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ON OPTIMAL COMPROMISE FOR MULTIDIMENSIONAL RESOURCE DISTRIBUTION

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Introduction

Let us consider a resource $W > 0$ is to be divided in parts $w_1,\ldots,w_n$ where ($\sum_1^n w_n = W$) and consumed in the following way: at the first step, an amount x_1,

$$\alpha_1 \leq x_1 \leq w_1 - b_1 \quad ,$$

can be consumed; at the second step, one can consume x_2,

$$\alpha_2 \leq x_2 \leq y_1 + w_2 - b_2 \quad ,$$

where

$$y_1 = w_1 - x_1 \quad ,$$

and, generally, at the kth step, consumption is x_k,

$$a_k \leq x_k \leq y_{k-1} + w_k - b_k \quad , \tag{1}$$

where a_k, b_k are some non-negative constants and y_k,

$$y_0 = 0 \quad , \quad y_k = \sum_{j=1}^{k} w_j - \sum_{j=1}^{k} x_j \quad ; \quad k = 1,\ldots,n.$$

Of course, all parameters a_k, b_k and w_k $(k = 1,\ldots,n)$ are such that there is at least one feasible resource distribution, say

$$x_1 = a_1, \ldots, x_n = a_n \quad .$$

The problem is to find the "best" distribution of resources. If it is possible to assume that the distribution $x_1, \ldots, x_n$ of resources gives us a total benefit

$$u(x) = \sum_1^n c_k x_k \quad ,$$

where $c_1, \ldots, c_n$ are some coefficients, then the finding of the best feasible distribution of resources becomes a classical linear programming problem.

But suppose there are, in some sense, <u>independent</u> consumers who are not very much interested in the <u>total</u> benefit, and the desirable purpose of each of them is to receive as much as possible. Suppose these consumers (who are not living in a jungle!) want to reach a compromise based on some <u>reasonable</u> demands.

Namely, suppose it is given that some demands $x_1^*, \ldots, x_n^*$ are considered as quite reasonable by all consumers, yet these demands are <u>not feasible</u>: $x_1^*, \ldots, x_n^*$ do not satisfy the constraints (1). For example, the total demand $\sum_1^n x_k^*$ can be much more than the total resources amount $W = \sum_1^n w_k$. The problem is how to distribute our resources $w_1,\ldots,w_n$ according to the demands $x_1^*,\ldots,x_n^*$, which contradict each other in the sense that if we satisfy one group of consumers, then we leave too little for others.

A solution of this problem may be based on minimization (in the proper sense) of some "distance" between resource distribution vector $x = (x_1,\ldots,x_n)$ and the demand vector $x^* = (x_1^*,\ldots,x_n^*)$. This is considered below.

We wish to say that our problem on a compromise for many consumers which was described above arose in the water resource distribution field.

Example. One can realize a problem of water storage and of water distribution during some n sequential periods of time. Suppose during each kth period, the storage receives the water amount w_k, and some amount x_k, $x_k \geq a_k$, is taken in such a way that the rest of water resource will not be less than $b_k \geq 0$; $k = 1,\ldots,n$. If we are given the desirable demands $x_1^*,\ldots,x_n^*$ for the water from this storage (x_k^* for the kth period of time), then we have to deal with the problem described above.

Example. Let us consider a big river basin which is divided, according to geographic or economic principles, in n sequential parts (along the main river). Let the total available amount of water (in the proper scale) at the kth region of this basin be w_k. Suppose at every region one can consume a corresponding water amount x_k, $x_k \geq a_k$, such that the rest has to be not less than some $b_k \geq 0$. Obviously, if at the first (k-1) parts it consumed amounts $x_1, \ldots, x_{k-1}$, then x_n is bounded with the value $\sum_{j=1}^{k} w_j - \sum_{j=1}^{k-1} x_j - b_n$. Under the desirable but non-realistic water demands $x_1^*, \ldots, x_n^*$ for all n regions of the river basin, we have to deal with the problem on a compromise concerning the actual water distribution.

In the general situation of the resources shortage, when it is reasonable to assume that

$$x_k \leq x_k^* \quad ; \qquad k = 1, \ldots, n, \tag{2}$$

we suggest that to determine a distance between distribution vector $x = (x_1, \ldots, x_n)$ and demand vector $x^* = (x_1^*, \ldots, x_n^*)$ as

$$R(x, x^*) = \sqrt{\sum_{1}^{n} \lambda_k (x_k - x_k^*)^2} \quad , \tag{3}$$

where $\lambda_1, \ldots, \lambda_n$ are some non-negative coefficients. The choice of the proper $\lambda_1, \ldots, \lambda_n$ may be considered as re-evaluation of different demands $x_1^*, \ldots, x_n^*$ under certain circumstances. For example, some demands x_k^* may be neglected completely (under the choice of $\lambda_k = 0$). But we suppose the choice of weight-coefficients $\lambda_1, \ldots, \lambda_n$ is such that all consumers agree to consider the corresponding metric $r(x, x^*)$ as the loss function, i.e. a vector x' is preferable with respect to a vector x" if

$$r(x',x^*) \leq r(x'',x^*) \quad .$$

According to this agreement, the most preferable distribtution vector will be

$$x^o = (x_1^o,\ldots,x_n^o) \quad ,$$

for which

$$r(x^o,x^*) = \min_x r(x,x^*) \quad , \tag{4}$$

where x runs all possible distribution, i.e. $x = (x_1,\ldots,x_n)$ satisfies to the constraints (1) and (2).

Unfortunately, all components $x_1^o,\ldots,x_n^o$ of such minimum points generally depend on <u>all</u> resource components $w_1,\ldots,w_n$, and if we have to choose the amount x_k only with our knowledge of $w_1,\ldots,w_k$; $x_1,\ldots,x_{k-1}$, then we actually choose the proper x_k^o; $k = 1,\ldots,n$.

But sometimes we can assume that all $w_1,\ldots,w_n$ actually already are known at the first step. Say for a water storage or a basin with one big river, the components $w_2,\ldots,w_n$ may be much less than w_1, and in this case we can assume approximately that

$$w_1 = W \quad ; \qquad w_2 = \cdots = w_n = 0 \quad . \tag{5}$$

Let us consider the case when all resource components $w_1,\dots,w_n$ are known from the very beginning, and we can assume minimum point $x = (x_1^o,\dots,x_n^o)$ of the loss function $r(x,x^*)$ as the <u>optimal compromise</u> for our resource distribution problem.

The demands $x_1^*,\dots,x_n^*$, which generally are implicit functions of actual resources $w_1,\dots,w_n$, in this case are known constants, and the minimization problem for loss function $r(x,x^*)$ of the type (3) is a problem of quadratic programming. Namely, after a variables substitution,

$$w_1 - (a_1 + b_1) \to w_1$$

$$w_2 - (a_2 + b_2 - b_1) \to w_2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$w_n - (a_n + b_n - b_{n-1}) \to w_n \quad ,$$

$$x_k - a_k \to x_k \quad ; \qquad k = 1,\dots,n,$$

the constraints (1) and (2) can be described as

$$0 \le x_k \le y_{k-1} + w_k \quad ; \qquad k = 1,\dots,n, \tag{1'}$$

where

$$y_0 = 0 \quad , \qquad y_k = \sum_{j=1}^{k} w_j - \sum_{j=1}^{k} x_j \quad ,$$

or in another form as

$$\sum_{j=1}^{k} x_j \leq \sum_{j=1}^{k} w_j \quad ; \qquad k = 1,\ldots,n. \tag{1''}$$

(Remember that $0 \leq x_k \leq x_k^*$, according to our assumption [2]).

1. Let us try to give some explicit formulas for the optimal $x_1^o,\ldots,x_n^o$, using the well-known Bellman's principle of dynamic programming. (See, for example, [2].)

Namely, let us begin with minimization of

$$\lambda_n (x_n - x_n^*)^2 \quad ,$$

where

$$0 \leq x_n \leq \min\,(y_{n-1} + w_n,\ x_n^*) \quad .$$

Obviously, x_n^o as a function of

$$\xi_n = y_{n-1} + w_n = \sum_{k=1}^{n} w_k = \sum_{k=1}^{n-1} x_k$$

is the following:

$$x_n = \begin{cases} \xi_n \ , & \xi_n \leq x_n^* \\ x_n^* \ , & \xi_n \geq x_n^* \end{cases} \quad .$$

(See Figure 1.)

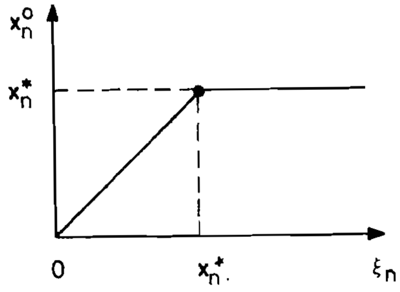


FIGURE 1

Remember that

$$y_{n-1} = \sum_{k=1}^{n-1} w_k - \sum_{k=1}^{n-1} x_k \quad .$$

Let us fix $x_1,\ldots,x_{n-2}$ and set

$$z_{n-1} = \sum_{k=1}^{n-1} w_k - \sum_{k=1}^{n-2} x_k + w_n \quad .$$

Then we have

$$x_n^o = \min\,(z_{n-1} - x_{n-1},\, x_n^*) \quad .$$

If $z_{n-1} > x_n^*$, then there is a feasible amount $x_{n-1} \geq 0$ such that $z_{n-1} - x_{n-1} \geq x_n^*$ and $x_n^o = x_n^*$. It holds true, for example, in the case of $w_n \geq x_n^*$, when the optimal x_{n-1}^o is obviously similar to x_n^o, namely,

$$x_{n-1}^o = \max\,(y_{n-2} + w_{n-1},\, x_{n-1}^*) \quad .$$

Suppose $\underline{w_n < x_n^*}$. If $z_{n-1} \geq x_n^* + x_{n-1}^*$, then

$$y_{n-2} + w_{n-1} > x^*_{n-1} \quad , \quad y_{n-1} + w_{n-1} - x^*_{n-1} + w_n > x^*_n \quad ,$$

and not only the last demand will be satisfied ($x^o_n = x^*_n$) but also $x^o_{n-1} = x^*_{n-1}$. In the case

$$0 \leq c_{n-1} = z_{n-1} - x^*_n < x^*_{n-1} \quad ,$$

we obviously have

$$\lambda_{n-1} (x_{n-1} - x^*_{n-1})^2 + \lambda_n (x^o_n - x^*_n)^2$$

$$= \begin{cases} \lambda_{n-1} (x_{n-1} - x^*_{n-1} - x^*_{n-1})^2 \quad , & \text{if } 0 \leq x_{n-1} \leq c_{n-1} \\ \lambda_{n-1} (x_{n-1} - x^*_{n-1})^2 + \lambda_n (c_{n-1} - x_{n-1})^2 \quad , & \\ & \text{if } c_{n-1} \leq x_{n-1} \leq x^*_{n-1} \end{cases}$$

and for small $\Delta x_{n-1} \geq 0$ the corresponding increment at the point $x_{n-1} = c_{n-1}$ of our utility function (under fixed $x_1,\ldots,x_{n-2}$) is

$$2\lambda_{n-1} (c_{n-1} - x^*_{n-1}) \Delta x + 0 (\Delta x^2) < 0 \quad ,$$

so the minimum point x^o_{n-1} is such that $x^o_{n-1} > c_{n-1}$. The same argument concerning $\Delta x_{n-1} < 0$ and $x_{n-1} = x^*_{n-1}$ gives us the inequality $x^o_{n-1} < x^*_{n-1}$. It means that if we cannot satisfy <u>both</u> of the demands x^*_n and x^*_{n-1}, then under the optimal compromise we have the strict inequalities

$$x^o_{n-1} < x^*_{n-1} \quad , \qquad x^o_n < x^*_n \quad .$$

In this case the function

$$f_{n-1}\,(x_{n-1}) = \lambda_{n-1}\,(x_{n-1} - x^*_{n-1})^2 + \lambda_n\,(x^o_n - x^*_n)^2$$

$$= \lambda_{n-1}\,(x_{n-1} - x^*_{n-1})^2 + \lambda_n\,(c_{n-1} - x_{n-1})^2 \quad ,$$

where

$$c_{n-1} = \sum_{k=1}^{n} w_k - \sum_{k=1}^{n-2} x_k - x^*_n$$

generally is not necessarily positive, but has the <u>absolute minimum point</u>

$$\tilde{x}_{n-1} = \frac{\lambda_{n-1} x^*_{n-1} + \lambda_n c_{n-1}}{\lambda_{n-1} + \lambda_n} \quad .$$

Because the considered function is monotone descreasing for $x_{n-1} \leq \tilde{x}_{n-1}$ and it is monotone increasing for $x_{n-1} \geq \tilde{x}_{n-1}$, we obviously obtain

$$x^o_{n-1} = \begin{cases} 0 \quad , & \tilde{x}_{n-1} \leq 0 \\ \tilde{x}_{n-1} \quad , & 0 \leq \tilde{x}_{n-1} \leq y_{n-2} + w_{n-1} \\ y_{n-2} + w_{n-1} \quad , & \tilde{x}_{n-1} \geq y_{n-2} + w_{n-1} \end{cases} \quad .$$

(Remember, as it was shown above, that under the conditions

$c_{n-1} < x^*_{n-1}$, $w_n < x^*_n$ the optimal amount x^o_{n-1} is strictly less than x^*_{n-1}.)

Let us consider x^o_{n-1} as a function of

$$\xi_{n-1} = y_{n-2} + w_{n-1} = \sum_{k=1}^{n-1} w_k - \sum_{k=1}^{n-2} x_k \quad ,$$

which is the total amount of available resources at the (n-1)th step. If the next demand x^*_n is comparatively high and ξ_{n-1} is too small, namely,

$$0 \le (x^*_n - w_n) - \frac{\lambda_{n-1}}{\lambda_n} x^*_{n-1}$$

and

$$\xi_{n-1} \le (x^*_n - w_n) - \frac{\lambda_{n-1}}{\lambda_n} x^*_{n-1} \quad ,$$

or that is the same as

$$\tilde{x}_{n-1} = \frac{\lambda_{n-1} x^*_{n-1} + \lambda_n (\xi_{n-1} + w_n - x^*_n)}{\lambda_{n-1} + \lambda_n} \le 0 \quad ;$$

then $x^o_{n-1} = 0$. It is easy to verify that the first inequality always implies

$$x^*_{n-1} - \frac{\lambda_n}{\lambda_{n-1}} (x^*_n - w_n) \le 0 \quad , \qquad \xi \ge \tilde{x}_{n-1} \quad .$$

The function x^o_{n-1} of ξ_{n-1} may be one of the following types

$$\text{(a)} \qquad x^{o}_{n-1} = \begin{cases} 0\ , & 0 \le \xi_{n-1} \le (x^*_n - w_n) - \dfrac{\lambda_{n-1}}{\lambda_n}\, x^*_{n-1} \\ \tilde{x}_{n-1}\ , & x^*_n - w_n - \dfrac{\lambda_{n-1}}{\lambda_n}\, x^*_{n-1} \le \xi \\ & \le x^*_{n-1} + x^*_n - w_n \\ x^*_{n-1}\ , & \xi_{n-1} \ge x^*_{n-1} + x^*_n - w_n \end{cases} \tag{6}$$

or

$$\text{(b)} \qquad x^{o}_{n-1} = \begin{cases} \xi\ , & 0 \le \xi_{n-1} \le x^*_{n-1} - \dfrac{\lambda_n}{\lambda_{n-1}}\,(x^*_n - w_n) \\ \tilde{x}_{n-1}\ , & x^*_{n-1} - \dfrac{\lambda_n}{\lambda_{n-1}}\,(x^*_n - w_n) \le \xi_{n-1} \\ & \le x^*_{n-1} + x^*_n - w_n \\ x^*_{n-1}\ , & \xi_{n-1} \ge x^*_{n-1} + x^*_n - w_n\ . \end{cases} \tag{7}$$

(See Figure 2.)

In particular, we obtain that under the condition of a non-extreme resources shortage, when we don't use extreme strategy 0 or ξ_{n-1} ("nothing" or "all"), the optimal amount x^{o}_{n-1} coincides with $\tilde{x}_{n-1}$, which is the <u>linear function</u> of ξ_{n-1} as well as of the parameters w_n; x^*_n, x^*_{n-1}, namely,

$$x^{o}_{n-1} = \tilde{x}_{n-1} = \frac{\lambda_{n-1} x^*_{n-1} + \lambda_n\,(\xi_{n-1} + w_n - x^*_n)}{\lambda_{n-1} + \lambda_n}\ , \tag{8}$$

where

$$\xi_{n-1} = y_{n-2} + w_{n-1} = \sum_{k=1}^{n-1} w_k - \sum_{k=1}^{n-2} x_k\ .$$

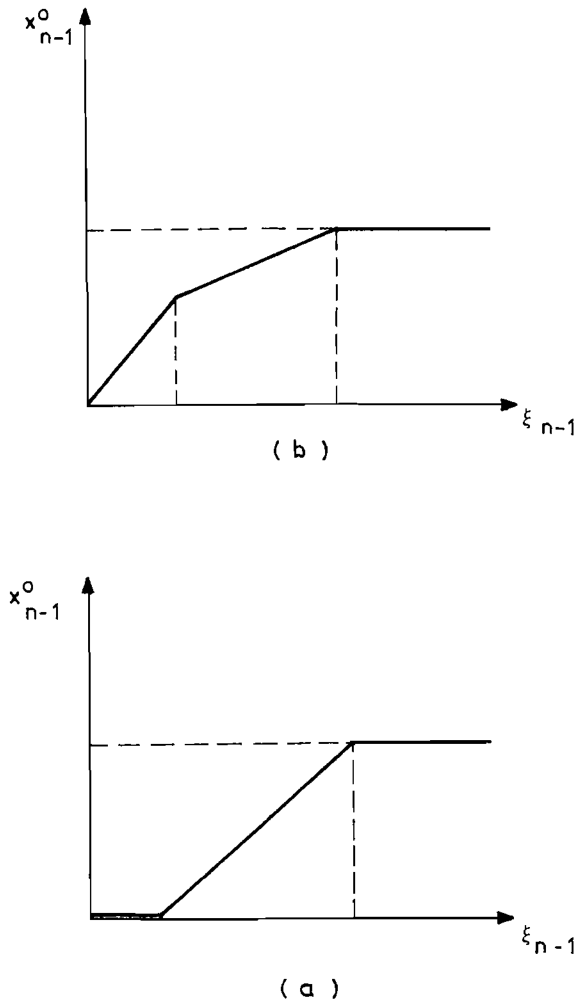


FIGURE 2

In a very similar way one can consider the structure of all other optimal components x_k^o; $k = n-2, n-1, \ldots, 1$, which are the minimum points of the corresponding functions

$$f_k(x_k) = \lambda_k (x_n - x_k^*)^2 + \sum_{j=k+1}^{n} \lambda_j (x_j^o - x_j^*)^2 \quad , \tag{9}$$

$$0 \leq x_k \leq \min(\xi_k, x_k^*) \quad ,$$

where

$$\xi_k = y_{k-1} + w_k = \sum_{i=1}^{k} w_i - \sum_{i=1}^{k-1} x_i$$

is the total resources (available at the <u>kth</u> step).

In particular, it is very easy to discover the following properties of optimal distributions.

If $x_k^o = x_k^*$, then $x_j^o = x_j^*$ for all $j > k$; moreover, if $x_k^o \neq 0$,

$$\lambda_k (x_k^* - x_k^o) \geq \max_{k<j\leq n} \lambda_j (x_j^* - x_j^o) \quad . \tag{10}$$

Indeed in the contrary case,

$$\lambda_k (x_k^* - x_k^o) < \lambda_i (x_i^* - x_i^o)$$

for some $i > k$, and if we take

$$x_k = x_k^o - \Delta x \quad , \quad x_i = x_i^o + \Delta x \quad , \quad x_j = x_j^o \ (j \neq i, k) \quad ,$$

then we could improve the optimal distribution, because

$$-\lambda_k \; (x_k^o - x_k^*)^2 + \lambda_i \; (x_i^o - x_i^*)^2 + \lambda_k \; (x_k^o - \Delta x - x_k^*)$$

$$+ \lambda_i \; (x_i^o + \Delta x - x_i^*)^2$$

$$= 2 \; [\lambda_k \; (x_k^* - x_k^o) - \lambda_i \; (x_i^* - x_i^o)] \; \Delta x + 0 \; (\Delta \tilde{x}) < 0$$

for sufficiently small $\Delta x > 0$.

Suppose that

$$x_k^o < x_k^* \quad ; \qquad k = 1, \ldots, n,$$

what one may expect normally under the resources shortage, then at each step $k = 1, \ldots, n$ there are three possibilities:

$$1) \quad x_k = 0 \qquad 2) \quad 0 < x_k < \xi_k \qquad 3) \quad x_k = \xi_n \quad .$$

(Remember that ξ_k is the total resources which are available at the <u>kth</u> step.)

According to the inequality (10), the decision $x_k^o = \xi_n$ may be optimal only if there is a possibility to satisfy other demands $x_{k+1}^*, \ldots, x_n^*$ (with their own resources $w_{k+1}, \ldots, w_n$) in such a way that

$$\max_{k<j\leq n} \lambda_j \; (x_j^* - x_j^o) \leq \lambda_k \; (x_k^* - \xi_k) \quad .$$

So we can check whether or not it is possible that $x_k^o = \xi_k$; $k = 1,2,\dots,n$. For example, in the case when

$$w_1 = W \quad ; \qquad w_2 \approx \cdots \approx w_n = 0$$

and the demands x_k^*; $k = 1,\dots,n$ don't decrease very much, we have $x_k^o < \xi_k$, $k = 1,2,\dots,n$.

On the other hand, it usually is not worth transferring a large amount $y_n = \xi_k - x_n$ to other consumers when the demand x_n^* is large. In particular, if $y_k^o > 0$ and

$$\lambda_k x_k^* > \max_{k<j\leq n} \lambda_j \, (x_j^* - x_j) \tag{11}$$

for some distribution $(x_{k+1},\dots,x_n)$ of the resources $y_k^o + w_{k+1},\dots,w_n$, then $x_n^o \neq 0$, because in the case $x_k^o = 0$ with taking back a sufficiently small amount Δy_k we can decrease our utility function $r(x,x^*)$.

Thus, one can sometimes find out (without any calculations) that the optimal distribution has to be the following:

$$0 < x_k^o < \xi_k \quad ; \qquad k = 1,2,\dots,n, \tag{12}$$

i.e. at each step k, one consumes something but not all available amount ξ_k.

In this case the minimum points x_n^o of the corresponding functions $f_k(x_k)$--see (9)--can be determined in an obvious way. Namely,

$$x_n^o = \xi_n \quad , \qquad x_{n-1}^o = \tilde{x}_{n-1}$$

is the absolute minimum point of the parabolic function $f_{n-1}(x_{n-1})$--see (8). The next function $f_{n-2}(x_{n-2})$ also is of the same parabolic type because $x_{n-1}^o = \tilde{x}_{n-1}$ is a linear function of $\xi_{n-1} = \sum_{k=1}^{n-1} w_k - \sum_{k=1}^{n-2} x_k$ (as well as of the parameters w_n, x_n^*, x_{n-1}^*), and under the condition (12): $0 < x_{n-1}^o < \xi_{n-2}$, the optimum x_{n-2}^o has to coincide with the absolute minimum point of the function $f_{n-2}(x_{n-2})$, and so on. By the same arguments, the optimum x_k^o for all other $K = n-2,\ldots,1$ coincides (under the condition [12]) with the absolute minimum point of the corresponding parabolic function

$$f_k(x_k) = \lambda_k (x_k - x_k^*)^2 + \sum_{j=k+1}^{n} (x_i^o - x_i^*)^2 \quad ,$$

where x_j^o, $j > k$ are the proper linear functions of $\xi_k = \sum_{i=1}^{k} w_i - \sum_{i=1}^{k-1} x_i$ as well as of the parameters $w_n,\ldots,w_{k+1}$; $x_n^*,\ldots,x_k^*$.

Remember that under the condition $x_n^o < x_n^*$ there are only three possibilities at each kth step: 1) $x_k^o = 0$ 2) $0 < x_k^o < \xi_k$ 3) $x_k^o = \xi_k$, where $\xi_k = \sum_{i=1}^{k} w_i - \sum_{i=1}^{k-1} x_i$ is the total available resource. So we can describe the type of our decision as the corresponding sequence $(d_1, d_2,\ldots,d_n)$, where d_k means "nothing"

($x_k^o = 0$), or "something" ($0 < x_k^o < \xi_k$), or "all" ($x_k^o = \xi_k$). Obviously if we know the type of the optimal distribution, then the optimal components x_k^o can be determined as $x_k^o = 0$, or as $x_k^o = \xi_k$, or as the absolute minimum point $x_k^o = \tilde{x}_k$ of the proper parabolic function $f_k(x_k)$--see (9)--with the already chosen $x_1^o,\dots,x_{k-1}^o$, which are the proper <u>linear functions</u> of $\xi_k = \sum_{i=1}^{k} w_i - \sum_{i=1}^{k-1} x_i$, as well as of the parameters $w_n,\dots,w_{k+1}$; $x_n^*,\dots,x_k^*$.

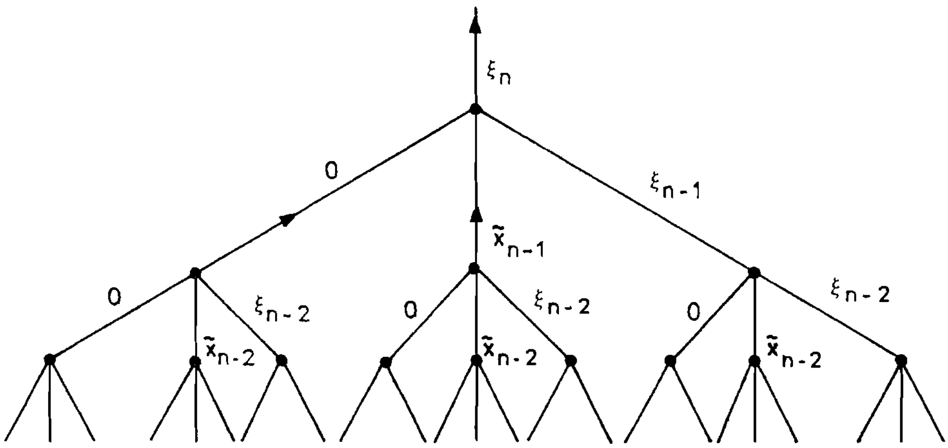


FIGURE 3

The tree of the possible decision types under the resources shortage when $x_k^o < x_k^*$ for all $k = 1,\dots,n$.

Example. For two consumers and resources $w_1 = W$, $w_2 = 0$ under the maximum demands $x_1^* = W$, $x_2^* = W$ for

$$r(x_1 x^*) = \sqrt{(x_1 - x_1^*)^2 + (x_2 - x_2^*)^2} \quad ,$$

we have

$$x_2^o = W - x_1 \quad , \qquad \tilde{x}_1 = \frac{W}{2} \quad .$$

Remember that $\tilde{x}_1$ is the absolute minimum point of the corresponding function

$$(x_1 - W)^2 + (x_2^o - W)^2 = (x_1 - W)^2 + x_1^2 \quad .$$

Because the value $\tilde{x}_1 = W/2$, it satisfies the constraints (1) and (2):

$$0 \leq \tilde{x}_1 \leq W \quad .$$

We obtain $x_1^o = \tilde{x}_1$ so our optimization principle gives us

$$x_1^o = W/2 \quad , \qquad x_2^o = W/2 \quad .$$

2. As it was described above, the optimal decision (even at the first step!) depends very much on all parameters $w_1,\ldots,w_n$. So a new problem arises in the case when the corresponding decision about a proper amount x_k has to be made with knowing only about $w_1,\ldots,w_k$ and $x_1,\ldots,x_{k-1}$.

Generally, in order to choose the components x_k^o (when $x_1^o,\dots,x_{k-1}^o$ have been chosen already and $w_1,\dots,w_{k-1}$ are known), it may be recommended to substitute the unknown parameters $w_{k+1},\dots,w_n$ with the appropriate estimate $w_{k+1}^*,\dots,w_n^*$ (which can be improved at the next (k+1)th step when x_k^o will be chosen and w_{k+1} will be known).

It is possible, for example, to use upper and lower boundaries for unknown resources. Namely, if we have some estimates

$$\underline{w}_k \leq w_k \leq \overline{w}_k \quad ; \qquad k = 1,\dots,n, \tag{13}$$

then we can obtain it as it was described above corresponding to optimal distribution vectors

$$\underline{x} = (\underline{x}_1,\dots,\underline{x}_n) \quad \text{and} \quad \overline{x} = (\overline{x}_1,\dots,\overline{x}_n)$$

with respect to the parameters

$$\underline{w} = (\underline{w}_1,\dots,\underline{w}_n) \quad \text{and} \quad \overline{w} = (\overline{w}_1,\dots,\overline{w}_n)$$

under the same demands $x_1^*,\dots,x_n^*$. By intuition it seems that

$$\underline{x}_k \leq x_k^o \leq \overline{x}_k \quad ; \qquad k = 1,\dots,n,$$

and actually it is true.

Let us show that for any parameters $\underline{w}_k \leq \overline{w}_k$; $k = 1,\dots,n$ the minimum points $\underline{x} = (\overline{x}_1,\dots,\overline{x}_n)$ and $\overline{x} = (\overline{x}_1,\dots,\overline{x}_n)$ of the loss function

$$r\ (x_1 x^*) = \sqrt{\sum_1^n \lambda_k\ (x_k - x_k^*)^2}$$

under the corresponding constraints (1) and (2) satisfy the inequalities $\underline{x}_k \leq \overline{x}_k$; $k = 1,\dots,n$.

Obviously, under the resources shortage, more precisely under the condition (2), the total consumption in the case of optimal distribution has to be as much as is available:

$$\sum_1^n x_k = \max\ .$$

So for the optimal distributions $\underline{x} = (\underline{x},\dots,\underline{x}_n)$ and $\overline{x} = (\overline{x}_1,\dots,\overline{x}_n)$ with respect to the resources $w = (w_1,\dots,w_n)$ and $\overline{w} = (\overline{w}_1,\dots,\overline{w}_n)$, we have

$$\sum_1^n \underline{x}_k \leq \sum_1^n \overline{x}_k\ .$$

If $\overline{x}_i < \underline{x}_i$ and $\Delta_i = \underline{x}_i - \overline{x}_i > 0$ for some i (say $i \varepsilon I$), then under $x_i = \overline{x}_i$ we have an extra positive amount $\Delta = \sum_{i \varepsilon I} \Delta_i$ (in comparison with $x_i = \underline{x}_i$, $i \varepsilon I$) which can be distributed among other variables x_j, $j \notin I$ (under the same resources $\underline{w}_1,\dots,\underline{w}_n$). Apparently, for the $\overline{x}_j \geq \underline{x}_j$ ($j \notin I$) we have

$$\sum_{j\varepsilon I} \overline{x}_j \geq \sum_{j\varepsilon I} \underline{x}_j + \Delta \ ,$$

so there is a partition $\Delta = \sum_{j \notin I} \Delta_j$ (with $\Delta_j \geq 0$) such that

$$x_j = \overline{x}_j - \Delta_j \geq \underline{x}_j \quad ; \quad j \notin I \ .$$

The distribution of the components

$$x_j = \overline{x}_i = \underline{x}_i - \Delta_i \ , \quad i\varepsilon I \ ; \quad x_j = \underline{x}_j + \Delta_j \ , \quad j \notin I$$

is feasible with $(x_j \leq \overline{x}_j !)$ under the resources $\underline{w}_1, \ldots, \underline{w}_n$. Because $\underline{x} = (\underline{x}_1, \ldots, \underline{x}_n)$ is the corresponding minimum point, we have

$$\sum_{i\varepsilon I} \lambda_i \ (x_i - \Delta_i - x_i^*) - \sum_{i\varepsilon I} \lambda_i \ (\underline{x}_i - x_i^*)^2$$

$$\geq \sum_{j \notin I} \lambda_j \ (\underline{x}_j - x_j^*)^2 - \sum_{j \notin I} \lambda_j \ (\underline{x}_j + \Delta_j - x_j^*)^2$$

$$> \sum_{j \notin I} \lambda_j \ (x_j - x_j^*)^2 - \sum_{j \notin I} \lambda_j \ (x_j + \Delta_j - x_j^*)^2 \ ,$$

because $x_j \geq \underline{x}_j$, $j \notin I$ and

$$(x_j - x_j^*)^2 - (x_j + \Delta_j - x_j^*)^2 \leq (\underline{x}_j - x_j^*)^2$$

$$- (\underline{x}_j + \Delta_j - x_j^*)^2$$

for any $\Delta_j \geq 0$; $x_j + \Delta_j \leq x_j^*$. (See Figure 4.) Thus for

$$\bar{x}_i = \underline{x}_i - \Delta_i \ (i \varepsilon I) \quad , \qquad \bar{x}_j = x_j + \Delta_j \ (j \notin I)$$

we obtain that

$$r\ (\bar{x}, x^*) = \sum_{i \varepsilon I} \lambda_i\ (\bar{x}_i - x_i^*)^2 + \sum_{j \notin I} \lambda_j\ (\bar{x}_j - x_j^*)^2$$

$$> \sum_{i \varepsilon I} \lambda_i\ (\underline{x}_i - x_i^*)^2 + \sum_{j \notin I} \lambda_j\ (\bar{x}_j - \Delta_j - x_j^*)^2$$

$$= r\ (x', x^*)$$

for feasible distribution (under the resources $\bar{w}_1,\ldots,\bar{w}_n$) $x' = (x_1',\ldots,x_n')$ with components

$$x_i' = \underline{x}_i > \bar{x}_i \quad , \qquad i \varepsilon I \quad ; \qquad x_j' = \bar{x}_j - \Delta_j \quad , \qquad i \varepsilon I \quad .$$

But it contradicts the fact that $\bar{x} = (\bar{x}_1,\ldots,\bar{x}_n)$ is the minimum point of $r(x, x^*)$ with respect to the resources $\bar{w}_1,\ldots,\bar{w}_n$, so our assumption on the strict inequalities $\bar{x}_i < \underline{x}_i$, $i \varepsilon I$ is not true and $\underline{x}_k \leq \bar{x}_k$ for all $k = 1,\ldots,n$.

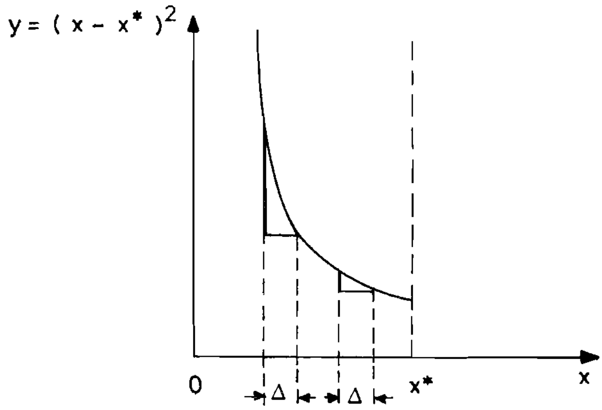


FIGURE 4

With knowledge of the inequalities, one can use the following resource distribution strategy in the case when at each kth step it is necessary to keep a good part of the total resources for the other consumers $k + 1,\dots,n$. Namely, one can choose the current amount x_k^o as the first component of the minimum point $(x_k^o,\dots,x_n^o)$ of the function

$$\sum_{j=k}^{n} \lambda_k \, (x_k - x_k^*)^2$$

under the constraints (1) and (2) with the already chosen $x_1^o,\dots,x_{k-1}^o$ and the corresponding parameters $w_j = \underline{w}_j$, $j = k + 1,\dots,n$, which are actually lower boundaries for the real resources. On the contrary, if one doesn't like to take great interest in other consumers, it is possible to use in a similar way the upper boundaries $w_j = \bar{w}_j^{(k)}$; $j = k+1,\ldots,n$.

Let us now consider $w_j,\ldots,w_n$ as random variables. In this case one can try to minimize a mean value of the loss function $r(x,x^*)$ and to find $\bar{x}^o = (\bar{x}_1^o,\ldots,\bar{x}_n^o)$ such that

$$Er\ (\bar{x}^{o}, x^*) = \min Er\ (x,x^*) \quad .$$

(Remember that each component x_k is a function of some data Γ_k including $x_1,\ldots,x_{k-1}$ and $w_1,\ldots,w_k$.)

Apparently the optimal decision at the last nth step is the same as it was above:

$$\bar{x}_n^o = x_n^o = \xi_n = \sum_{k=1}^{n} w_k - \sum_{k=1}^{n-1} x_k \quad .$$

According to the well-known Bellman's principle of dynamic programming, let us minimize the conditional expectation

$$f_{n-1}\ (x_{n-1}) = E\ \{\lambda_{n-1}\ (x_{n-1} - x_{n-1}^*)^2 + \lambda_n\ (\bar{x}_n^o - x_n^*)^2/\Gamma_{n-1}\}$$

$$= \lambda_{n-1}\ (x_{n-1} - x_{n-1}^*)^2 + \lambda_n\ (\eta_n - x_n^*)^2$$

$$+ \lambda_n\ E\ \{(\bar{x}_n^o - \eta_n)^2/\Gamma_{n-1}\} \quad ,$$

where

$$\eta_n = E\ \{\bar{x}_n^o/\Gamma_{n-1}\} \quad .$$

Obviously, $\bar{x}_n^o - \eta_n$ is the function only of w_n

$$(\bar{x}_n^o - \eta_n = w_n - E\{w_n/\Gamma_{n-1}\}) \quad ,$$

so η_n coincides with the optimal distribution component $x_n^o(w_1,\ldots,w_{n-1}, \bar{w}_n)$ with respect to the resources $w_1,\ldots,w_{n-1}$, $\bar{w}_{n-1} = E\{w_n/\Gamma_{n-1}\}$, and the same conclusion we have to make concerning $\bar{x}_{n-1}^o$, namely, $\bar{x}_{n-1}^o = x_{n-1}^o(w_1,\ldots,w_{n-1}, \bar{w}_n)$ is the optimal distribution component with respect to the parameters $w_1,\ldots,w_{n-1}$, w_n, where

$$\bar{w}_n - E\{w_n/\Gamma_{n-1}\} \quad .$$

Generally, $x_{n-1}^o(w_1,\ldots,w_{n-1}, \bar{w}_n)$ is the non-linear function of $\xi_{n-1} = \sum_{k=1}^{n-1} w_k - \sum_{k=1}^{n-2} x_k$ as well as of $\bar{w}_n$. See formulas (6) and (7).

Suppose that for all <u>possible parameters</u> $w = (w_1,\ldots,w_n)$, which may be from some known set W in n-dimensional vector space R^n, the corresponding optimal distributions $x^o = (x_1^o,\ldots,x_n^o)$ have to be of the <u>same type</u>. (It may be any of $\mathcal{N} = 3^{n-1}$ different types. For example, it may be of the type [12].) In this case $\bar{x}_n^o$ and $\bar{x}_{n-1}^o$ are known <u>linear functions</u> of the variables ξ_n and ξ_{n-1}, $\bar{w}_n$. Obviously, the conditional expectations

$$\eta_n = E\{\bar{x}_n^o/\Gamma_{n-2}\} \quad \text{and} \quad \eta_{n-1} = E\{\bar{x}_{n-1}^o/\Gamma_{n-2}\}$$

are of similar types and coincide with the corresponding optimal distribution components

$$x_n^o = x_n^o\ (w_1,\ldots,w_{n-2},\ \bar{w}_{n-1},\ \bar{w}_n)$$

and

$$x_{n-1}^o = x_{n-1}^o\ (w_1,\ldots,w_{n-2},\ \bar{w}_{n-1})$$

with respect to the resources $w_1,\ldots,w_{n-2}$, where

$$\bar{w}_{n-1} = E\ \{w_{n-1}/\Gamma_{n-2}\}\quad ,\qquad \bar{w}_n = E\ \{w_n/\Gamma_{n-2}\}\quad .$$

We have

$$f_{n-2}\ (x_{n-2}) = E\ \{\lambda_{n-2}\ (x_{n-2} - x_{n-2}^*)^2 + \sum_{j=n-1}^{n} \lambda_j\ (\bar{x}_j^o - x_n^*)^2\}$$

$$= \lambda_{n-2}\ (x_{n-2} + x_{n-2}^*)^2 + \sum_{j=n-1}^{n} \lambda_j\ (\eta_j - x_j^*)^2$$

$$+ \sum_{j=n-1}^{n} \lambda_j\ E\ \{(\bar{x}_j^o - \eta_j)^2/\Gamma_{n-2}\}\quad .$$

Here the last term is the <u>constant</u> (because $\bar{x}_j^o$ is the proper linear function, in particular, the linear functions of the variable x_{n-2}), so $\eta_j = x_j^o\ (w_1,\ldots,w_{n-2},\ w_{n-1},\ \bar{w})$ are the optimal distribution components with respect to the resources $w_1,\ldots,w_{n-2}$, $\bar{w}_{n-1}$, $\bar{w}_n$, and $\bar{x}_{n-2}^o$ also has to be the optimal component with respect to parameters $w_1,\ldots,w_{n-2}$, $\bar{w}_{n-1}$, $\bar{w}_n$.

Now it seems clear that at each kth step the optimal amount $\bar{x}_k^o$ as the minimum point of the function

$$f_k(x_k) = E\{\lambda_k (x_k - x_k^*)^2 + \sum_{j=k+1}^{n} \lambda_j (\bar{x}_j^o - x_j^*)^2 / \Gamma_k\}$$

coincides with the optimal distribution component with respect to the parameters $w_1, \ldots, w_k, \bar{w}_{k+1}, \ldots, \bar{w}_n$, where

$$\bar{w}_j = E\{w_j / \Gamma_k\} \quad ; \quad j = k+1, \ldots, n, \tag{14}$$

namely,

$$\bar{x}_k^o = x_k^o (w_1, \ldots, w_k, \bar{w}_{k+1}, \ldots, \bar{w}_n) \quad ; \quad k = 1, \ldots, n. \tag{15}$$

(Remember we assumed above that for all possible parameters $w_1, \ldots, w_n$ the corresponding optimal distributions are the same type!)

Note that in the case when, for different groups of parameters $(w_1, \ldots, w_n)$ the corresponding optimal distributions $(x_1^o, \ldots, x_n^o)$ are of different types and $\bar{x}_j^o$ are non-linear functions of the variables w_k, $k < j$, the optimal distributions $(x_1^o, \ldots, \bar{x}_n^o)$--concerning mean value of the loss function--is more complicated than it was described above.

It is worthy to note also that the mean value criterion is not uniformly good for any probability distributions of the parameters $(w_1, \ldots, w_n)$.

Example. Let us consider two demands x_1^*, x_2^* for resources with independent components w_1, w_2. Suppose $w_2 \neq 0$ with a very small probability p (say, $p = 0.001$), so we almost can be sure that $w_2 = 0$.

In order to make this more clear, let us assume that $E\,w_2 \geq x_2^*$. Then on the basis of mean value criterion we have to choose $\bar{x}_1^o = w_1$ and $\bar{x}_2^o = 0$ (with a big probability I-p). Obviously, such a decision is not good in the case when with a good guarantee the second demand x_2^* has to be partly satisfied.

3. Suppose that under a condition of unknown resources $w_1,\ldots,w_n$ the corresponding demands $x_1^*,\ldots,x_n^*$ are given in the form

$$x_n^* = \sum_{j=1}^{k} \alpha_{kj}^* w_j \quad ; \qquad x = 1,\ldots,n,$$

where α_{nj}^*, $0 \leq \alpha_{nj}^* \leq 1$, are some coefficients (i.e. the k<u>th</u> consumer demands α_{kj}^*<u>th</u> part of the resource w_j, $j \leq k$). Of course, these coefficients α_{kj}^* may be <u>non-feasible</u>. Namely, it may be $\sum_{k=j}^{n} \alpha_{nj}^* > 1$, and the problem is to find <u>feasible</u> coefficients α_{nj}:

$$\alpha_{nj} \geq 0 \quad , \qquad \sum_{k=j}^{n} \alpha_{nj} \leq 1 \qquad\qquad (16)$$

which are <u>optimal</u> in some reasonable sense for the resource distribution

$$x_n = \sum_{j=1}^{k} \alpha_{nj} w_j \quad ; \qquad k = 1,\ldots,n.$$

Note that the conditions (16) for <u>arbitrary</u> parameters $w_1,\ldots,w_n$ are equivalent to the following:

$$\sum_{k=1}^{m} x_n = \sum_{k=1}^{m} \left(\sum_{j=1}^{k} \alpha_{nj} w_j \right)$$

$$= \sum_{j=1}^{m} \left(\sum_{k=j}^{m} \alpha_{kj} \right) w_j \leq \sum_{j=1}^{m} w_j \quad ; \qquad m = 1,\ldots,n.$$

According to our general principle of optimality, we propose as an <u>optimal compromise</u> the resource distribution

$$x_k^o = \sum_{j=1}^{k} \alpha_{kj}^o w_j \quad ; \qquad k = 1,\ldots,n$$

with the coefficients α_{nj}^o for which

$$\sum_{k=1}^{n} \lambda_k \sum_{j=1}^{k} (\alpha_{kj}^o - \alpha_{kj}^*)^2 = \sum_{j=1}^{n} \sum_{k=j}^{n} \lambda_k \, (\alpha_{nj}^o - \alpha_{nj}^*)^2 = \min \quad .$$

Obviously, the optimal coefficients α_{nj}^o, $k = j,\ldots,n$ for any $j = 1,\ldots,n$ can be determined from the condition

$$\sum_{k=1}^{n} \lambda_k \, (\alpha_{kj}^o - \alpha_{kj}^*)^2 = \min \quad . \tag{17}$$

Let us fix $j = 1,\ldots,n$. Under the substitution

$$\alpha_{kj} \to \alpha_{k-j+1} \quad ,$$

$$\lambda_k \to \lambda_{k-j+1} \quad , \qquad k = j,\ldots,n,$$

let us consider $\alpha = (\alpha_1,\ldots,\alpha_m)$ as vectors in m-dimensional space with the inner product

$$(\alpha,\beta) : \sum_{k=1}^{m} \lambda_k \alpha_k \beta_n$$

and the metric

$$||\alpha - \beta|| = \sqrt{\sum_{k=1}^{m} \lambda_k \ (\alpha_n - \beta_n)^2} \quad .$$

Let S_m be a simplex of all vectors $\alpha = (\alpha_1,\ldots,\alpha_m)$ which satisfy the constraints (1), namely,

$$\alpha_k \geq 0 \quad , \qquad \sum_{k=1}^{m} \alpha_k \leq 1 \quad . \tag{16'}$$

The problem is to find a vector $\alpha^{\circ} = (\alpha_1^{\circ},\ldots,\alpha_m^{\circ}) \ \varepsilon S_m$ such that

$$||\alpha^{\circ} - \alpha^*|| = \min_{\alpha\varepsilon S_m} ||\alpha - \alpha^*|| \quad . \tag{17'}$$

Of course, $\alpha^{\circ}_{\cdot} = \alpha^*$ if $\alpha^* \ \varepsilon S_m$. Suppose $\alpha^* \notin S_m$. Let us consider a half-line from the point α^*, which is perpendicular to the hyperplane $L = \{\alpha ! \ \sum_{k=1}^{m} \alpha_n = 1 \ _s\}$. It is all points β with coordinates

$$\beta_n = \alpha_k^* - \frac{t}{\lambda_k} \quad ; \qquad k = 1,\ldots,m \ (t \geq 0). \tag{18}$$

Let $\Pi = (\Pi_1,\dots,\Pi_m)$ be a projection of the point α^* onto the hyperplane L:

$$\Pi_k = \alpha_k^* - \frac{t_\Pi}{\lambda_k} , \qquad k = 1,\dots,n$$

$$t_\Pi = \frac{\sum_{k=1}^{m} \alpha_k^* - 1}{\sum_{k=1}^{m} 1/\lambda_k} , \tag{19}$$

and $S'_m = L \cap S_m$ be a set of all vectors $\alpha = (\alpha_1,\dots,\alpha_m)$ such that

$$\sum_{k=1}^{m} \alpha_n = 1 , \qquad \alpha_n \geq 0 .$$

Obviously, if $\Pi \varepsilon S'_m$, i.e.

$$\alpha_j^* \geq \frac{\sum_{k=1}^{m} \alpha_k^* - 1}{\lambda_j \sum_{k=1}^{m} 1/\lambda_k} \qquad \text{for all } j = 1,\dots,m ,$$

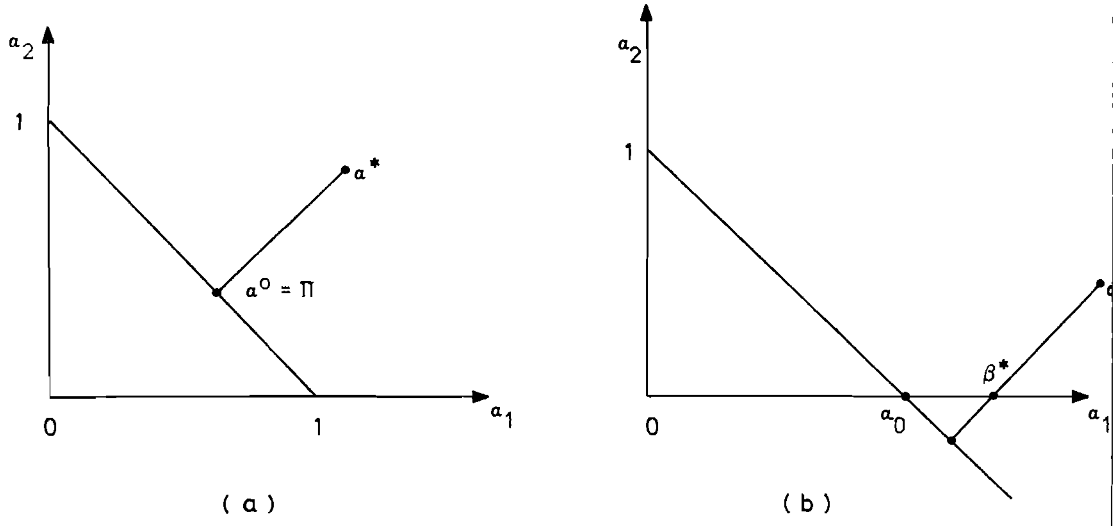
then $\alpha^0 = \Pi$. (See Figure 5.)

Suppose $\Pi \notin S'_m$. This means that some of the coordinates $\Pi_1,\dots,\Pi_m$ are <u>negative</u>. Let us consider

$$\beta_n^* = \alpha_k^* - \frac{t^*}{\lambda_k} , \qquad k = 1,\dots,m , \tag{20}$$

where

$$t^* = \min \lambda_k \alpha_k^* .$$



α_2
1
α^*
$\alpha^0 = \Pi$
0
1
α_1
(a)
α_2
1
β^*
0
α_0
α_1
(b)

FIGURE 5

Obviously, $\sum_{k=1}^{m} \beta_n^* > 1$ and the hyperplane L separates β^* and S_m. Because α^o is the minimum point of

$$\begin{aligned}\min_{\alpha\varepsilon S_m} ||\alpha - \beta||^2 &= \min_{\alpha\varepsilon S_m'} ||\alpha - \beta||^2 \\ &= \min_{\alpha\varepsilon S_m'} ||\alpha - \Pi||^2 + ||\beta - \Pi||^2 \\ &= \min_{\alpha\varepsilon S_m} ||\alpha - \beta||^2 + ||\beta - \Pi||^2\end{aligned}$$

for <u>any</u> β of the type (3), $\sum_{k=1}^{m} \beta_n \geq 1$, we can determine α^o from the condition

$$||\alpha^o - \beta^*|| = \min_{\alpha\varepsilon S_m} ||\alpha - \beta^*|| \quad .$$

From the very beginning we could set coordinates in such a way that

$$\lambda_1\alpha_1^* \geq \cdots \geq \lambda_m\alpha_m^* \quad .$$

Then

$$\min_{1\leq k\leq m} \lambda_k\alpha_k^* = \lambda_m\alpha_m^* \quad ,$$

i.e. $\beta^* = (\beta_1^*,\ldots,\beta_{m-1}^*, 0)$. Obviously,

$$\min_{\alpha\varepsilon S_m} \sum_{k=1}^{m} \lambda_k (\alpha_k - \beta_k^*)^2 = \min_{\alpha\varepsilon S_m} \sum_{k=1}^{m-1} \lambda_k (\alpha_k - \beta_k^*)^2 + \lambda_k\alpha_k$$

$$= \min_{\alpha\varepsilon S_{m-1}} \sum_{k=1}^{m-1} \lambda_k (\alpha_k - \beta_k^*)^2 \quad ,$$

where S_{m-1} denotes the set of all vectors $\alpha = (\alpha_1,\ldots,\alpha_{m-1})$ in (m-1)-dimensional space for which

$$\alpha_k \geq 0 \quad ; \qquad \sum_{k=1}^{m-1} \alpha_n \leq 1 \quad . \tag{16"}$$

Thus $\alpha_m^o = 0$ and the question on $\alpha_1^o,\ldots,\alpha_{m-1}^o$ can be considered in the same way as it was done above, because the problem now is to find a vector $\alpha^o = (\alpha_1^o,\ldots,\alpha_{m-1}^o) \varepsilon S_{m-1}$ such that

$$||\alpha^o - \beta^*|| = \min_{\alpha\varepsilon S_{m-1}} ||\alpha - \beta|| \quad . \tag{17"}$$

Similarly, as it was above, we can determine $\alpha^o = (\alpha_1^o,\ldots,\alpha_{m-1}^o)$ at this <u>second</u> step or reduce our problem to the case of (m-2) unknown components $\alpha_1^o,\ldots,\alpha_{m-2}^o$. Not more than in m steps can we determine all components $\alpha_1^o,\ldots,\alpha_m^o$.

According to the formulas (19) and (20), at every step we have to reduce α_k^*, β_k^* etc. to zero or with subtraction of a value which is proportional to the corresponding $1/\lambda_k$. Thus the optimal $\alpha_1^o,\ldots,\alpha_n^o$ are the following:

$$\alpha_k^o = \max\left(0,\ \alpha_k^* - \frac{t^o}{\lambda_k}\right) \quad ; \qquad k = 1,\ldots,m, \qquad (21)$$

where the constant t^o can be determined from the condition

$$\sum_{k=1}^{m} \alpha_k^o = 1 \quad ; \qquad (22)$$

i.e. there is the crucial level t^o such that

$$\alpha_k^o = \begin{cases} 0 \ , & \text{if } \alpha_k^* \leq \dfrac{t^o}{\lambda_k} \\ \alpha_k^* - \dfrac{t^o}{\lambda_k} \ , & \text{if } \alpha_k^* \geq \dfrac{t^o}{\lambda_k} \ . \end{cases}$$

The same result holds true in the case when, in addition to the constraints (16), we assume--according to (2)--that

$$\alpha_k \leq \alpha_k^* \quad ; \qquad k = 1,\ldots,m,$$

because α_k^o; $k = 1,\ldots,m$ of the type (21) satisfies these "extra" constraints.