GENERALIZED SHAPLEY VALUES BY SIMPLICIAL SAMPLING

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Characteristic function representation of n-person cooperative games precludes the modelling of structural properties of a game other than the relationship between coalition structure and the worth of a game. This means that the Shapley value, a measure of expected return to a player from playing the game, is restricted as a solution concept to only those games satisfying the condition that all coalitions of the same cardinality are equiprobable. By contrast, as we demonstrate below, Shapley's three axioms are satisfied for Shapley-like measures based on richer characterizations of a game. In particular, we extend the Shapley value to a class of abstract games for which the roles that players assume are determinants of the likelihood of particular coalitions and for which the original Shapley value can be found as a special case.

In Section 1 we briefly consider Shapley's axioms and two possible derivations of the Shapley value. Section 2 discusses two recent formal attempts to extend the Shapley value to games for which the structure of roles is important. Section 3 presents our notion of a "clique" structure as a formalization of relationships among roles and describes our extension of the Shapley value. Since calculation of Shapley values (especially generalized ones) is computationally problematic for games involving sizeable numbers of players, we describe in Section 4 a sampling approach on (deformed) simplices to estimate (generalized) Shapley values.

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SECTION 1

Shapley defined a game to be a superadditive set function (the characteristic function v) from the power set of a universe, U, of players to the real line. An abstract game is the class of games πv define on the set of one-to-one mappings, $\Pi(U)$, of U onto itself when

$$\pi v(\pi S) = v(S) \quad (\text{all } S \subseteq U).$$

An abstract game exhibits the property that the worth of a coalition is invariant under the identities of the players that form it, depending rather on the roles they assume. For any game v, a set $N \subset U$ and all its supersets are called carriers of v if:

$$v(S \cap N) = v(S) \quad (\text{all } S \subset U);$$

that is, for any v, the set of players can be partitioned into sets of real and dummy players, the dummies having no effect on the worth of a coalition.

Shapley sought to construct a value $\phi[v]$ of a game v which satisfied three axioms:

Axiom 1: $\phi_{\pi_i}[\Pi v] = \phi_i[v] \quad (\text{all } \pi \in \Pi(U))$

That is, value depends only on role and not on which players assume the roles. Value is thus an intrinsic property of the abstract game.

Axiom 2: $\sum_N \phi_i[v] = v(N) \quad (\text{all } N, \text{ carriers of } v)$

Axiom 2 requires that $\phi[v]$ exhibit joint efficiency. Combined with the definition of a carrier it also implies that the value of the game for dummy players is zero.

Axiom 3: $\phi[v+w] = \phi[v] + \phi[w]$

for any two games v and w; i.e. the value of any game must be independent of the play of any other game.

Shapley demonstrated that for a game v a unique value $\phi[v]$ exists satisfying axioms 1 to 3 and having the form:

$$1.1 \qquad \phi_i[v] = \sum_{\substack{S \ni i \\ S \subset N}} \frac{(s-1)!\,(n-s)!}{n!} V_i(S);$$

s is the cardinality of S, n is the number of players (including dummies) playing v and $V_i(S)$ is the "marginal characteristic function"

$$1.2 \qquad V_i(S) = v(S) - v(S \setminus \{i\}) .$$

Expression (1.1) has commonly been interpreted in a probability framework. That is, since all coalitions of size s are equally likely 1/ ,

$$1.3 \qquad \frac{(s-1)!\,(n-s)!}{n!} = \binom{n-1}{s-1}^{-1}$$

is just the probability that any coalition S is realized. $\phi_i[v]$ is player i's expected contribution to a coalition where the expectation is taken with respect to the distribution of coalitions.

Shapley proposed,as well, a bargaining model of coalition formation that would yield the value $\phi[v]$ as the expected outcome. Each of the n! orders of the players may be thought of as generated by the successive arrivals of the players at some given point to form the present coalition N. Player i is awarded $V_i(S)$ only if the players $S \setminus \{i\}$ have arrived before him. For any order t let

(1.4) $$S_t(i) = \{ j \in N \mid t(j) \leq t(i) \}$$

where t(i) is the position index of player i. If order t occurs player i receives $V_i(S_t(i))$. Since Shapley's (implicit) assumption of equiprobable coalitions is clearly equivalent to assuming that all orders have the same probability $\frac{1}{n!}$, player i's expected marginal contribution to all coalitions in which he participates can thus be written

(1.5) $$\phi_i[v] = \sum_{t=1}^{n!} \frac{1}{n!} V_i(S_t(i)) .$$

(1.5) is an alternative representation of Shapley's value, that is particularly suitable for the generalizations to be discussed below.

An apparent weakness of the Shapley value is its restriction to games for which all coalitions of the same size (and equivalently, all orders) are equally likely. There are many examples of games which do not meet this condition because of relationships among the roles (rather than the personalities of players) enhancing the likelihood of some coalitions while diminishing that of others. A foremost example are the inter- and intra-party relations of legislative representatives, which in most countries will make a majority of coalition structures extremely unlikely. It is thus of interest to investigate the value of such games. The usual vehicle to do this is expression (1.5) above, but with differentiated probabilities ρ_t of orders replacing $\frac{1}{n!}$, i.e.

(1.6) $$\phi_i^G[v] = \sum_{t=1}^{n!} \rho_t V_i(S_t(i))$$

Before discussing our own approach in section 3, we consider two models of Shapley-like values (Kilgour 1974, Owen 1971) that proceed along these lines.

SECTION 2

The first author, Kilgour (1974), explicitly introduces (1.6), but his main tool is a redefinition of the characteristic function. His goal is to determine the effect on value of a subset $Q \subset N$ of quarrelling players, no two of which will join the same coalition; this behaviour can be described by a characteristic function that is strictly additive if more than one quarreller participates in a "coalition". Let v be the game without quarrelling; then $[v,Q] \equiv v^*$ represents the game with quarrelling, where v^* is defined by:

$$(2.1) \qquad v^*(S) = v(S) \text{ if } |S \cap Q| = 1$$

$$(2.2) \qquad v^*(S \cup \{k\}) = v(S) + v(\{k\}), \text{ if } |S \cap Q| = 1,\ k \in Q$$

We note that incrementally constructed coalitions are needed if v^* is to be determinate; i.e., if $S \cap Q = \phi$ and k and j are quarrellers, then $v^* |S \cup \{k\} \cup \{j\})$ will in general depend on the order in which k and j join S.

Kilgour's value $\phi[v,Q]$ is not a true generalization of the original Shapley value, since it satisfies Shapley's joint efficiency axiom only under v^* but not under v. This is so because quarrelling reduces the payoff to cooperation for many "coalitions"; in particular, v(N) is not attainable for essential games and hence

$$\sum_{i=1}^{n} \phi_i[v,Q] < v(N) \quad , \quad |Q| \geq 2$$

violating axiom 2.

Owen's (1971) approach to the problem motivated our own extension of the Shapley value. Like Kilgour, Owen focuses on games for which information about relationships among players influences the probabilities of different orderings, but he models a continuous concept of "affinity" between players, rather than Kilgour's absolute repulsion within a certain subset of them.

He does this by introducing a geometric framework, in which players are assigned to points, p^i, on a d-sphere ($d \leq n-2$), with the points chosen such that the distances m between them (geodesic or Euclidean) directly reflect the relative mutual attractions between players. For instance, two players assigned antipodal points are least attracted to each other. It is now possible to derive theoretically the probabilities ρ_t (in 1.6) of different orderings, which depend on the system of affinities between players as follows:

Each point, z, on the sphere produces an ordering t if

$$m(p^i,z) < m(p^j,z) < \ldots < m(p^k,z)$$

implies $\quad t(i) < t(j) < \ldots < t(k)$,

If the sphere has full dimensionality $d = n-2$, i.e., if the player points p^i are arranged affinely independently, there exists $\binom{n}{2}$ distinct hyperplanes through the sphere's center, which are orthogonal to the $\binom{n}{2}$ segments between each pair of player points. These hyperplanes slice the sphere into exactly n! regions (more precisely, n! spherical polytopes), and all points in the interior of each region produce the same unambiguous strict order. The probability ρ_t of the order t is then defined as the ratio of the measure of the region producing t to the measure of the whole sphere.[2/] Given the probabilities ρ_t could actually be extracted, their application to (1.6) would yield a generalized Shapley value.

Owen's value is shown to have two properties deemed desirable:

1. An ordering and its reversal are equiprobable.
2. The exclusion from the game of a set of players will not affect the probabilities of the relative orderings of the remaining players.

The desirability of property (1) is a natural consequence of the possibility that an issue initiating a game may be stated either positively or negatively; in addition it ensures the equality of the power and the blocking index. Property (2) implies independence of the degree of affinity between any two players from whoever else plays the game.

Owen's basic idea of using a measure of attraction between players, and the associated spherical framework are appealing on the theoretical level. Computability and empirical use are quite different matters and here the prospects are not good.

First, the information requirements of Owen's value are high. It is surprisingly difficult to find enough independent criteria to place n points affinely independently into n-space (where they then define an (n-2) - sphere), as n becomes larger. It is equally frustrating to try to define a mapping that distributes lower-dimensional clusters of n points onto an (n - 2) - sphere in any meaningful fashion. Secondly, even if that goal could be attained, it is virtually impossible to compute the volumes of n! (n - 3) - dimensional polytopes on the surface of that sphere, for $n > 4$.

To avoid these difficulties Owen suggests that degenerate spheres of dimensionality 1, 2 and possibly 3 could be used for an approximate derivation of n-person values. The trouble with this approach is that a great majority of orderings are immediately excluded from consideration which leads to intolerable distortions. 3/

As another avenue to circumvent the computational impasse of Owen's full dimensional value we tried our sampling approach (see Section 4) adapted to spheres. There are various ways of drawing uniformly distributed sample points z on an (n - 2)-sphere, but none of them is computationally simple, and at least one method becomes numerically unstable in higher dimensions. The spherical environment furthermore requires, for each sample point, the calculation of n Euclidean distances in n-space, a non-trivial computational burden.

In summary Owen's value represents a genuine generalization of Shapley's value but its actual use is severely impeded by informational and computational obstacles.

SECTION 3

The goals set for our modification of the Shapley value, $\phi[v, C, \lambda]$ are:

(a) $\phi[v, C, \lambda]$ (see (3.1) below) should require only a modest amount of information beyond the characteristic function.

(b) $\phi[v, C, \lambda]$ should be a true generalization of Shapley's value, i.e., it should satisfy Shapley's three axioms and the original Shapley value should emerge as a special case.

(c) $\phi[v, C, \lambda]$ should be easy to approximate computationally.

Points (a) and (b) will be answered in the course of this section, point (c) in the next.

As mentioned in Section 1, Shapley defines a game as its characteristic function v. We generalize the notion of a game to a triple $[v, C, \lambda]$, where v is the characteristic function, C is a partition of the set of players N, called a clique structure, and λ is a collusion parameter, a scalar. Players belonging to a clique $C \in C$, $C \subset N$ are postulated to have mutual affinity (measured by $0 \leq \lambda < 1$) but not to players belonging to other cliques.

Since clique membership can be signified for each player by a single number and because the same collusion parameter λ is assumed to apply to all cliques, the information requirements (given the char. function) are $n + 1$ numbers, which compares favourably with Owen's $\binom{n}{2}$ distances. Information about cliques is furthermore easily available, and thus goal (a) is met.

Shapley's axiom 1 remains satisfied by our assuming that clique membership is a property of roles, rather than of personalities of players.

$$j \in C \text{ implies } \pi j \in C^{\pi} \quad , \quad \text{all } \pi \in \Pi(U)$$

(see also footnote 1/).

(Our implicit use of axioms 2 and 3 is identical to Shapley's and thus they remain untouched). If the clique structure is trivial ($\lambda = 0$ for any C, or $C = \{N\}$, or $C = \{\{1\}\ \{2\},\ldots,\{n\}\}$) the game is essentially described by v alone, and the concomitant value is Shapley's original one. The above implements goal (b).

Parallel to Owen (1971), we aim at assigning higher probabilities of formation to certain coalitions (of given size); in our case the selected coalitions will be those which contain relatively fewer incomplete cliques. The natural route is again to operate on orders t of players, i.e. to find appropriately differentiated probabilities ρ_t and to apply them to

$$(3.1) \qquad \phi_i[v,C,\lambda] = \sum_{t=1}^{n!} \rho_t V_i(S_t(i))$$

which is the generalized Shapley value (compare formula 1.5). Theorem (3.2) below digresses briefly to establish a firm, albeit partial foundation for this indirect line of attack, which is also used, but not proved, by Kilgour and Owen (see footnote 2).

Definition: An order t of n players is clique-preserving if the members of every clique appear contiguously in t.

Definition: A partial clique is a strict, nonempty subset of a clique.

Theorem 3.2: If clique-preserving orders have probability $\rho_t > \frac{1}{n!}$, then coalitions that include at most one partial clique are more likely than coalitions of the same size containing more than one partial clique.

Proof: Let S be any coalition of size s; the probability of S is just

$$\alpha(S)\ \frac{1+\varepsilon}{n!} + [\beta(s) - \alpha(S)]\ \frac{1-\delta}{n!}$$

where

$\beta(s) = s!(n-s)!$ is the total number of orders for which the first s elements are contained in S, $\alpha(S)$ is the number of such orders which are also clique-preserving and $\frac{1+\varepsilon}{n!}$, $\frac{1-\delta}{n!}$ are the probabilities of clique-preserving and non-clique preserving orders, respectively ($\varepsilon > 0$ by assumption for any nontrivial clique structure, and $\varepsilon > 0$ implies $\delta > 0$).

Now, if a particular coalition S_1 contains more than one partial clique, $\alpha(S_1) = 0$ and thus $Pr(S_1) = \beta(s)\ \frac{1-\delta}{n!}$. If another coalition S_2 of the same size s contains at most one partial clique, $\alpha(S_2) > 0$ and $Pr(S_2) = \alpha(S_2)\ \frac{\varepsilon+\delta}{n!} + \beta(s)\frac{1-\delta}{n!}$; thus $Pr(S_2) > Pr(S_1)$. Q.E.D.

Returning to the development of games and values with clique structure, we now introduce a geometric representation of such games that allow the measurement (and later the computation) of the probabilities ρ_t of n! orders t:

Rather than points on a sphere, we assign each player $i \in N$ a vertex p^i, $i = 1, 2, \ldots, n$ of a simplex

$$S^P = \{ z \in R^n \mid z = \sum_{i=1}^{n} p^i x_i \ , \ \sum x_i = 1,\ x_i \geq 0 \}$$

Collecting the p^i's as columns of a matrix P, one can write:

$$S^P = \{z \in R^n \mid z = Px \ , \ \sum x_i = 1 \ , \ x_i \geq 0\}$$

We shall call P the basis of S^P, which spans or generates S^P. The x_i's are barycentric coordinates of z, w.r.t. P.

The connection between points $z \varepsilon S^P$ and orders t of players is as follows:
Each point z produces an order t if

$$z_i > z_j > \ldots > z_k \quad \text{implies}$$

$$t(i) < t(j) < \ldots < t(k) \quad ;$$

the z_i are the Cartesian coordinates of z and the t(i) the position indices of players i in the order t. If P is the identity matrix $I = (e^1, e^2, \ldots, e^n)$ (where the $p^i = e^i$ are unit vectors) S^P is the unit simplex

$$S^I = \{x \varepsilon R^n \mid x = Ix , \quad \sum x_i = 1, \; x_i \geq 0\};$$

for $x \varepsilon S^I$, the barycentric and Cartesian coordinates of x obviously coincide.

S^I depicts games without or with a trivial clique structure, and its use leads to the original Shapley value. Indeed, the simplex S^I splits into n! subsimplices, such that all points x in the interior of each subsimplex produce the same unambiguous strict order. The subsimplices thus defined are obviously congruent and considering a probability mass uniformly distributed over S^I, each subsimplex represents the same probability, i.e. $\rho_t = \frac{1}{n!}$ for all t. Applying these ρ_t in (3.1) clearly yields the plain Shapley value.

Now the clique structure $\mathcal{C}$ is brought into play: For each clique $C \varepsilon \mathcal{C}$, the points p^i associated with players i in the clique are moved toward their common centroid (which lies, if the clique contains c players, in the center of an (c - 1)- face of S^I). How much they are moved depends on the size of the collusion parameter λ.

For example, let N = {1,2,3}, C = {{1,2},{3}}, $\lambda = \frac{1}{2}$. Then

$$p^1 = [(1-\lambda)e^1 + \lambda \frac{e^1+e^2}{2}] = \begin{bmatrix} 0.75 \\ 0.25 \\ 0 \end{bmatrix}$$

$$p^2 = [(1-\lambda)e^2 + \lambda \frac{e^1+e^2}{2}] = \begin{bmatrix} 0.25 \\ 0.75 \\ 0 \end{bmatrix}$$

$$p^3 = e^3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The associated basis is

$$P = [p^1, p^2, p^3] = \begin{bmatrix} 0.75 & 0.25 & 0 \\ 0.25 & 0.75 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

P is a doubly stochastic matrix (i.e. both rows and columns sum to 1), that represents a linear, nonsingular, symmetric contraction mapping, with det $P = \frac{1}{2} \leq 1$. If applied to S^I, it yields

$$S^P = \{z \in R^n \mid z = Px, \ \sum x_i = 1, \ x_i \geq o\}$$

Fig. 3.1 depicts both S^I and S^P and the regions associated with the orders ρ_t.

Fig. 3.1

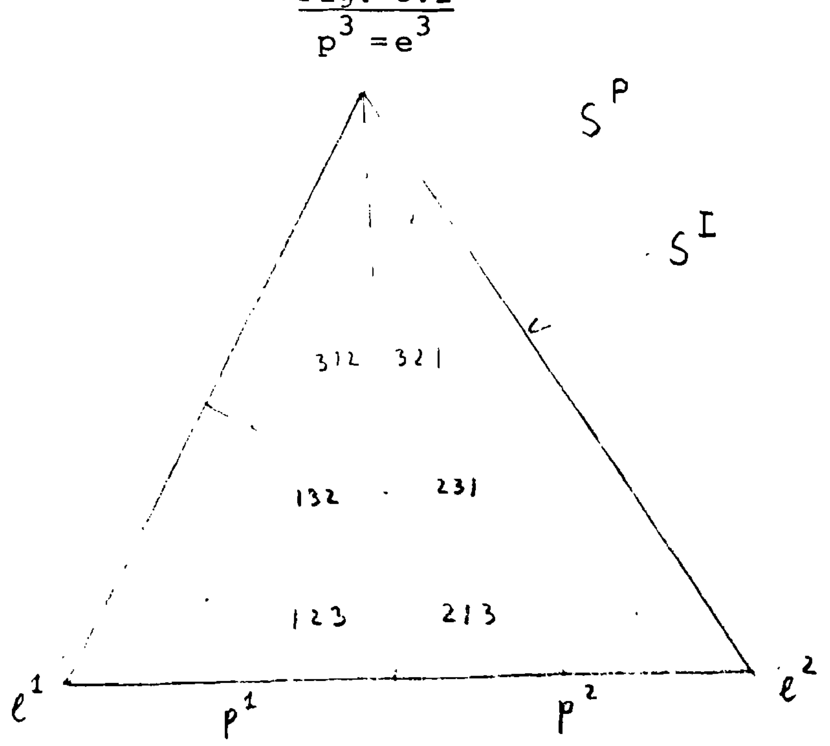

Fig. 3.1 also shows that the sections of S^P containing the non-clique-preserving orders (1 3 2) and (2 3 1) are much smaller than the others having clique-preserving orders and further, that their measure can be made arbitrarily small as $\lambda \to 1$.

The above exemplified procedure can clearly be carried out for any number of players and any clique structure, with the mapping P retaining the indicated properties. The next theorem gives a summary.

<u>Theorem 3.3</u>. Let S^I be a unit simplex with $x \in S^I$ being uniformly distributed. Let P be a doubly stochastic matrix representing a linear, nonsingular, symmetric contraction mapping defining $z = Px \in S^P$. Then

(a) the transformed density on the contracted simplex S^P is uniform;

(b) the mathematical expectations of $x \in S^I$ and $z \in S^P$ are equal

(c) the probability of non-clique-preserving orders is smaller on S^P than on S^I, given the clique structure defining P is not trivial.

<u>Proof</u>: (a) follows from the fact that the Jacobean of the inverse mapping $x = P^{-1}z$ is nonzero and constant.

(b) The mean of a uniform distribution on any simplex equals the (ordinary) mean of the simplex' vertices. $Ex \in S^I$ thus equals $[\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}]$. The vertices of S^P are just the columns of P, and since P is doubly stochastic it follows by (a) that $E z \in S^P$ is also $[\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}]$

(c) The result of applying the transformation P to $x \in S^I$ is that coordinates of $z = P x \in S^P$ corresponding to players belonging to the same clique are nearer their common mean (and therefore closer <u>together</u>), while these means themselves remain invariant for all cliques. Thus for any three players i, j and k, where k does not belong to the clique of i and j, it follows that

$$\Pr (z_i < z_k < z_j) < \Pr (x_i < x_k < x_j)$$

Q.E.D.

The theoretical framework discussed sofar is clearly capable of generating reasonable and consistent variations in the probabilities p_t of the orders of players, which could be used to calculate modified Shapley values. A blemish is still present, however: The probabilities culled from S^P do not satisfy Owen's property 1; as Fig. 3.1 shows, sections of S^P representing some orders may neither be congruent nor equal in measure to the sections associated with the reversals of these orders. Fortunately, an easy remedy is available: Since carriers of a game can be arbitrarily enlarged, one simply adds dummy players to the smaller cliques (if any) until all cliques are of equal size. This evens out heterogeneous clique structures (which are responsible for the asymmetries violating property 1) and in consequence the contracted simplex S^P becomes centrally symmetric.

Using the example given above, $\{3\} \in C$ is augmented by dummy player 4, resulting in $C' = \{\{1,2\}, \{3,4\}\}$.

The associated matrix P is then

$$P = \begin{bmatrix} 0.75 & 0.25 & 0 & 0 \\ 0.25 & 0.75 & 0 & 0 \\ & & 0.75 & 0.25 \\ 0 & 0 & 0.25 & 0.75 \end{bmatrix}$$

which, when applied to a suitably enlarged S^I yields a simplex S^P whose 4! sections appear in symmetric pairs, e.g., the order (1 2 3 4) is represented by a polytope congruent to the one containing the reverse order (4 3 2 1).

Again, this symmetrization clearly generalizes without difficulty to clique structures with any number and size of cliques. The sampling procedure discussed in the next section uses this approach computationally.

SECTION 4.

Exact calculation of Shapley values for large games has always presented a problem due to the combinatorially large numbers of probabilities of coalitions (or orders) that have to be evaluated. Owen (1975) and Manne and Shapley (1962) have given approximation procedures that have been used to find values for the U.S. electoral college. The exact calculation of generalized Shapley values is even more difficult because the volumes on n! high-dimensional polytopes would have to be computed (this holds for both Owen's and our generalization).

A strikingly simple remedial idea is to adapt a sampling approach to the problem, thus making (generalized) Shapley values easily accessible to any desired and - affordable - degree of accuracy.

For our simplicial model the procedure is as follows: After the clique structure C has been symmetrized by the addition of dummies (if any) the game contains $m \geq n$ players. A uniformly random m - vector $x \in S^I$ (the (m-1)- dimensional unit simplex) is drawn[4/] and transformed into $z = Px \in S^P$ by the m by m contraction matrix P, that was derived from C' and the collusion parameter $0 \leq \lambda < 1$. A reordering of the players 1,2,...,n,...m according to the values of the coordinates of the vector z yields an order t, which is used to evaluate the order-dependent marginal characteristic function $V_i(S_t(i))$ for each player $i \in N$ (dummy players always get zero and can be ignored at this point). Each player i then receives the indicated number of tokens, the draw of another $x \in S^I$ is made, etc. After the allotted number of sample draws is exhausted the tokens each player has received are toted up and the approximate (generalized) Shapley value of each player is obtained by dividing his holdings by the total number of tokens disbursed.

Following standard statistical theory, confidence intervals can be derived for each player's value ϕ_i independently. If the sample size is $k \geq 30$, the 95% confidence interval is

$$\phi_i \pm 1.96 \left[\frac{\phi_i(1-\phi_i)}{k}\right]^{\frac{1}{2}}$$

The expected accuracy of values approximated in the above fashion thus increases rather slowly with the square root of sample size, but it is surprisingly unaffected by large numbers of players. E.g., our trial solutions for the 50 U.S. state electoral college game, with sample sizes of 3000, were remarkably close to the values found by Mann and Shapley (1962), despite the relatively insignificant computational effort required.

In Table 1, we give a well-documented APL-code called VALUE, that uses our simplicial model and the above sampling approach to calculate generalized Shapley values for the special case of weighted majority games with simple majority. We chose this case because its simple 0-1 characteristic function can be found solely on the basis of voting strengths of players (a mere n-vector).

Table 2 shows 3 sample computations with VALUE, of the clique-structured game "My aunt and I": "My aunt" (player 1) has two votes and forms a clique with her nephew "I" (player 2), who has one vote; two other players (3 and 4) with one vote each stand by themselves. If the clique {1,2} does not collude ($\lambda' = 0$), the precise (Shapley) value is

$\phi[0] = [\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}]$. With $\lambda = 0.5$ the power distribution becomes about

$\phi[0.5] = [0.6, 0.2, 0.1, 0.1]$, and the limiting result, with players 1 and 2 always acting together, should be

$\phi[0.9999] = [\frac{2}{3}, \frac{1}{3}, 0, 0]$.

TABLE 1

```
      ∇VALUEΔHOW[⎕]∇
      ∇ VALUEΔHOW
[1]    'THE FUNCTION VALUE APPROXIMATES GENERALIZED SHAPLEY'
[2]    'VALUES FOR GAMES INVOLVING CLIQUES, FOR THE SPECIAL'
[3]    'CASE OF WEIGHTED VOTING GAMES WITH SIMPLE MAJORITY.'
[4]    'LOCAL INPUTS: V, N-VECTOR OF VOTES,'
[5]    'C, MATRIX REPRESENTING CLIQUE STRUCTURE, E.G., FOR 3 '
[6]    'PLAYERS 1,2 3, 1 AND 2 IN CLIQUE, C=(2 2)⍴1 2 3 0'
[7]    'INPUTS ENTERED ON REQUEST: 0≤L<1, SCALAR COLLUSION'
[8]    'PARAMETER (L=0 MEANS NO COLLUSION)'
[9]    'S, SCALAR, SAMPLE SIZE'
[10]   'OUTPUT: S, N×4 MATRIX; 1ST COL.: PLAYER NUMBERS,'
[11]   '2ND COL.:VOTES, 3RD COL.: VALUES SUMMING TO TOTAL VOTES,'
[12]   '4TH COL.: VALUES SUMMING TO UNITY'
      ∇

      ∇VALUE[⎕]∇
      ∇ S←C VALUE V;I;J;K;L;M;N;P;T
[1]   ⍝CLIQUE MATRIX; ADDITION OF DUMMIES
[2]    C←(⍴C)⍴(,C)+K\(⍴V)+⍳+/K←0=,C
[3]   ⍝CONTRACTION MATRIX (LOOP 1)
[4]    L←⎕,0/⍞←'ENTER COLLUSION PARAMETER'
[5]    P←K∘.=K←⍳×/⍴C×I←1
[6]   Δ1:K←P[;T←C[I;]]
[7]    P[;T]←(K×1-L)+(+/K÷⍴T)∘.×(⍴T)⍴L
[8]    →Δ1×⍳(1↑⍴C)≥I←I+1
[9]   ⍝SAMPLING FOR VALUE (LOOP 2)
[10]   S←N≠N←⍳⍴V×I←1
[11]   K←⎕,0/⍞←'ENTER SAMPLE SIZE'
[12]  ⍝MAJORITY
[13]   M←⌊1++/V÷2
[14]  Δ2: ⍝SAMPLE POINT GENERATION
[15]   J←J÷+/J←(1↑⍴P)?1000
[16]  ⍝CONTRACTION OF SAMPLE POINT, ORDER
[17]   T←N[⍋(P+.×J)[N]]
[18]  ⍝SEQUENTIAL VOTING, PIVOTAL PLAYER
[19]   J←T[+/1,M>+\V[T]]
[20]  ⍝VALUE ACCUMULATION
[21]   S[J]←S[J]+1
[22]   →Δ2×⍳K≥I←I+1
[23]  ⍝PLAYERS, VOTES, VALUES (2 NORMALIZATIONS)
[24]   S←⍉(4,⍴S)⍴N,V,(S÷(+/S)÷+/V),S÷+/S
      ∇
```

TABLE 2

```
       C
 1 2
 3 0
 4 0
       V
2 1 1 1


       C VALUE V
ENTER COLLUSION PARAMETER
□:
       0
ENTER SAMPLE SIZE
□:
       500
 1      2      2.63  0.526
 2      1      0.7   0.14
 3      1      0.84  0.168
 4      1      0.83  0.166


       C VALUE V
ENTER COLLUSION PARAMETER
□:
       0.5
ENTER SAMPLE SIZE
□:
       500
 1      2      2.95  0.59
 2      1      1.02  0.204
 3      1      0.55  0.11
 4      1      0.48  0.096


       C VALUE V
ENTER COLLUSION PARAMETER
□:
       0.99999
ENTER SAMPLE SIZE
□:
       500
 1      2      3.54  0.708
 2      1      1.46  0.292
 3      1      0     0
 4      1      0     0
```

FOOTNOTES

Footnote 1/. It is important to realize that the equiprobability of coalitions of the same size in Shapley's value is not a consequence of axiom 1, as is often erroneously assumed (e.g. Owen: Political Games, p.346) but is implicit in the postulate that the characteristic function is sufficient to describe the game. Shapley (p.311. A Value for a n-person game) tacitly invokes the principal of insufficient reason to arrive at (1.3). Axiom 1, in contrast, brings about only equal sharing of the spoils of a coalition among players in symmetric games, out of which more general games are then constructed, with the help of axiom 3.

A move to introduce additional information to differentiate the probabilities of coalitions is thus a true generalization of Shapley's value, since no violation of the 3 axioms occurs.

Footnote 2/. It is easy to verify in a graphic example with 3 players on a circle, that the above framework assigns higher probabilities to ordering in which players that are close (in affinity and on the sphere) appear contiguously. A general proof of this proposition might be constructed using displaced dual cones, but Owen does not do so; he also takes for granted another, albeit intuitively suggestive result, namely that higher probabilities of orderings with clusters of friendly players increase the likelihood of coalitions containing these clusters (see theorem 3.2.).

Footnote 3/. For instance, in large majority games, of which we tested several computationally on a half circle as suggested by Owen, some single player toward the middle of the affinity spectrum has an impossibly high power spike, i.e. his value is up to 10 times his voting weight, while his equally deserving neighbors with similar numbers of votes receive small values.

Footnote 4/. For our simplicial approach, uniformity of the distribution of sample points is not essential, as long as the distribution is centrally symmetric. In contrast, an adaptation of the sampling procedure to Owen's spheres depends vitally on the uniformity of sample points on the sphere, because there the n! sections are not arrayed around a center (as in the simplex), but are distributed like countries on a globe. See also Section 2.

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