent risk measures and is invariant with respect to the above-mentioned three operations.

5. Consistency of second order stochastic dominance with PCRM

The concept of the second order stochastic dominance (SSD) is fundamental in a theory of decision-making in economics [14]. A random value X dominates a random value Y in the sense of SSD that is denoted by $X \ge_{SSD} Y$, if the inequality $E[u(X)] \ge E[u(X)]$ is true corresponding expectations of any concave nondecreasing function $u(.)$. This concept guarantees that the expected utility of X in the sense of Neumann–Morgenstern [15] will be not less than that of Y for any concave nondecreasing utility function $u(\cdot)$, i.e., when a person is avoiding of risk. This concept was found to be important since, in practice, it is almost impossible to construct such a utility function in an explicit form, especially if it is necessary to find the consensus of opinions of a group of investors who have different ideas of utility.

As it is well known from [16], the relation $X \ge_{SSD} Y$ is equivalent to the following one

$$E[(\eta - X)^+] \le E[(\eta - Y)^+] \ \ \forall\ \eta \in R, \qquad (15)$$

where $E[\cdot]$ is the expectation and $(.)^+$ are nonnegative components.

For the case of discretely distributed random variables on a set of elementary scenarios-events with initial probabilities p_0, relation (15) is written in the form (16):

$$< (\eta\mathbf{1}-x)^+, p_0> \le < (\eta\mathbf{1}-y)^+, p_0> \quad \forall \eta \in R, \qquad (16)$$

where $(z=(z_1,\ldots,z_n))^+$ denotes vector $(\max\{0, z_1\},\ldots,\max\{0, z_n\})$.

Some sufficient conditions of consistency of SSD with PCRM can be described by the following proposition [17].

Theorem 3. Let the matrix B and vector c from the representation of a PCRM in the form (1), (2) be of the form,

$$B = \begin{pmatrix} I \\ S_2 \end{pmatrix},\ c = \begin{pmatrix} \beta p_0 \\ c_2 \end{pmatrix}, \tag{17}$$

where I is the identity matrix (dimension $n \times n$), S_2 is matrix with identical columns (dimension $n \times k$, $k \geq 2$) , βp_0 is vector p_0 multiplied by some number $\beta \geq 0$, and c is an arbitrary vector of dimension k. Then measure (1), (2) is consistent with SSD.

Corollary 3. WCR, CVaR, and SCRM measures from Examples 1–2 and 4 are consistent with SSD.

As a proof, we note that, condition (17) is true for all these measures.

Note that the property of consistency of CVaR with SSD was firstly mentioned in [18]. Using this fact, it is easy to obtain similar properties for the other two measures. In this case, reasoning in [18] was based on the following representation of CVaR, proved in [6]:

$$\mathrm{CVaR}_\alpha(x)\mathrm{CVaR}_\alpha = \min_\zeta \left[-\zeta + \frac{1}{\alpha} E[(\zeta - x)^+] \right]. \tag{18}$$

Consider a necessary condition of consistency of coherent risk measure with SSD [17].

Theorem 4. The necessary condition of consistency of a coherent risk measure represented in the form (1) with SSD is the following condition

$$p_0 \in P. \tag{19}$$

6. Portfolio optimization problems with PCRM

Portfolio optimization problem is considered to be a basic classical problem of financial mathematics of great interest for financial applications. In the pioneering paper [1], the following two criteria of constructing an efficient portfolio are considered: the portfolio expected return and its dispersion as a risk measure. Later on, other risk functions were studied instead of dispersion in a number of works (see, for example, [2–4]). After publishing [5], the great interest was shown in portfolio optimization problems in which a coherent risk measure would be used as the risk factor since this measure has theoretically attractive properties. For example, CVaR was used in [6, 7]. Moreover, in these papers, the optimal portfolio problem was reduced to some LP problem by representation of CVaR in the form (18). This allows for efficient solving large-scale problems required for practical applications.

Next, as it is shown in [10], portfolio optimization problems based on return-risk ratio can also be reduced to the corresponding LP problems for the class of PCRM. This is true for both types of such problems: the portfolio PCRM minimization under guaranteed return expectation and the portfolio return expectation maximization under PCRM constraint.

Let us consider these results. Let the set of distributions of returns for all possible assets z_j, j=1,…, k of a portfolio be represented in the form of $n\times k$ matrix H, whose j-th the column describes return distribution of the j-th asset. A vector $u=(u_1,\dots,u_k)$ that describes the structure of the portfolio is considered as a variable and, therefore, we have $\sum_1^k u_i = 1$, $u_i \geq 0$, i=1,…,k.

Consider a portfolio PCRM minimization problem under guaranteed return expectation. If the expectation should be not less than μ, a set of constraints M imposed on the structure of the portfolio u is defined by

$$M=\{<u, \mathbf{1}> =1, <Hu, p_0> \geq \mu, u \geq 0\},$$

where p_0 is a vector of scenario probabilities. As is easily seen, $M=\{u: Au\leq b, u\geq 0\}$, where matrix A and vector b are of the form

$$A=\begin{pmatrix} 11...1 \\ -1-1...-1 \\ -p_0^T H \end{pmatrix},\ b=\begin{pmatrix} 1 \\ -1 \\ -\mu \end{pmatrix}. \tag{20}$$

Let a PCRM be given in the form (1)-(2), where the set of scenario probabilities is $P=\{p: Bp\leq c, p\geq 0\}$ and

$$B=\begin{pmatrix} 11...1 \\ -1-1...-1 \\ B_0 \end{pmatrix},\ c=\begin{pmatrix} 1 \\ -1 \\ c_0 \end{pmatrix}. \tag{21}$$

Note that, in equalities (20) and (21), the first two rows represent standard constraints on structural components of the portfolio and the corresponding probabilities: $\sum_1^k u_i = 1$, $\sum_1^n p_i = 1$. A matrix B_0 and a vector c_0 describe the set P that corresponds to a concrete risk measure.

As it is easily seen from the previous discussion and relations (3), (4), and (14), for the cases of WCR, CVaR$_\alpha$, and SCRM, they are, respectively, of the form:

$$B_0,\ c_0 \text{ are absent (WCR)}, \tag{22}$$

$$B_0 = I,\ c_0 = (1/\alpha)\, p_0\ (\text{CVaR}_\alpha), \tag{23}$$

$$\mathrm{B}_0 = \mathrm{I}, \mathrm{c}_0 = \left(\sum_1^m (\lambda_j/\alpha_j)\right) p_0,\ (\text{SCRM}). \tag{24}$$

The problem of minimization of a polyhedral coherent risk measure under constraints on expected guaranteed return can be formulated in the following form

$$\min_{Au\leq b, u\geq 0} \rho = \min_{Au\leq b, u\geq 0} \max_{Bp\leq c, p\geq 0} <\mathrm{Hu}, \mathrm{p}>, \tag{25}$$

where component ($-Hu$) describes a distribution of portfolio losses, the inner subproblem of the right side of the equality describes the portfolio risk measure, and the outer subproblem describes minimization of the measure on the portfolio structure that takes into account the constraint on the portfolio return expectation. This problem can be reduced to a LP problem for all class of PCRM [10].

Theorem 5. The solution of optimal portfolio problem (25) is the component u of the solution (v, u) of the following LP problem:

$$
\begin{gathered}
\min_{(v,u)} < c, v > \\
-B^T v - Hu \le 0 \\
Au \le b \\
v \ge 0, u \ge 0
\end{gathered}
\tag{26}
$$

In this case, taking into account expressions (22)–(24), problem (26) can be immediately written in an explicit form for the cases of WCR, CVaR, and SCRM. We note that a similar result was obtained for the WCR measure in [11]. In [6, 7], problem (25) for the case of CVaR is reduced to the same LP problem (in a different form).

As has been already mentioned, although the WCE from Example 3 belongs to the class of PCRM, the main difficulty in studying this risk measure consists of the representation of implicitly described set (5) in the form of a set of extreme supports (2). If such a reduction (sufficiently labor-consuming) is obtained, then Theorem 5 can be used for studying the problem of minimization of this risk measure. Otherwise, it can be formulated in the form of a problem of fractional-linear programming with integer variables and then can be solved by the branch and bound method [19].

Remark 1. It may be shown that risk measures based on the absolute deviation and the semideviation from the average return from [3, 4] under some conditions also belong to the class of polyhedral coherent risk measures. In this case, by analogy with (22)–(24), the matrix B_0 and vector c_0 can be explicitly written for these measures.

Let us consider the portfolio return expectation maximization problem when values of the risk measure being used are constrained by some level $\rho_0 > 0$.

$$
\begin{gathered}
\max < Hu, p_0 >, \\
\textstyle\sum_1^n u_i = 1, u_i \ge 0, \\
\rho(Hu) \le \rho_0,
\end{gathered}
\tag{27}
$$

where, as before, u describes the structure of a portfolio, Hu is a portfolio return, and a PCRM is represented in the form (1), (2). Then the following proposition holds [10].

Theorem 6. The solution of problem (27), (1), (2) is the component u of the solution (v, u) of the following LP problem

$$\begin{aligned}
&\max_{(v,u)} < u, H^T p_0 >, \\
&-B^T v - Hu \le 0, \\
&< c, v > \le \rho_0, \\
&\textstyle\sum_1^n u_i = 1, \\
&v \ge 0, u \ge 0.
\end{aligned} \tag{28}$$

A similar situation also takes place for several constraints on polyhedral coherent risk measures. For example, if m such measures

$$\begin{aligned}
&\rho_i(x) = \max < -x, p > \\
&B_i p \le c_i \\
&p \ge 0, i = 1,...,m,
\end{aligned} \tag{29}$$

are given and the optimal portfolio problem is considered in the form

$$\begin{aligned}
&\max < Hu, p_0 >, \\
&\textstyle\sum_1^n u_i = 1, u_i \ge 0, \\
&\rho_i(Hu) \le \rho_i^0, i = 1,...,m.
\end{aligned} \tag{30}$$

Theorem 7. The solution of problem (30), (29) is a component u of a solution (v_1, v_2,…, v_m, u) of the following LP problem.

$$\begin{aligned}
&\max_{(v_1,...,v_m,u)} < H^T p_0, u > \\
&-B_1^T v_1 - Hu \le 0 \\
&< c_1, v_1 > \le \rho_1^0 \\
&.................... \\
&-B_m^T v_m - Hu \le 0 \\
&< c_m, v_m > \le \rho_m^0 \\
&\textstyle\sum_1^n u_i = 1 \\
&v_1 \ge 0,...,v_m \ge 0, u \ge 0.
\end{aligned} \tag{31}$$

In this case, taking into account (22)–(24), problems (26), (28) and (31) are explicitly written for cases of WCR, CVaR, and SCRM.

Note that similar problem with CVaR constraints was already considered in [6, 7] and was reduced to appropriate LP problems. Theorems 6 and 7 are applicable to the entire class of PCRM. To use them we need to represent the corresponding risk measure in the form (1)-(2).

In [20], for problems that use the WCE risk measures from Example 3 when an explicit description of the set P in the form (2) is absent, a combined algorithm is given that also solves LP problems, estimates obtained solutions, adds new constraints, etc. It can be used to solve similar problems.

7. Some generalization of PCRM and portfolio optimization

Generally speaking, reducing portfolio optimization problems to appropriate LP problems does not demand a coherent property of risk measures, polyhedral property is sufficient [21].

Definition 2. Function

$$\delta(x) = <-x, a> + \sigma(x), \tag{32}$$

$$\sigma(x) = \max\{<-Ax, p> / Bp \le c, p \ge 0\}, \tag{33}$$

is called *polyhedral risk measure* (PRM), where <·,·> is scalar multiplication, A and B – some matrixes of dimensions $n\times n$ and $n\times m$ respectively, a and c are some vectors, $a\in R^n$, $c\in R^m$, and set $\{p \ge 0: Bp \le c\}$ is nonempty and bounded,.

Remark 2. Condition $p \ge 0$ in (33) is not essential. A more general situation can be reduced to the case by appropriate transformation of variables.

This definition permits to expand a class of risk measures in comparison with PCRM one, since PCRM class is a special case of the current one under conditions $a = \mathbf{0}$, $A = I$ and $\{B\, p \le c, p \ge 0\} \subseteq S^n$.

Example 5. Consider functions of the form:

$$d\,(x; r) = -E\,[x] + r\rho(x), \tag{34}$$

where $r \ge 0$, and $\rho(x)$ is PCRM from (1)-(2). Then its parameters also are described as:
$a = p_0$, $A = r\,I$, and B and c are inherited from (2).

Risk measures designed by the absolute deviation and by the semideviation on return expectation [3]-[4] are in the class.

Example 6. Consider a risk measure designed by the semideviation on return expectation [4]:

$$\delta_S(x; r) = -E[x] + r\,E[(E[x] - x)^+] = <-x, p_0> + r\rho_S(x), \tag{35}$$

where

$$\rho_S(x) = \max\{<-A_1\, x, p> / I\, p \le p_0, p \ge 0\},$$

$$\rho_S(x) = \max\{< A_1 x, p > / Ip \le p_0, p_0 \ge 0, A_1 = I - \begin{pmatrix} p_0^T \\ \dots \\ p_0^T \end{pmatrix}. \tag{36}$$

Then parameters from (35)-(36) are described as: $a = p_0$, $A = r\,A_1$, $B = I$, $c = p_0$.

Example 7. Consider now a measures designed by the absolute deviation [3],

$$\delta_A(x; r) = -E[x] + r\,E[\,|x - E[x]|\,] = <-x, p_0> + r\rho_A(x), \tag{37}$$

where $\rho_A(x) = \max\{<-A_1 x, p> / -p_0 \le Ip \le p_0\}$, $r \ge 0$.
It's obvious representation is:
$\rho_A(x) = 2\rho_S(x)$. Then the following equality holds

$$\delta_A(x; r) = \delta_S(x; r) + r\rho_S(x) = <-x, p_0> + 2\,r\rho_S(x), \tag{38}$$

where function $\rho_S\,(x)$ is described by (36). Then $a = p_0$, $A = 2rA_1$, $B = I$, $c = p_0$.

Obvious properties of the measure follow directly from the definition of PRM.

Theorem 8. PRM satisfies axioms *A2* and *A3*, i.e., it is positively homogeneous and subadditive.

Introduce the following notations: $M = \{p \geq 0: Bp \leq c\}$ is the appropriate admissible set, $M(x) = \{p \in M: \delta(x) = \langle -x, a\rangle + \langle -Ax, p\rangle\}$ is a set of those p on which the maximum is attained. Consider now conditions of performance of axioms $A1$ and $A4$ [21].

Theorem 9. A necessary and sufficient condition of coherent property of PRM is described by the following relation

$$A^T M + a \subseteq S^n. \quad (39)$$

Axiom $A4$ is fulfilled under the following condition:

$$A^T M + a \subseteq R^n_+. \quad (40)$$

Corollary 4. Function $d(x; r)$ from (34) satisfies to axioms $A2$-$A4$. Function $\delta(x) = -\lambda E[x] + (1-\lambda)\rho(x)$, where $0 \leq \lambda \leq 1$ and $\rho(\cdot)$ is PCRM, is PCRM as well.

Corollary 5. $\delta_S(x; r) = -E[x] + rE[(E[x]-x)^+]$ from (35) at $0 \leq r \leq 1$ is PCRM.

Remark 3. As to $\delta_S(x; r)$ at $r > 1$ it, generally speaking, does not satisfy axiom $A4$ since (39) is not true. Properties of function $\delta_A(x; r)$ easily follow from properties $\delta_S(x; r)$ by known equality $\delta_A(x; r/2) = \delta_S(x; r)$.

Remark 4. Functions $\delta(\cdot)$ from Corollary 1 and $\delta_S(x; r)$ for $0 \leq r \leq 1$ can be presented as PCRM in standard form of (1)-(2).

Consider now some operations for PRM.

Theorem 10. Multiplication to non-negative numbers, addition, operation of maximum and the infimal convolution are invariant for PRM class, and the following calculation holds:

$$\lambda\delta(x) = \langle -x, \lambda a\rangle + \max\{\langle -Ax, p\rangle \,/\, Bp \leq \lambda c, p \geq 0\};$$

$$\delta_1(x) + \delta_2(x) = \langle -x, a_1 + a_2\rangle + \max\{\langle -x, q\rangle \,/\, q \in A_1^T M_1 + A_2^T M_2\}; \quad (41)$$

$$\max\{\delta_1(x), \delta_2(x)\} = \max\{\langle -x, q\rangle \,/\, q \in \mathrm{co}\{A_1^T M_1 + a_1, A_2^T M_2 + a_2\}\}; \quad (42)$$

$$\delta_1(x) \,\square\, \delta_2(x) = \max\{\langle -x, q\rangle / q \in \{A_1^T M_1 + a_1\} \cap \{A_2^T M_2 + a_2\}\}, \quad (43)$$

if $\{A_1^T M_1 + a_1\} \cap \{A_2^T M_2 + a_2\} \neq \varnothing$.

The theorem can be easily proven by using a standard technique of the convex analysis.

Consider now portfolio optimization problems using PRM (32)-(33). As before, return distributions of portfolio components z_j, $j=1,\ldots,k$ are presented by matrix H of dimension $n\times k$, which *j-th* column describes return distribution of *j-th* component. A vector $u = (u_1,\ldots, u_k)$ describes portfolio structure: $\sum_1^k u_i = 1$, $u_i \geq 0$, $i=1,\ldots, k$. Let also constraints on a portfolio return expectation by μ_0 and a portfolio PRM by δ_0 be given. Then the following theorems hold [21].

Theorem 11. The solution of the portfolio PRM minimization problem under guaranteed return expectation by μ_0 is the component u of the solution (u, v) of the following LP problem:

$$\min_{\substack{\sum_1^k u_i=1,u\geq 0\\ <u,H^T p_0>\geq \mu_0}} \delta(Hu) = \min_{\substack{\sum_1^n u_i=1,u\geq 0\\ AHu+B^T v\geq 0, v\geq 0\\ <u,H^T p_0>\geq \mu_0}} \left[<-H^T a,u>+<c,v> \right] \tag{44}$$

Theorem 12. The solution of the portfolio return expectation maximization problem under PRM constraint by δ_0 is the component u of the solution (u, v) of the following LP problem:

$$\begin{aligned} &\max <H^T p_0, u> \\ &\textstyle\sum_1^n u_i = 1, u \geq 0, \\ &AHu + B^T v \geq 0, v \geq 0, \\ &-<H^T a, u> + <c, v> \leq \delta_0. \end{aligned} \tag{45}$$

Theorem 13. The following reformulation of problem (45) is true under m PRM constraints:

$$\max_{\substack{\sum_1^k u_i=1,u\geq 0\\ \delta_1(Hu)\leq \delta_1^0\\ \dots\dots\dots\dots\\ \delta_m(Hu)\leq \delta_m^0}} <H^T p_0,u> = \max_{\substack{\sum_1^n u_i=1,u\geq 0\\ A_1 Hu+B^T v_1\geq 0, v_1\geq 0\\ -<H^T a_1,u>+<c_1,v_1>\leq \delta_1^0\\ \dots\dots\dots\dots\dots\dots\dots\dots\\ A_m Hu+B_m^T v_m\geq 0, v_m\geq 0\\ -<H^T a_m,u>+<c_m,v_m>\leq \delta_m^0}} <H^T p_0,u>$$

where matrixes A_i, B_i and vectors a_i, c_i are parameters of measures $\delta_i(\cdot)$ for i=1,…, m.

8. Induced uncertainty and portfolio optimization problems

In representation of $\rho(\cdot)$ as (1)-(2), an interpretation of set P from (2) is important. If vector p_0 of scenario probabilities is known, and decisions are made on the basis of expected return $E_{p_0}[X]$– risk measure $\rho(X)$ then the set P from (2) is used, more likely for reinsurance from possible losses. The example of such situation is CVaR measure when a decision-maker is guided by conditional probabilities α-tail of the return distribution (for some α).

Essentially different situation occurs when matrix H of component distributions for scenarios remains known, but vector $p_0 = (p_1^0, ..., p_n^0)$ of scenario probabilities is not (only its estimation by inclusion $p_0 \in P$ is available). As a rule, determining probabilities of future scenarios is an essentially more complicated problem than developing scenarios of future events. Especially it is true for rare events [22]. Sometimes it is possible to estimate such probabilities of scenarios with in some bounds, or, more mathematically, as inclusion: $p_0 \in P$, where P is a polyhedron.

However such a situation adds basic difficulties to the problem because when probabilities p_0 are unknown, it is impossible to operate even with average return values $E_{p_0}[X]$. Consider a formal technique suitable for studying such situations. P and Q**.** be two formal sets of scenarion probabilities**.**

Definition 3. The following values are called ***lower*** and ***upper estimations*** of random variable X on sets of scenario probabilities P and Q:

$$LE_P(X) = \inf\{E_p[X] \,/\, p \in P\}, \tag{46}$$

$$UE_Q(X) = \sup\{E_p[X] \,/\, p \in Q\}. \tag{47}$$

The following interval could be called ***estimation*** of X on sets of scenario probabilities P and Q:

$$ES_{P,Q}(X) = [LE_P(X),\ UE_Q(X)].$$

We will estimate a random variable X by ratio of parameters $LE_P(X)$ and $UE_Q(X)$, or, in previous terms, by ratio

$(UE_Q(X), -LE_P(X)) = (g(X), \rho(X))$, where

$$\rho(X) = \max\{E_p[-X] \,/\, p \in P\} = -\min\{E_p[X] \,/\, p \in P\}, \tag{48}$$

$$g(X) = \max\{E_p[X] \,/\, p \in Q\}. \tag{49}$$

Remark 5. It is obvious, that if $Q = \{p_0\}$ in (49) and P in (48) are from (1), the problem is reduced to estimation of ($E_{p_0}[X]$, $\rho(X)$) ratio, i.e. expected return and risk measure. If $P = Q$ and it is an estimation of unknown scenario probabilities by the set, X is estimated by pair $(LE_P(X), UE_P(X))$.

Definition 4. Functions $g(X)$ and $\rho(X)$ are called ***return*** and ***risk*** functions of random variable X *respectively*. They are ***polyhedral*** if sets P and Q are polyhedral, i.e. described as a convex hull of finite number of points.

Theorem 14. Function $g(X)$ satisfies the following axioms:

$A1$ ") $g(X+c\mathbf{1}) = g(X) + c$ for $c \in R$;

$A2$) $g(\mathbf{0})=0$, $g(\lambda X) = \lambda\, g(X)$ (positive uniformity);

$A3$) $g(X+Y) \le g(X) + g(Y)$ (subadditivity);

$A4$) $g(X) \le g(Y)$ if $X \le Y$ (monotony).

The proof follows from representation $g(\cdot)$ by (49) and appropriate properties of set Q.

Note that properties of function $g(\cdot)$ are practicable. In particular, its subadditivity means that a portfolio diversification, reducing risk of a financial stream, also reduces a potentially possible return as well.

We will formulate now return-risk portfolio optimization problems by functions $g(\cdot)$ and $\rho(\cdot)$, and reduce them to LP problems.

Recall that functions $g(\cdot)$ and $\rho(\cdot)$ from (48), (49) are polyhedral if the sets of scenario probabilities P and Q are of the following type:

$$P = \{p: B\, p \le c, p \ge 0\} = \text{co } \{p_i, i = 1,\dots, l\}. \quad (50)$$

$$Q = \{p: C\, p \le d, p \ge 0\} = \text{co } \{q_i, i = 1,\dots, s\} \quad (51)$$

Remark 6. A technique for designing risk measures induced by an initial polyhedral coherent risk measure and an estimation of unknown scenario probabilities by a set, i.e. appropriate P_i and Q sets can be found in [22].

As before, return distributions of portfolio components z_j, j=1, …, k are given by matrix H of dimension $n\times k$, which j-th column describes return distribution of j-th component. According to an available information about the process modeled, a set of scenario probabilities P and Q be given, and risk $\rho(\cdot)$ and return $g(\cdot)$ functions be calculated by (48)-(51).

Consider now portfolio optimization problem by return $g(\cdot)$ and risk $\rho(\cdot)$ ratio. We start with portfolio risk minimization problem under guaranteed return $g(\cdot) \ge g_0$:

$$\begin{aligned} &\min \rho(Hu). \\ &\sum u_i = 1, u \ge 0, \\ &g(Hu) \ge g_0. \end{aligned} \quad (52)$$

Theorem 15. The solution of optimal portfolio problem (52) is the component u of the solution (v, u) of the following problem:

$$\begin{aligned} &\min_{1\le i\le s}\{\min_{(u,v)} < c, v >\}, \\ &\sum u_i = 1, u \ge 0, v \ge 0, \\ &-B^T v - Hu \le 0, \\ &< Hu, q_i > \ge g_0, \end{aligned} \quad (53)$$

where q_i, i =1,…, s are extreme points of set Q from (51) and agreement:

$$\begin{aligned} &\min_{(u,v)} < c, v > = +\infty, \\ &\sum u_i = 1, u \ge 0, v \ge 0, \\ &-B^T v - Hu \le 0, \\ &< Hu, q_i > \ge g_0 \end{aligned}$$

if admissible set of the subproblem is empty.

The proof of the theorem is similar to previous ones. We should proceed from initial LP problem

$\rho(Hu) = \max\{E_p[-Hu] \,/\, p\in P\}$

to its dual and perform simple transformations of standard LP technique. Besides, the following reasonings are used for reduction of nonlinear constraints from the left part of equality (53) to linear constraints:

$$C\, p \le d, p \ge 0, < H\, u, p > \ge g_0 \Leftrightarrow \exists\, \lambda_i \ge 0, \Sigma\, \lambda_i = 1, < H\, u, \Sigma\, \lambda_i\, p_i > \ge g_0 \;\Leftrightarrow$$

$$\Leftrightarrow \exists\, q_i : < H\,u, q_i > \geq g_0.$$

Therefore, problem (52) is reduced to an LP problem.

Portfolio return maximization problem under constraint of portfolio risk as $\rho(\cdot) \leq \rho_0$ is formulated as:

$$\begin{aligned} &\max g(Hu), \\ &\sum u_i = 1, u \geq 0, \\ &\rho(Hu) \leq \rho_0. \end{aligned} \qquad (54)$$

Theorem 16. The solution of optimal portfolio problem (54) is the component u of the solution (v, u) of the following problem:

$$\begin{aligned} &\max_{1 \leq j \leq s} \{\max_{(u,v)} < Hu, q_j >\}, \\ &\sum u_i = 1, u \geq 0, v \geq 0, \\ &-B^T v - Hu \leq 0, \\ &< c, v > \leq \rho_0, \end{aligned} \qquad (55)$$

where q_j, $j = 1, \ldots, s$ are extreme points of set Q from (51).

The proof of Theorem 16 is similar to the previous one. We should proceed from the initial LP problem

$\rho(Hu) = \max\{E_p[-Hu] \,/\, p \in P\}$

to its dual and perform simple transformations. Besides, the internal problem from the right part of (55) is concave, therefore, the maximum is reached on extreme points of set P. Therefore, problem (54) is reduced to LP problems.

In some situations, we can have different measures of risk $\rho_i(.)$ which use different sets P_i constructed by an initial set P for various reasons, for example reinsurance (see above described interpretation of PCRM).

Consider the problem statement if return function $g(.)$ is described by (49), (51), and risk measures are described as

$$\rho_i(x) = \max\{E_p[-X] \,/\, p \in P_i\}, \qquad (56)$$

where

$$P_i = \{p: B_i\, p \leq c_i, p \geq 0\} = \mathrm{co}\,\{\, p_j^i, j = 1, \ldots, s_i \}, \; 1 \leq i \leq m, \qquad (57)$$

and constraints of these risk measures are given by ρ_i^0, $1 \leq i \leq m$ respectively.

Then an appropriate portfolio return maximization problem under constraints on these risk functions is formulated as follows:

$$
\begin{aligned}
&\max g(Hu),\\
&\sum u_i = 1, u \ge 0,\\
&\rho_1(Hu) \le \rho_1^0\\
&\dots\dots\dots\dots\dots\\
&\rho_m(Hu) \le \rho_m^0.
\end{aligned}
\qquad (58)
$$

Then the following theorem can be formulated, and proved similarly to previous ones.

Theorem 17. The solution of optimal portfolio problem (49), (51), (56)-(58) is the component u of the solution (u, v_1, …, v_m) of the following problem:

$$
\begin{aligned}
&\max_{1 \le j \le s} \{\max_{(u, v_1, \dots, v_m)} < u, H^T q_j >\},\\
&\sum u_i = 1, u \ge 0, v \ge 0\\
&- B_1^T v_1 - Hu \le 0\\
&< c_1, v_1 > \le \rho_1^0\\
&\dots\dots\dots\dots\dots\dots\dots\\
&- B_m^T v_m - Hu \le 0\\
&< c_m, v_m > \le \rho_m^0\\
&v_1 \ge 0, \dots, v_m \ge 0
\end{aligned}
\qquad (59)
$$

where q_j, $1 \le j \le s$ are extreme points of set Q from (51).

9. Investment allocation problems under risk of catastrophic floods

Let us consider the applicability of the proposed mathematical technique for optimal investment under catastrophic floods. In particular, here we follow the methodology for catastrophic risk management developed at IIASA on the basis of stochastic programming modeling and search technique [23-28].

Let considеer a collection of base scenarios S, which are inputs in modeling system of floods occurrence and inundations characterized by probabilities p_0. For such scenarios in regions of a river basins, the historical data on occurrence of flooding and their distribution, or a collection of events triggering the flooding, with their further modeling on corresponding systems and regional models can be used.

Consider the following problem. Let there be a number of investment objects that have not only different returns, but also different damage levels caused by flooding and inundation. Moreover, their returns as well as their damages essentially depend on location of these objects in a region. For instance, closeness to water of recreation objects is appealing to tourists (correspondingly, their investment returns) but in two cases their potential damages by flooding.

Which objects should be chosen and how should these objects be located in region in order to optimize the total investment portfolio with respect to return-risk ratio?

We assume similar to [23-28] that the region is divided into cells, and a set of all potential investment objects (indexed by, k) located in each possible cells (indexed by, m) is considered as portfolio components ($k \times m$ components).

Now for each portfolio component $i \in 1,..., k \times m$ and each scenario $j \in 1,..., s$, determine (by analysis of economic data, expert estimating, etc.) its return r_{ij} and its potential damage d_{ij} from flooding as shares of investment cost. According to scenario j *to determine* d_{ij}, it is necessary to simulate its flooding zone, water levels, flooding duration, etc., then according to characteristics of object i (its classification: 1, …, k) and its location (1, …, m) to estimate the damage level d_{ij} from flooding as a share of the cost of object i.

For the estimation of probabilistic damage distributions and estimation of the efficiency of flood mitigation plans special computer systems are developed. The examples of such systems are: freely disseminated HEC-FDA [29] and MIKE 11 GIS – FAT system [30]. HEC-FDA system supports integrated probabilistic hydrological and economic analysis flood mitigation plans. Usually the system uses flood scenarios of 50 %, 20 %, 10 %, 4 %, 2 %, 1 %, 0.4 %, 0.2 % probability of exceeding. Profiles of corresponding flood waves are imported from the system of calculation of high waters HEC-RAS (or Mike-11) through a database. For the calculation of damages catalogues of «depth – damage (percent of the lost value)» functions, defined for certain classes of economic structures (residential, commercial, industrial, agricultural, infrastructure) are used.

Thus, let for each component $i \in 1,..., k \times m$ and for each scenario $j \in 1,..., s$ return r_{ij} and potential damages d_{ij} from flooding be known. Then, matrix H of distributions of portfolio components has the following form [31]:

$$H = R - D, \tag{60}$$

where $R = \{r_{ij}\}_{i=k\times m, j=s}$, $D = \{d_{ij}\}_{i=k\times m, j=s}$.

Now this is a situation which completely falls under the technique of portfolio optimization according to theorems 5-7 and 15-17.

In the case, of known very long historical data on flooding, and climatic, hydrological, and landscape changes small enough to be neglected to a certain extent, the situation corresponds to optimal investment decisions with known distributions of random variables. Then optimal portfolio decision by return-risk ratio is described by theorems 5-7.

A more realistic is the case of incomplete information: only inclusion $p_0 \in P$ is available. This situation is described by theorems 15-17, which allow solving corresponding problems by LP technique although they demand some additional calculations.

A number of calculations have been performed for investment allocation problems in Ukrainian part of Tisza river basin. Optimal portfolio decisions were effectively found by LP tools of the MATLAB 7 system.

10. Conclusions

The class of PCRM, sufficiently broad because it contains well-known coherent risk measures and is invariant for some operations, is described in this paper. The use of risk measures from this class allows reducing portfolio optimization problems to LP problems. The study of this class allows to systematize a number of results obtained earlier as special cases, and to obtain new results for portfolio optimization problems within a unified approach. Some generalization of PCRM notion is considered as well.

The reduction of portfolio optimization problems to LP models makes it possible to efficiently solve them by standard LP methods even for practically important large-scale problems.

As an example of applications, investment allocation problems under risk of catastrophic floods are considered. Appropriate portfolio decisions were effectively found by using data for the Ukrainian part of the Tisza river basin.

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