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Fuzzy Random Noncooperative Two-level Linear Programming through Absolute Deviation Minimization Using Possibility and Necessity

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Foreword

In this paper, assuming noncooperative behavior of the decision makers, we consider solution methods for decision making problems in hierarchical organizations under fuzzy random environments. Taking into account vagueness of judgments of decision makers, fuzzy goals are introduced into the formulated fuzzy random noncooperative two-level linear programming problems. Considering the possibility and necessity measure that each objective function fulfills the corresponding fuzzy goal, we transform the fuzzy random two-level linear programming problems to minimize each objective function with fuzzy random variables into stochastic two-level programming problems to maximize the degree of possibility and necessity that each fuzzy goal is fulfilled. Through the use of absolute deviation minimization in stochastic programming, the transformed stochastic two-level programming problems can be reduced to deterministic two-level programming problems. It should be emphasized here that the absolute deviation minimization model is suitable for risk-averse decision makers and it is more tractable than the variance minimization model. For the transformed problems, extended concepts of Stackelberg solutions are introduced and computational methods are also presented. It is significant to note that the extended Stackelberg solutions can be obtained through the combined use of the variable transformation method and the Kth best algorithm for two-level linear programming problems. A numerical example is provided to illustrate the proposed methods.

Abstract

This paper considers fuzzy random two-level linear programming problems under non-cooperative behavior of the decision makers. Having introduced fuzzy goals of decision makers together with the possibility and necessity measure, following absolute deviation minimization, fuzzy random two-level programming problems are transformed into deterministic ones. Extended Stackelberg solutions are introduced and computational methods are also presented.

Keywords: Two-level linear programming problem; fuzzy random variables; Stackelberg solutions; possibility; necessity; absolute deviation minimization.

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About the Authors

Masatoshi Sakawa joined the Integrated Modeling Environment in April 2009. His research and teaching activities are in the area of systems engineering, especially mathematical optimization, multiobjective decision making, fuzzy mathematical programming and game theory. In addition to over 300 articles in national and international journals, he is an author and coauthor of 5 books in English and 14 books in Japanese. At present Dr. Sakawa is a Professor at Hiroshima University, Japan and is working with the Department of Artificial Complex Systems Engineering. Dr. Sakawa received BEng, MEng, and DEng degrees in applied mathematics and physics at Kyoto University, in 1970, 1972, and 1975 respectively. From 1975 he was with Kobe University, where from 1981 he was an Associate Professor in the Department of Systems Engineering. From 1987 to 1990 he was Professor of the Department of Computer Science at Iwate University and from March to December 1991 he was an Honorary Visiting Professor at the University of Manchester Institute of Science and Technology (UMIST), Computation Department, sponsored by the Japan Society for the Promotion of Science (JSPS). He was also a Visiting Professor of the Institute of Economic Research, Kyoto University from April 1991 to March 1992. In 2002 Dr. Sakawa received the Georg Cantor Award of the International Society on Multiple Criteria Decision Making.

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Contents

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1 Introduction

In the real world, we often encounter situations where there are two or more decision makers in an organization with a hierarchical structure, and they make decisions in turn or at the same time so as to optimize their objective functions. Decision making problems in decentralized organizations are often modeled as Stackelberg games [46], and they are formulated as two-level mathematical programming problems [45, 44]. In the context of two-level programming, the decision maker at the upper level first specifies a strategy, and then the decision maker at the lower level specifies a strategy so as to optimize the objective with full knowledge of the action of the decision maker at the upper level. In conventional multi-level mathematical programming models employing the solution concept of Stackelberg equilibrium, it is assumed that there is no communication among decision makers, or they do not make any binding agreement even if there exists such communication. Computational methods for obtaining Stackelberg solutions to two-level linear programming problems are classified roughly into three categories: the vertex enumeration approach [5], the Kuhn-Tucker approach [3, 4, 5, 15], and the penalty function approach [51]. The subsequent works on two-level programming problems under noncooperative behavior of the decision makers have been appearing [34, 35, 14, 36, 9, 11] including some applications to aluminum production process [33], pollution control policy determination [2], tax credits determination for biofuel producers [10], pricing in competitive electricity markets [12], supply chain planning [39] and so forth.

However, to utilize two-level programming for resolution of conflict in decision making problems in real-world decentralized organizations, it is important to realize that simultaneous considerations of both fuzziness [41, 42, 43] and randomness [48, 6, 47] would be required. Fuzzy random variables, first introduced by Kwakernaak [25], have been developing [24, 37, 29], and an overview of the developments of fuzzy random variables was found in [13]. Studies on linear programming problems with fuzzy random variable coefficients, called fuzzy random linear programming problems, were initiated by Wang and Qiao [50], Qaio, Zhang and Wang [38] as seeking the probability distribution of the optimal solution and optimal value. Optimization models for fuzzy random

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linear programming problems were first considered by Luhandjula et al. [30, 32] and further developed by Liu [27, 28] and Rommelfanger [40]. A brief survey of major fuzzy stochastic programming models was found in the paper by Luhandjula [31]. As we look at recent developments in the fields of fuzzy random programming, we can see continuing advances [16, 18, 20, 17, 19, 22, 40, 21, 1, 52].

Under these circumstances, in this paper, assuming noncooperative behavior of the decision makers, we consider solution methods for decision making problems in hierarchical organizations under fuzzy random environments. Taking into account vagueness of judgments of decision makers, fuzzy goals are introduced into the formulated noncooperative two-level linear programming problems involving fuzzy random variables. Considering the possibility and necessity measure that each objective function fulfills the corresponding fuzzy goal, we transform the fuzzy random two-level linear programming problems to minimize each objective function with fuzzy random variables into stochastic two-level programming problems to maximize the degree of possibility and necessity that each fuzzy goal is fulfilled. Through the use of absolute deviation minimization [23] in stochastic programming, the transformed stochastic two-level programming problems can be reduced to deterministic two-level programming problems. It is significant to note that the absolute deviation minimization model is suitable for risk-averse decision makers and it is more tractable than the variance minimization model. For the transformed problems, extended concepts of Stackelberg solutions are introduced and computational methods are also presented. It is shown that extended Stackelberg solutions can be obtained through the combined use of the variable transformation method by Charnes et al. [8] and the Kth best algorithm by Bialas et al. [5].

2 Fuzzy random two-level linear programming problems

Fuzzy random variables, first introduced by Kwakernaak [25], have been defined in various ways [25, 37, 24, 29]. For example, as a special case of fuzzy random variables given by Kwakernaak, Kruse and Meyer [24] defined a fuzzy random variable as follows.

Definition 1 (Fuzzy random variable) *Let (Ω, B, P) be a probability space, $F(\mathcal{R})$ the set of fuzzy numbers with compact supports and X a measurable mapping $\Omega \to F(\mathcal{R})$. Then X is a fuzzy random variable if and only if given $\omega \in \Omega$, $X_\alpha(\omega)$ is a random interval for any $\alpha \in (0,1]$, where $X_\alpha(\omega)$ is an α-level set of the fuzzy set $X(\omega)$.*

Although there exist some minor differences in several definitions of fuzzy random variables, fuzzy random variables could be roughly understood to be a random variable whose observed values are fuzzy sets.

In this paper, we deal with fuzzy random noncooperative two-level linear programming problems formulated as:

$$\left.\begin{array}{ll}\underset{\text{DM1}}{\text{minimize}} & z_1(\boldsymbol{x}_1,\boldsymbol{x}_2) = \tilde{\bar{\boldsymbol{C}}}_{11}\boldsymbol{x}_1 + \tilde{\bar{\boldsymbol{C}}}_{12}\boldsymbol{x}_2 \\ \underset{\text{DM2}}{\text{minimize}} & z_2(\boldsymbol{x}_1,\boldsymbol{x}_2) = \tilde{\bar{\boldsymbol{C}}}_{21}\boldsymbol{x}_1 + \tilde{\bar{\boldsymbol{C}}}_{22}\boldsymbol{x}_2 \\ \text{subject to} & A_1\boldsymbol{x}_1 + A_2\boldsymbol{x}_2 \le \boldsymbol{b} \\ & \boldsymbol{x}_1 \ge \boldsymbol{0}\,,\ \boldsymbol{x}_2 \ge \boldsymbol{0} \end{array}\right\}. \tag{1}$$

It is significant to note here that randomness and fuzziness of the coefficients are denoted by the "dash above" and "wave above" i.e., "$^-$" and "$^\sim$", respectively. In this formulation, $\boldsymbol{x}_1$ is an n_1 dimensional decision variable column vector for the decision maker at the upper level (DM1), $\boldsymbol{x}_2$ is an n_2 dimensional decision variable column vector for the decision maker at the lower level (DM2), $z_1(\boldsymbol{x}_1, \boldsymbol{x}_2)$ is the objective function for DM1 and $z_2(\boldsymbol{x}_1, \boldsymbol{x}_2)$ is the objective function for DM2. Elements $\tilde{\bar{C}}_{ljk}$, $k = 1, 2, \ldots, n_j$ of coefficient vectors $\tilde{\bar{\boldsymbol{C}}}_{lj}$, $l = 1, 2$, $j = 1, 2$ are fuzzy random variables characterized by the membership function:

$$\mu_{\tilde{\bar{C}}_{ljk}}(\tau) = \begin{cases} \max\left\{0, 1 - \dfrac{\bar{d}_{ljk} - \tau}{\beta_{ljk}}\right\} & , \quad \text{if } \tau \leq \bar{d}_{ljk} \\ \max\left\{0, 1 - \dfrac{\tau - \bar{d}_{ljk}}{\gamma_{ljk}}\right\} & , \quad \text{otherwise,} \end{cases}$$

where $\bar{d}_{ljk}$ is a random variable that takes an observed value d_{ljks_l} under a scenario $s_l \in \{1, 2, \ldots, S_l\}$ whose probability is p_{ls_l}, and parameters β_{ljk} and γ_{ljk}, representing left and right spreads of $\mu_{\tilde{\bar{C}}_{ljk}}(\cdot)$, are positive numbers. This definition of fuzzy random variables was first appeared in the literature by Katagiri et al. [20]. Figure 1 illustrates an example of the membership function of a fuzzy random variable $\tilde{\bar{C}}_{ljk}$.

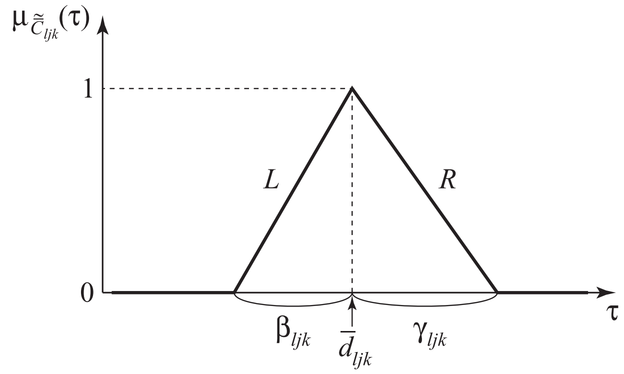

Figure 1: An example of a membership function of a fuzzy random variable.

Fuzzy random two-level linear programming problems formulated as (1) are often seen in actual decision making situations. For example, consider a supply chain planning [39] where the distribution center (DM1) and the production part (DM2) hope to minimize the distribution cost and the production cost respectively. Since coefficients of these objective functions are often affected by the economic conditions varying at random, they can be regarded as random variables. In addition, since observed values of them are often ambiguous and estimated by fuzzy numbers, they are expressed by fuzzy random variables. Hence, the supply chain planning problem can be formulated as a two-level linear programming problem involving fuzzy random variable coefficients.

Observing that each coefficient $\tilde{\bar{C}}_{ljk}$ is a fuzzy random variable defined as a random variable whose observed values are L-R fuzzy numbers, each objective function $\tilde{\bar{\boldsymbol{C}}}_l \boldsymbol{x} = \tilde{\bar{\boldsymbol{C}}}_{l1} \boldsymbol{x}_1 + \tilde{\bar{\boldsymbol{C}}}_{l2} \boldsymbol{x}_2$ is also a fuzzy random variable whose observed values are fuzzy numbers characterized by the membership function

$$\mu_{\tilde{\bar{C}}_l \boldsymbol{x}}(\upsilon) = \begin{cases} \max\left\{0, 1 - \dfrac{\bar{\boldsymbol{d}}_l \boldsymbol{x} - \upsilon}{\boldsymbol{\beta}_l \boldsymbol{x}}\right\} & , \text{ if } \upsilon \leq \bar{\boldsymbol{d}}_l \boldsymbol{x} \\ \max\left\{0, 1 - \dfrac{\upsilon - \bar{\boldsymbol{d}}_l \boldsymbol{x}}{\boldsymbol{\gamma}_l \boldsymbol{x}}\right\} & , \text{ otherwise.} \end{cases}$$

An example of a membership function of an objective function for DMl is shown in Figure 2.

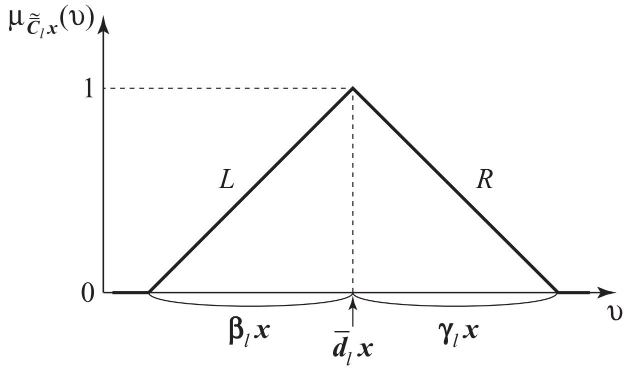

Figure 2: An example of a membership function of an objective function for DMl.

It should be emphasized here that problem (1) is not a well-defined problem due to both fuzziness and randomness of the coefficients, and it cannot be minimized in the sense of deterministic two-level linear programming. Therefore, it is necessary to interpret the problem from some point of view and to transform the problem into the deterministic equivalent one. Realizing this difficulty, in this paper, we assume that decision makers prefer to maximize the degree of possibility or necessity that objective function values satisfy fuzzy goals.

2.1 Fuzzy goals

Considering vague natures of decision makers' judgments, it is natural to assume that decision makers may have vague or fuzzy goals for each of the objective functions. In a minimization problem, a goal stated by decision makers may be to achieve "substantially less than or equal to some value." This type of statement can be quantified by eliciting a corresponding membership function. In this paper, in view of the linearity of the formulated problems, the fuzzy goals $\tilde{G}_l$ such as "$z_l(\boldsymbol{x}_1, \boldsymbol{x}_2)$ should be substantially less than or equal to a certain value" are assumed to be quantified by the linear membership functions:

$$\mu_{\tilde{G}_l}(y) = \begin{cases} 1 & , \text{ if } y \leq z_l^1 \\ \dfrac{y - z_l^0}{z_l^1 - z_l^0} & , \text{ if } z_l^1 < y \leq z_l^0 \\ 0 & , \text{ if } y > z_l^0. \end{cases} \tag{2}$$

Figure 3 illustrates a possible shape of the membership function for the fuzzy goal.

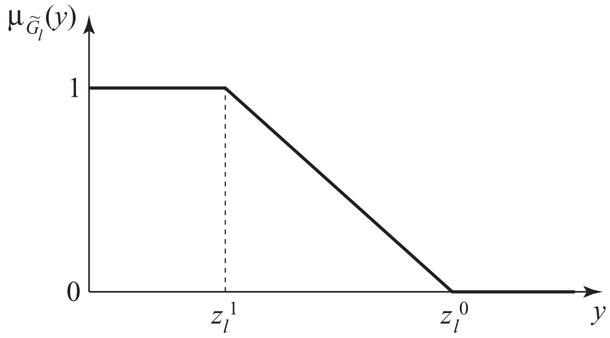

Figure 3: An example of a membership function $\mu_{\tilde{G}_l}(\cdot)$ of a fuzzy goal $\tilde{G}_l$.

2.2 Possibility and necessity

Having determined the fuzzy goals of the decision makers, if we regard $\mu_{\bar{\tilde{\boldsymbol{C}}}_l\boldsymbol{x}}(\cdot)$ as a possibility distribution on the basis of the concept of possibility measure, the degree of possibility $\Pi_{\bar{\tilde{\boldsymbol{C}}}_l\boldsymbol{x}}(\tilde{G}_l)$ that the fuzzy goal $\tilde{G}_l$ is fulfilled under the possibility distribution $\mu_{\bar{\tilde{\boldsymbol{C}}}_l\boldsymbol{x}}(\cdot)$ is given by:

$$\Pi_{\bar{\tilde{\boldsymbol{C}}}_l\boldsymbol{x}}(\tilde{G}_l) = \sup_y \min\left\{\mu_{\bar{\tilde{\boldsymbol{C}}}_l\boldsymbol{x}}(y),\ \mu_{\tilde{G}_l}(y)\right\},\ \ l = 1, 2. \tag{3}$$

Figure 4 illustrates the degree of possibility that the fuzzy goal $\tilde{G}_l$ is fulfilled under the possibility distribution $\mu_{\bar{\tilde{\boldsymbol{C}}}_l\boldsymbol{x}}(\cdot)$.

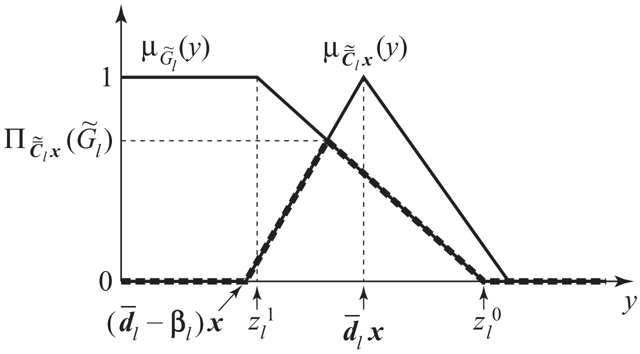

Figure 4: The degree of possibility $\Pi_{\bar{\tilde{\boldsymbol{C}}}_l\boldsymbol{x}}(\tilde{G}_l)$

Using the degree of possibility, problem (1) to minimize each objective function $\bar{\tilde{\boldsymbol{C}}}_l\boldsymbol{x}$ can be transformed into the following stochastic two-level programming problem to maximize the degree of possibility for each objective function $\Pi_{\bar{\tilde{\boldsymbol{C}}}_l\boldsymbol{x}}(\tilde{G}_l)$:

$$\left.\begin{array}{ll} \underset{\text{DM1}}{\text{maximize}} & \Pi_{\bar{\tilde{\boldsymbol{C}}}_1\boldsymbol{x}}(\tilde{G}_1) \\ \underset{\text{DM2}}{\text{maximize}} & \Pi_{\bar{\tilde{\boldsymbol{C}}}_2\boldsymbol{x}}(\tilde{G}_2) \\ \text{subject to} & A_1\boldsymbol{x}_1 + A_2\boldsymbol{x}_2 \leq \boldsymbol{b} \\ & \boldsymbol{x}_1 \geq \boldsymbol{0},\ \boldsymbol{x}_2 \geq \boldsymbol{0} \end{array}\right\}. \tag{4}$$

On the other hand, if decision makers are more risk-averse or wish to avoid a risk, decision making using possibility may be inappropriate since the obtained solution becomes too optimistic. In such a situation, decision making using necessity seems to be suitable for pessimistic decision makers. To be more specific, the degree of necessity $N_{\tilde{\bar{C}}_l \boldsymbol{x}}(\tilde{G}_l)$ that the fuzzy goal $\tilde{G}_l$ is fulfilled under the possibility distribution $\mu_{\tilde{\bar{C}}_l \boldsymbol{x}}(\cdot)$ is given by:

$$N_{\tilde{\bar{C}}_l \boldsymbol{x}}(\tilde{G}_l) = \inf_y \max \left\{ 1 - \mu_{\tilde{\bar{C}}_l \boldsymbol{x}}(y),\ \mu_{\tilde{G}_l}(y) \right\},\ \ l = 1, 2. \tag{5}$$

Figure 5 illustrates the degree of necessity that the fuzzy goal $\tilde{G}_l$ is fulfilled under the possibility distribution $N_{\tilde{\bar{C}}_l \boldsymbol{x}}(\tilde{G}_l)$.

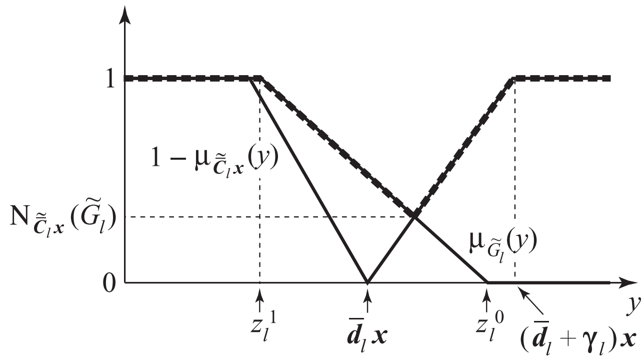

Figure 5: The degree of necessity $N_{\tilde{\bar{C}}_l \boldsymbol{x}}(\tilde{G}_l)$

Quite similar to the possibility case, using the degree of necessity, problem (1) can be transformed into the following stochastic two-level programming problem to maximize the degree of necessity for each objective function $N_{\tilde{\bar{C}}_l \boldsymbol{x}}(\tilde{G}_l)$:

$$\left.\begin{array}{ll} \underset{\text{DM1}}{\text{maximize}} & N_{\tilde{\bar{C}}_1 \boldsymbol{x}}(\tilde{G}_1) \\ \underset{\text{DM2}}{\text{maximize}} & N_{\tilde{\bar{C}}_2 \boldsymbol{x}}(\tilde{G}_2) \\ \text{subject to} & A_1 \boldsymbol{x}_1 + A_2 \boldsymbol{x}_2 \leq \boldsymbol{b} \\ & \boldsymbol{x}_1 \geq \boldsymbol{0},\ \boldsymbol{x}_2 \geq \boldsymbol{0} \end{array}\right\}. \tag{6}$$

For each of the objective functions in (4) and (6), if we set

$$z_l^0 = \max_{s_l \in \{1,2,\ldots,S_l\}} \max_{\boldsymbol{x} \in X} \sum_{j=1}^{2} \sum_{k=1}^{n_j} d_{ljks_l} x_{jk} \tag{7}$$

$$z_l^1 = \min_{s_l \in \{1,2,\ldots,S_l\}} \min_{\boldsymbol{x} \in X} \sum_{j=1}^{2} \sum_{k=1}^{n_j} d_{ljks_l} x_{jk}, \tag{8}$$

the degree of possibility (3) and the degree of necessity (5) can be rewritten as:

$$\Pi_{\tilde{\bar{C}}_l \boldsymbol{x}}(\tilde{G}_l) = \frac{\displaystyle\sum_{j=1}^{2} \sum_{k=1}^{n_j} \left\{ \beta_{ljk} - \bar{d}_{ljk} \right\} x_{jk} + z_l^0}{\displaystyle\sum_{j=1}^{2} \sum_{k=1}^{n_j} \beta_{ljk} x_{jk} - z_l^1 + z_l^0} \tag{9}$$

$$N_{\tilde{\bar{\boldsymbol{C}}}_l\boldsymbol{x}}(\tilde{G}_l) = \frac{-\displaystyle\sum_{j=1}^{2}\sum_{k=1}^{n_j}\bar{d}_{ljk}x_{jk} + z_l^0}{\displaystyle\sum_{j=1}^{2}\sum_{k=1}^{n_j}\gamma_{ljk}x_{jk} - z_l^1 + z_l^0} \tag{10}$$

where denominators in (9) and (10) are assumed to be positive since each of β_{ljk}, γ_{ljk} and x_{jk}, $l = 1, 2$, $j = 1, 2$, $k = 1, 2, \ldots, n_j$ is nonnegative and $z_l^1 < z_l^0$.

In this way, it follows that both of the problems (4) and (6) are stochastic two-level programming problems whose objective functions $\Pi_{\tilde{\bar{\boldsymbol{C}}}_l\boldsymbol{x}}(\tilde{G}_l)$ and $N_{\tilde{\bar{\boldsymbol{C}}}_l\boldsymbol{x}}(\tilde{G}_l)$ are linear fractional and vary randomly depending on random variables $\bar{d}_{ljk}$.

3 Stackelberg solutions through absolute deviation minimization

In this section, assuming the decision makers are risk-averse, we reduce the transformed stochastic two-level programming problems (4) and (6) to deterministic two-level programming problems through the absolute deviation minimization model [23]. It is significant to note that the absolute deviation minimization model is suitable for risk-averse decision makers and more tractable than the variance minimization model.

3.1 Possibility case

Following the absolute deviation minimization model, the maximization of $\Pi_{\tilde{\bar{\boldsymbol{C}}}_l\boldsymbol{x}}(\tilde{G}_l)$ is replaced with the minimization of its absolute deviation $\mathrm{E}\left[\left|\Pi_{\tilde{\bar{\boldsymbol{C}}}_l\boldsymbol{x}}(\tilde{G}_l) - \mathrm{E}\left[\Pi_{\tilde{\bar{\boldsymbol{C}}}_l\boldsymbol{x}}(\tilde{G}_l)\right]\right|\right]$ as follows:

$$\left.\begin{array}{ll} \underset{\text{DM1}}{\text{minimize}} & Z_1^{\Pi,AD}(\boldsymbol{x}_1,\boldsymbol{x}_2) = \mathrm{E}\left[\left|\Pi_{\tilde{\bar{\boldsymbol{C}}}_1\boldsymbol{x}}(\tilde{G}_1) - \mathrm{E}\left[\Pi_{\tilde{\bar{\boldsymbol{C}}}_1\boldsymbol{x}}(\tilde{G}_1)\right]\right|\right] \\ \underset{\text{DM2}}{\text{minimize}} & Z_2^{\Pi,AD}(\boldsymbol{x}_1,\boldsymbol{x}_2) = \mathrm{E}\left[\left|\Pi_{\tilde{\bar{\boldsymbol{C}}}_2\boldsymbol{x}}(\tilde{G}_2) - \mathrm{E}\left[\Pi_{\tilde{\bar{\boldsymbol{C}}}_2\boldsymbol{x}}(\tilde{G}_2)\right]\right|\right] \\ \text{subject to} & A_1\boldsymbol{x}_1 + A_2\boldsymbol{x}_2 \le \boldsymbol{b} \\ & \boldsymbol{x}_1 \ge \boldsymbol{0},\ \boldsymbol{x}_2 \ge \boldsymbol{0} \end{array}\right\}. \tag{11}$$

Since each $\bar{d}_{ljk}$ is a random variable that takes an observed value d_{ljks_l} under a scenario $s_l \in \{1, 2, \ldots, S_l\}$ whose probability is p_{s_l}, $\mathrm{E}\left[\left|\Pi_{\tilde{\bar{\boldsymbol{C}}}_l\boldsymbol{x}}(\tilde{G}_l) - \mathrm{E}\left[\Pi_{\tilde{\bar{\boldsymbol{C}}}_l\boldsymbol{x}}(\tilde{G}_l)\right]\right|\right]$ in (11) are rewritten as:

$$\begin{aligned} &\mathrm{E}\left[\left|\Pi_{\tilde{\bar{\boldsymbol{C}}}_l\boldsymbol{x}}(\tilde{G}_l) - \mathrm{E}\left[\Pi_{\tilde{\bar{\boldsymbol{C}}}_l\boldsymbol{x}}(\tilde{G}_l)\right]\right|\right] \\ &\quad = \mathrm{E}\left[\left|\frac{\displaystyle\sum_{j=1}^{2}\sum_{k=1}^{n_j}\left\{\beta_{ljk} - \bar{d}_{ljk}\right\}x_{jk} + z_l^0}{\displaystyle\sum_{j=1}^{2}\sum_{k=1}^{n_j}\beta_{ljk}x_{jk} - z_l^1 + z_l^0} - \mathrm{E}\left[\frac{\displaystyle\sum_{j=1}^{2}\sum_{k=1}^{n_j}\left\{\beta_{ljk} - \bar{d}_{ljk}\right\}x_{jk} + z_l^0}{\displaystyle\sum_{j=1}^{2}\sum_{k=1}^{n_j}\beta_{ljk}x_{jk} - z_l^1 + z_l^0}\right]\right|\right] \\ &\quad = \sum_{s_l=1}^{S_l} p_{ls_l}\left|\frac{\displaystyle\sum_{j=1}^{2}\sum_{k=1}^{n_j}\left\{\beta_{ljk} - d_{ljks_l}\right\}x_{jk} + z_l^0}{\displaystyle\sum_{j=1}^{2}\sum_{k=1}^{n_j}\beta_{ljk}x_{jk} - z_l^1 + z_l^0} - \sum_{s_l=1}^{S_l} p_{ls_l}\frac{\displaystyle\sum_{j=1}^{2}\sum_{k=1}^{n_j}\left\{\beta_{ljk} - d_{ljks_l}\right\}x_{jk} + z_l^0}{\displaystyle\sum_{j=1}^{2}\sum_{k=1}^{n_j}\beta_{ljk}x_{jk} - z_l^1 + z_l^0}\right| \end{aligned}$$

$$= \sum_{s_l=1}^{S_l} p_{ls_l} \left| \frac{\displaystyle\sum_{j=1}^{2}\sum_{k=1}^{n_j}\left\{\sum_{s_l=1}^{S_l} p_{ls_l} d_{ljk} - d_{ljks_l}\right\} x_{jk} + z_l^0}{\displaystyle\sum_{j=1}^{2}\sum_{k=1}^{n_j} \beta_{ljk} x_{jk} - z_l^1 + z_l^0} \right| .$$

If we introduce the auxiliary variables

$$\begin{aligned} r_{ls_l}^{+} &= \frac{1}{2}\left[\left|\sum_{j=1}^{2}\sum_{k=1}^{n_j}\left\{\sum_{s_l=1}^{S_l} p_{ls_l} d_{ljk} - d_{ljks_l}\right\} x_{jk} + z_l^0\right| \right. \\ &\qquad \left. + \left(\sum_{j=1}^{2}\sum_{k=1}^{n_j}\left\{\sum_{s_l=1}^{S_l} p_{ls_l} d_{ljk} - d_{ljks_l}\right\} x_{jk} + z_l^0\right)\right] \\ r_{ls_l}^{-} &= \frac{1}{2}\left[\left|\sum_{j=1}^{2}\sum_{k=1}^{n_j}\left\{\sum_{s_l=1}^{S_l} p_{ls_l} d_{ljk} - d_{ljks_l}\right\} x_{jk} + z_l^0\right| \right. \\ &\qquad \left. - \left(\sum_{j=1}^{2}\sum_{k=1}^{n_j}\left\{\sum_{s_l=1}^{S_l} p_{ls_l} d_{ljk} - d_{ljks_l}\right\} x_{jk} + z_l^0\right)\right], \end{aligned}$$

$r_{ls_l}^{+} \geq 0$, $r_{ls_l}^{-} \geq 0$ and the following relations hold:

$$\begin{aligned} r_{ls_l}^{+} + r_{ls_l}^{-} &= \left|\sum_{j=1}^{2}\sum_{k=1}^{n_j}\left\{\sum_{s_l=1}^{S_l} p_{ls_l} d_{ljk} - d_{ljks_l}\right\} x_{jk} + z_l^0\right| \\ r_{ls_l}^{+} - r_{ls_l}^{-} &= \sum_{j=1}^{2}\sum_{k=1}^{n_j}\left\{\sum_{s_l=1}^{S_l} p_{ls_l} d_{ljk} - d_{ljks_l}\right\} x_{jk} + z_l^0 \\ r_{ls_l}^{+} \cdot r_{ls_l}^{-} &= 0. \end{aligned}$$

In this way, through absolute deviation minimization using possibility, (11) can be reduced to the following deterministic two-level programming problem:

$$\left.\begin{array}{l} \underset{\text{DM1}}{\text{minimize}}\ Z_1^{\Pi,AD}(\boldsymbol{x}_1,\boldsymbol{x}_2) = \dfrac{\displaystyle\sum_{s_1=1}^{S_1} p_{1s_1}\left(r_{1s_1}^{+} + r_{1s_1}^{-}\right)}{\displaystyle\sum_{j=1}^{2}\sum_{k=1}^{n_j} \beta_{1jk} x_{jk} - z_1^1 + z_1^0} \\ \underset{\text{DM2}}{\text{minimize}}\ Z_2^{\Pi,AD}(\boldsymbol{x}_1,\boldsymbol{x}_2) = \dfrac{\displaystyle\sum_{s_2=1}^{S_2} p_{2s_2}\left(r_{2s_2}^{+} + r_{2s_2}^{-}\right)}{\displaystyle\sum_{j=1}^{2}\sum_{k=1}^{n_j} \beta_{2jk} x_{jk} - z_2^1 + z_2^0} \\ \text{subject to}\ \ A_1\boldsymbol{x}_1 + A_2\boldsymbol{x}_2 \leq \boldsymbol{b} \\ \qquad \displaystyle\sum_{j=1}^{2}\sum_{k=1}^{n_j}\left\{\sum_{s_l=1}^{S_l} p_{ls_l} d_{ljk} - d_{ljks_l}\right\} x_{jk} - r_{ls_l}^{+} + r_{ls_l}^{-} = -z_l^0, \\ \qquad\qquad\qquad l = 1,2,\ s_l = 1,2,\ldots,S_l \\ \qquad r_{ls_l}^{+} \cdot r_{ls_l}^{-} = 0,\ l = 1,2,\ s_l = 1,2,\ldots,S_l \\ \qquad \boldsymbol{x}_1 \geq \mathbf{0},\ \boldsymbol{x}_2 \geq \mathbf{0},\ r_{ls_l}^{+} \geq 0,\ r_{ls_l}^{-} \geq 0,\ l = 1,2,\ s_l = 1,2,\ldots,S_l \end{array}\right\}. \qquad (12)$$

It should be noted here that for noncooperative two-level programming problems, DM1 first specifies a decision and then DM2 determines a decision so as to optimize the objective function of self with full knowledge of the decision of DM1. According to this rule, DM1 also makes a decision so as to optimize the objective function of self. The solution defined as the procedure is called a Stackelberg solution.

Realizing that (12) is a deterministic two-level programming problem, we are now ready to introduce the extended concepts of Stackelberg solution for the original fuzzy random two-level linear programming problem (1).

Definition 2 (AD-P-Stackelberg solution) *A feasible solution* $(\boldsymbol{x}_1^*, \boldsymbol{x}_2^*) \in X$ *is called an AD-P-Stackelberg solution, meaning a Stackelberg solution through absolute deviation minimization using possibility, if* $(\boldsymbol{x}_1^*, \boldsymbol{x}_2^*)$ *is an optimal solution to the following two-level linear fractional programming problem:*

$$\left.\begin{array}{ll}
\underset{\boldsymbol{x}_1}{\textit{minimize}} & Z_1^{\Pi,AD}(\boldsymbol{x}_1, \boldsymbol{x}_2) \\
\textit{where } \boldsymbol{x}_2 \textit{ solves} & \\
\underset{\boldsymbol{x}_2}{\textit{minimize}} & Z_2^{\Pi,AD}(\boldsymbol{x}_1, \boldsymbol{x}_2) \\
\textit{subject to} & A_1\boldsymbol{x}_1 + A_2\boldsymbol{x}_2 \leq \boldsymbol{b} \\
& \displaystyle\sum_{j=1}^{2}\sum_{k=1}^{n_j}\left\{\sum_{s_l=1}^{S_l} p_{ls_l} d_{ljk} - d_{ljks_l}\right\} x_{jk} - r_{ls_l}^+ + r_{ls_l}^- = -z_l^0, \\
& \qquad\qquad l = 1,2,\ s_l = 1,2,\ldots,S_l \\
& r_{ls_l}^+ \cdot r_{ls_l}^- = 0,\ l = 1,2,\ s_l = 1,2,\ldots,S_l \\
& \boldsymbol{x}_1 \geq \mathbf{0},\ \boldsymbol{x}_2 \geq \mathbf{0},\ r_{ls_l}^+ \geq 0,\ r_{ls_l}^- \geq 0,\ l = 1,2,\ s_l = 1,2,\ldots,S_l
\end{array}\right\}. \tag{13}$$

Observing that (13) is a two-level linear fractional programming problem when complementary conditions are relaxed, it is now appropriate to consider some effective computational methods for obtaining AD-P-Stackelberg solutions.

Following the definition of Stackelberg solutions, for any feasible decision $\hat{\boldsymbol{x}}_1$ given by DM1, DM2 is assumed to select a decision $\boldsymbol{x}_2(\hat{\boldsymbol{x}}_1)$ which is an optimal solution to the following problem:

$$\left.\begin{array}{ll}
\text{minimize} & \dfrac{\displaystyle\sum_{s_2=1}^{S_2} p_{2s_2}\left(r_{2s_2}^+ + r_{2s_2}^-\right)}{\displaystyle\sum_{k=1}^{n_2}\beta_{22k}x_{2k} - z_2^1 + z_2^0 + \sum_{k=1}^{n_1}\beta_{21k}\hat{x}_{1k}} \\
\text{subject to} & A_2\boldsymbol{x}_2 \leq \boldsymbol{b} - A_1\hat{\boldsymbol{x}}_1 \\
& \displaystyle\sum_{k=1}^{n_2}\left(\sum_{s_l=1}^{S_l} p_{ls_l} d_{l2k} - d_{l2ks_l}\right) x_{2k} - r_{ls_l}^+ + r_{ls_l}^- \\
& \displaystyle = -z_l^0 - \sum_{k=1}^{n_1}\left(\sum_{s_l=1}^{S_l} p_{ls_l} d_{l1k} - d_{l1ks_l}\right)\hat{x}_{1k},\ l = 1,2,\ s_l = 1,2,\ldots,S_l \\
& r_{ls_l}^+ \cdot r_{ls_l}^- = 0,\ l = 1,2,\ s_l = 1,2,\ldots,S_l \\
& \boldsymbol{x}_2 \geq \mathbf{0},\ r_{ls_l}^+ \geq 0,\ r_{ls_l}^- \geq 0,\ l = 1,2,\ s_l = 1,2,\ldots,S_l
\end{array}\right\}. \tag{14}$$

The optimal solution $\boldsymbol{x}_2(\hat{\boldsymbol{x}}_1)$ to (14) is called a rational reaction for $\hat{\boldsymbol{x}}_1$. Let us denote the set of rational reactions for $\hat{\boldsymbol{x}}_1$ by $RR(\hat{\boldsymbol{x}}_1)$. Then, DM1 should select a solution $(\boldsymbol{x}_1, \boldsymbol{x}_2)$

to optimize $Z_1^{\Pi,AD}(\boldsymbol{x}_1,\boldsymbol{x}_2)$ from among the inducible region $IR=\{(\boldsymbol{x}_1,\boldsymbol{x}_2)\mid(\boldsymbol{x}_1,\boldsymbol{x}_2)\in X,\ \boldsymbol{x}_2\in RR(\boldsymbol{x}_1)\}$. To be more explicit, DM1 selects an optimal solution to the following problem:

$$\left.\begin{array}{ll}\text{minimize} & Z_1^{\Pi,AD}(\boldsymbol{x}_1,\boldsymbol{x}_2)\\ \text{subject to} & A_1\boldsymbol{x}_1+A_2\boldsymbol{x}_2\le\boldsymbol{b}\\ & \displaystyle\sum_{j=1}^{2}\sum_{k=1}^{n_j}\left\{\sum_{s_l=1}^{S_l}p_{ls_l}d_{ljk}-d_{ljks_l}\right\}x_{jk}-r_{ls_l}^{+}+r_{ls_l}^{-}=-z_l^0,\\ & \qquad\qquad l=1,2,\ s_l=1,2,\ldots,S_l\\ & r_{ls_l}^{+}\cdot r_{ls_l}^{-}=0,\ l=1,2,\ s_l=1,2,\ldots,S_l\\ & \boldsymbol{x}_1\ge\boldsymbol{0},\ \boldsymbol{x}_2\in RR(\boldsymbol{x}_1),\ r_{ls_l}^{+}\ge0,\ r_{ls_l}^{-}\ge0,\ l=1,2,\ s_l=1,2,\ldots,S_l\end{array}\right\}. \tag{15}$$

It should be emphasized here that the optimal solution to (15) is an AD-P-Stackelberg solution.

For two-level linear fractional programming problems with linear fractional objective functions and linear constraints, it is shown that Stackelberg solutions exist at some extreme point of the feasible region [7]. Although (13) is not a two-level linear fractional programming problem, if complementary conditions $r_{ls_l}^{+}\cdot r_{ls_l}^{-}=0,\ l=1,2,\ s_l=1,2,\ldots,S_l$ in (13) are relaxed, it should be emphasized here that the resulting relaxed problem becomes a two-level linear fractional programming one. Hence, from the property of Stackelberg solutions to two-level linear fractional programming problems, we can omit complementary conditions from (13) since these conditions automatically hold at any extreme point of the feasible region. In this way, we can consider the following relaxed problem:

$$\left.\begin{array}{l}\underset{\boldsymbol{x}_1}{\text{minimize}}\ Z_1^{\Pi,AD}(\boldsymbol{x}_1,\boldsymbol{x}_2)=\dfrac{\displaystyle\sum_{s_1=1}^{S_1}p_{1s_1}\left(r_{1s_1}^{+}+r_{1s_1}^{-}\right)}{\displaystyle\sum_{j=1}^{2}\sum_{k=1}^{n_j}\beta_{1jk}x_{jk}-z_1^1+z_1^0}\\ \text{where } \boldsymbol{x}_2 \text{ solves}\\ \underset{\boldsymbol{x}_2}{\text{minimize}}\ Z_2^{\Pi,AD}(\boldsymbol{x}_1,\boldsymbol{x}_2)=\dfrac{\displaystyle\sum_{s_2=1}^{S_2}p_{2s_2}\left(r_{2s_2}^{+}+r_{2s_2}^{-}\right)}{\displaystyle\sum_{j=1}^{2}\sum_{k=1}^{n_j}\beta_{2jk}x_{jk}-z_2^1+z_2^0}\\ \text{subject to}\ \ A_1\boldsymbol{x}_1+A_2\boldsymbol{x}_2\le\boldsymbol{b}\\ \qquad\displaystyle\sum_{j=1}^{2}\sum_{k=1}^{n_j}\left\{\sum_{s_l=1}^{S_l}p_{ls_l}d_{ljk}-d_{ljks_l}\right\}x_{jk}-r_{ls_l}^{+}+r_{ls_l}^{-}=-z_l^0,\\ \qquad\qquad l=1,2,\ s_l=1,2,\ldots,S_l\\ \qquad\boldsymbol{x}_1\ge\boldsymbol{0},\ \boldsymbol{x}_2\ge\boldsymbol{0},\ r_{ls_l}^{+}\ge0,\ r_{ls_l}^{-}\ge0,\ l=1,2,\ s_l=1,2,\ldots,S_l\end{array}\right\}. \tag{16}$$

Observing that (16) is a two-level linear fractional programming problem, we can construct the following computational method for obtaining AD-P-Stackelberg solutions through the combined use of the variable transformation method by Charnes and Cooper [8] and the Kth best algorithm for two-level linear programming problems by Bialas et al. [5].

The computational method for obtaining AD-P-Stackelberg solutions

Step 1: Let $i := 1$. Removing the objective function of DM2 from (16), solve the following problem:

$$\left.\begin{array}{ll} \text{minimize} & Z_1^{\Pi,AD}(\boldsymbol{x}_1,\boldsymbol{x}_2) = \dfrac{\sum_{s_1=1}^{S_1} p_{1s_1}\left(r_{1s_1}^+ + r_{1s_1}^-\right)}{\sum_{j=1}^{2}\sum_{k=1}^{n_j}\beta_{1jk}x_{jk} - z_1^1 + z_1^0} \\ \text{subject to} & A_1\boldsymbol{x}_1 + A_2\boldsymbol{x}_2 \le \boldsymbol{b} \\ & \sum_{j=1}^{2}\sum_{k=1}^{n_j}\left\{\sum_{s_l=1}^{S_l} p_{ls_l}d_{ljk} - d_{ljks_l}\right\} x_{jk} - r_{ls_l}^+ + r_{ls_l}^- = -z_l^0, \\ & \qquad l=1,2,\ s_l=1,2,\ldots,S_l \\ & \boldsymbol{x}_1 \ge \boldsymbol{0},\ \boldsymbol{x}_2 \ge \boldsymbol{0},\ r_{ls_l}^+ \ge 0,\ r_{ls_l}^- \ge 0,\ l=1,2,\ s_l=1,2,\ldots,S_l \end{array}\right\}. \tag{17}$$

Observing that (17) is a linear fractional programming problem and the denominator of the objective function is positive as discussed in (9), it can be transformed into an equivalent linear programming problem by the variable transformation method by Charnes and Cooper [8]. To be more specific, introducing the variable transformation

$$t = \frac{1}{\sum_{j=1}^{2}\sum_{k=1}^{n_j}\beta_{1jk}x_{jk} - z_1^1 + z_1^0}$$

and letting $\boldsymbol{y}_1 = t\cdot\boldsymbol{x}_1$, $\boldsymbol{y}_2 = t\cdot\boldsymbol{x}_2$, $\boldsymbol{q}^+ = t\cdot\boldsymbol{r}^+$, $\boldsymbol{q}^- = t\cdot\boldsymbol{r}^-$, (17) is transformed into the following linear programming problem:

$$\left.\begin{array}{ll} \text{minimize} & \sum_{s_1=1}^{S_1} p_{1s_1}\left(q_{1s_1}^+ + q_{1s_1}^-\right) \\ \text{subject to} & A_1\boldsymbol{y}_1 + A_2\boldsymbol{y}_2 - \boldsymbol{b}t \le \boldsymbol{0} \\ & \sum_{j=1}^{2}\sum_{k=1}^{n_j}\left\{\sum_{s_l=1}^{S_l} p_{ls_l}d_{ljk} - d_{ljks_l}\right\} y_{jk} - q_{ls_l}^+ + q_{ls_l}^- + z_l^0 t = 0, \\ & \qquad l=1,2,\ s_l=1,2,\ldots,S_l \\ & \sum_{j=1}^{2}\sum_{k=1}^{n_j}\beta_{1jk}y_{jk} - (z_1^1 - z_1^0)t = 1 \\ & \boldsymbol{y}_1 \ge \boldsymbol{0},\ \boldsymbol{y}_2 \ge \boldsymbol{0},\ \boldsymbol{q}^+ \ge \boldsymbol{0},\ \boldsymbol{q}^- \ge \boldsymbol{0},\ t \ge 0 \end{array}\right\}. \tag{18}$$

Observing that (18) is a linear programming problem, we can obtain an optimal solution by the simplex method. Using the optimal solution to (18) denoted by $(\boldsymbol{y}_{1[1]}^T, \boldsymbol{y}_{2[1]}^T, (\boldsymbol{q}_{[1]}^+)^T, (\boldsymbol{q}_{[1]}^-)^T, t_{[1]})^T$, we can obtain

$$(\boldsymbol{x}_{1[1]}^T, \boldsymbol{x}_{2[1]}^T)^T := (\boldsymbol{y}_{1[1]}^T/t_{[1]}, \boldsymbol{y}_{2[1]}^T/t_{[1]})^T$$

which is an extreme point of the feasible region of (17) as shown in [49]. Let W be a set of feasible extreme points to be searched and U a set of feasible extreme points that had been searched. Let $W := \{(\boldsymbol{x}_{1[1]}^T, \boldsymbol{x}_{2[1]}^T)^T\}$ and $U := \emptyset$. Go to step 2.

Step 2: In order to check whether the present extreme point $(\boldsymbol{x}_{1[i]}^T, \boldsymbol{x}_{2[i]}^T)^T$ exists in the inducible region IR, i.e., $\boldsymbol{x}_{2[i]}$ is a rational reaction for $\boldsymbol{x}_{1[i]}$ or not, we solve the following problem:

$$\left.\begin{array}{ll}\text{minimize} & \dfrac{\sum_{s_2=1}^{S_2} p_{2s_2}\left(r_{2s_2}^+ + r_{2s_2}^-\right)}{\sum_{k=1}^{n_2}\beta_{22k}x_{2k} - z_2^1 + z_2^0 + \sum_{k=1}^{n_1}\beta_{21k}x_{1k[i]}} \\ \text{subject to} & A_2\boldsymbol{x}_2 \le \boldsymbol{b} - A_1\boldsymbol{x}_{1[i]} \\ & \sum_{k=1}^{n_2}\left(\sum_{s_l=1}^{S_l} p_{ls_l}d_{l2k} - d_{l2ks_l}\right)x_{2k} - r_{ls_l}^+ + r_{ls_l}^- \\ & = -z_l^0 - \sum_{k=1}^{n_1}\left(\sum_{s_l=1}^{S_l} p_{ls_l}d_{l1k} - d_{l1ks_l}\right)x_{1k[i]}, \\ & \qquad l = 1,2,\ s_l = 1,2,\ldots,S_l \\ & \boldsymbol{x}_2 \ge \boldsymbol{0},\ r_{ls_l}^+ \ge 0,\ r_{ls_l}^- \ge 0,\ l = 1,2,\ s_l = 1,2,\ldots,S_l\end{array}\right\}. \quad (19)$$

Observing that (19) is also a linear fractional programming problem and the denominator of the objective function is positive, we can apply the variable transformation method by Charnes and Cooper [8] to (19). Namely, introducing the variable transformation

$$u = \frac{1}{\sum_{k=1}^{n_2}\beta_{22k}x_{2k} - z_2^1 + z_2^0 + \sum_{k=1}^{n_1}\beta_{21k}x_{1k[i]}}$$

and letting $\boldsymbol{w}_2 := u \cdot \boldsymbol{x}_2$, $\boldsymbol{o}^+ := u \cdot \boldsymbol{r}^+$, $\boldsymbol{o}^- := u \cdot \boldsymbol{r}^-$, (19) is transformed into the following linear programming problem:

$$\left.\begin{array}{ll}\text{minimize} & \sum_{s_2=1}^{S_2} p_{2s_2}\left(o_{2s_2}^+ + o_{2s_2}^-\right) \\ \text{subject to} & A_2\boldsymbol{w}_2 - (\boldsymbol{b} - A_1\boldsymbol{x}_{1[i]})u \le \boldsymbol{0} \\ & \sum_{k=1}^{n_2}\left(\sum_{s_l=1}^{S_l} p_{ls_l}d_{l2k} - d_{l2ks_l}\right)w_{2k} - o_{ls_l}^+ + o_{ls_l}^- \\ & + \left(z_l^0 + \sum_{k=1}^{n_1}\left(\sum_{s_l=1}^{S_l} p_{ls_l}d_{l1k} - d_{l1ks_l}\right)x_{1k[i]}\right)u = 0, \\ & \qquad l = 1,2,\ s_l = 1,2,\ldots,S_l \\ & \sum_{k=1}^{n_2}\beta_{22k}w_{2k} - \left(z_2^1 - z_2^0 - \sum_{k=1}^{n_1}\beta_{21k}x_{1k[i]}\right)u = 1 \\ & \boldsymbol{w}_2 \ge \boldsymbol{0},\ \boldsymbol{o}^+ \ge \boldsymbol{0},\ \boldsymbol{o}^- \ge \boldsymbol{0},\ u \ge 0\end{array}\right\}. \quad (20)$$

Observing that (20) is a linear programming problem, we can obtain an optimal solution $(\boldsymbol{w}_{2[i]}^T, (\boldsymbol{o}_{[i]}^+)^T, (\boldsymbol{o}_{[i]}^-)^T, u_{[i]})^T$ by the simplex method. If $\boldsymbol{w}_{2[i]}/u_{[i]}$ is equal to $\boldsymbol{x}_{2[i]}$, then the current extreme point $(\boldsymbol{x}_{1[i]}^T, \boldsymbol{x}_{2[i]}^T)^T$ exists in IR, i.e., it is an AD-P-Stackelberg solution and the algorithm is terminated. Otherwise, go to step 3.

Step 3: Let $W_{[i]}$ be a set of feasible extreme points which is adjacent to $(\boldsymbol{x}_{1[i]}^T, \boldsymbol{x}_{2[i]}^T)^T$ and satisfies $Z_1^{\Pi,AD}(\boldsymbol{x}_1, \boldsymbol{x}_2) \ge Z_1^{\Pi,AD}(\boldsymbol{x}_{1[i]}, \boldsymbol{x}_{2[i]})$. Let $U := U \cup \{(\boldsymbol{x}_{1[i]}^T, \boldsymbol{x}_{2[i]}^T)^T\}$ and $W := (W \cup W_{[i]})\backslash U$, and go to step 4.

Step 4: Let $i := i + 1$. Choose an extreme point $(\boldsymbol{x}_{1[i]}^T, \boldsymbol{x}_{2[i]}^T)^T$ such that

$$Z_1^{\Pi,AD}(\boldsymbol{x}_{1[i]}, \boldsymbol{x}_{2[i]}) = \min_{(\boldsymbol{x}_1^T, \boldsymbol{x}_2^T)^T \in W} Z_1^{\Pi,AD}(\boldsymbol{x}_1, \boldsymbol{x}_2),$$

and return to step 2.

It should be noted here that the proposed computational method uses nothing but the variable transformation method, the simplex method and the pivot operation for obtaining an AD-P-Stackelberg solution.

3.2 Necessity case

Quite similar to the possibility case, following the absolute deviation minimization model, the maximization of $N_{\tilde{\bar{\boldsymbol{C}}}_l \boldsymbol{x}}(\tilde{G}_l)$ in (6) is replaced with the minimization of its absolute deviation $\mathrm{E}\left[\left|N_{\tilde{\bar{\boldsymbol{C}}}_l \boldsymbol{x}}(\tilde{G}_l) - \mathrm{E}\left[N_{\tilde{\bar{\boldsymbol{C}}}_l \boldsymbol{x}}(\tilde{G}_l)\right]\right|\right]$ as follows:

$$\left.\begin{array}{ll} \underset{\text{DM1}}{\text{minimize}} & Z_1^{N,AD}(\boldsymbol{x}_1, \boldsymbol{x}_2) = \mathrm{E}\left[\left|N_{\tilde{\bar{\boldsymbol{C}}}_1 \boldsymbol{x}}(\tilde{G}_1) - \mathrm{E}\left[N_{\tilde{\bar{\boldsymbol{C}}}_1 \boldsymbol{x}}(\tilde{G}_1)\right]\right|\right] \\ \underset{\text{DM2}}{\text{minimize}} & Z_2^{N,AD}(\boldsymbol{x}_1, \boldsymbol{x}_2) = \mathrm{E}\left[\left|N_{\tilde{\bar{\boldsymbol{C}}}_2 \boldsymbol{x}}(\tilde{G}_2) - \mathrm{E}\left[N_{\tilde{\bar{\boldsymbol{C}}}_2 \boldsymbol{x}}(\tilde{G}_2)\right]\right|\right] \\ \text{subject to} & A_1 \boldsymbol{x}_1 + A_2 \boldsymbol{x}_2 \leq \boldsymbol{b} \\ & \boldsymbol{x}_1 \geq \boldsymbol{0},\ \boldsymbol{x}_2 \geq \boldsymbol{0} \end{array}\right\}, \tag{21}$$

where $\mathrm{E}\left[\left|N_{\tilde{\bar{\boldsymbol{C}}}_l \boldsymbol{x}}(\tilde{G}_l) - \mathrm{E}\left[N_{\tilde{\bar{\boldsymbol{C}}}_l \boldsymbol{x}}(\tilde{G}_l)\right]\right|\right]$ is rewritten as:

$$\begin{aligned} & \mathrm{E}\left[\left|N_{\tilde{\bar{\boldsymbol{C}}}_l \boldsymbol{x}}(\tilde{G}_l) - \mathrm{E}\left[N_{\tilde{\bar{\boldsymbol{C}}}_l \boldsymbol{x}}(\tilde{G}_l)\right]\right|\right] \\ = \ & \mathrm{E}\left[\left|\frac{-\sum_{j=1}^{2}\sum_{k=1}^{n_j} \bar{d}_{ljk} x_{jk} + z_l^0}{\sum_{j=1}^{2}\sum_{k=1}^{n_j} \gamma_{ljk} x_{jk} - z_l^1 + z_l^0} - \mathrm{E}\left[\frac{-\sum_{j=1}^{2}\sum_{k=1}^{n_j} \bar{d}_{ljk} x_{jk} + z_l^0}{\sum_{j=1}^{2}\sum_{k=1}^{n_j} \gamma_{ljk} x_{jk} - z_l^1 + z_l^0}\right]\right|\right] \\ = \ & \sum_{s_l=1}^{S_l} p_{ls_l} \left| \frac{-\sum_{j=1}^{2}\sum_{k=1}^{n_j} d_{ljks_l} x_{jk} + z_l^0}{\sum_{j=1}^{2}\sum_{k=1}^{n_j} \gamma_{ljk} x_{jk} - z_l^1 + z_l^0} - \sum_{s_l=1}^{S_l} p_{ls_l} \frac{-\sum_{j=1}^{2}\sum_{k=1}^{n_j} d_{ljks_l} x_{jk} + z_l^0}{\sum_{j=1}^{2}\sum_{k=1}^{n_j} \gamma_{ljk} x_{jk} - z_l^1 + z_l^0} \right| \\ = \ & \sum_{s_l=1}^{S_l} p_{ls_l} \left| \frac{\sum_{j=1}^{2}\sum_{k=1}^{n_j} \left\{ \sum_{s_l=1}^{S_l} p_{ls_l} d_{ljk} - d_{ljks_l} \right\} x_{jk} + z_l^0}{\sum_{j=1}^{2}\sum_{k=1}^{n_j} \gamma_{ljk} x_{jk} - z_l^1 + z_l^0} \right|. \end{aligned}$$

Introducing the auxiliary variables $r^+_{ls_l}$ and $r^-_{ls_l}$ as defined in the possibility case, (21) can be reduced to the following deterministic two-level programming problem:

$$\left.\begin{array}{ll}
\underset{\text{DM1}}{\text{minimize}} & Z_1^{N,AD}(\boldsymbol{x}_1,\boldsymbol{x}_2) = \dfrac{\displaystyle\sum_{s_1=1}^{S_1} p_{1s_1}\left(r^+_{1s_1}+r^-_{1s_1}\right)}{\displaystyle\sum_{j=1}^{2}\sum_{k=1}^{n_j}\gamma_{1jk}x_{jk} - z_1^1 + z_1^0} \\
\underset{\text{DM2}}{\text{minimize}} & Z_2^{N,AD}(\boldsymbol{x}_1,\boldsymbol{x}_2) = \dfrac{\displaystyle\sum_{s_2=1}^{S_2} p_{2s_2}\left(r^+_{2s_2}+r^-_{2s_2}\right)}{\displaystyle\sum_{j=1}^{2}\sum_{k=1}^{n_j}\gamma_{2jk}x_{jk} - z_2^1 + z_2^0} \\
\text{subject to} & A_1\boldsymbol{x}_1 + A_2\boldsymbol{x}_2 \leq \boldsymbol{b} \\
& \displaystyle\sum_{j=1}^{2}\sum_{k=1}^{n_j}\left\{\sum_{s_l=1}^{S_l} p_{ls_l}d_{ljk} - d_{ljks_l}\right\}x_{jk} - r^+_{ls_l} + r^-_{ls_l} = -z_l^0, \\
& \qquad\qquad\qquad\qquad l=1,2,\ s_l=1,2,\ldots,S_l \\
& r^+_{ls_l}\cdot r^-_{ls_l} = 0,\ l=1,2,\ s_l=1,2,\ldots,S_l \\
& \boldsymbol{x}_1 \geq \mathbf{0},\ \boldsymbol{x}_2 \geq \mathbf{0},\ r^+_{ls_l}\geq 0,\ r^-_{ls_l}\geq 0,\ l=1,2,\ s_l=1,2,\ldots,S_l
\end{array}\right\}. \qquad (22)$$

Observing that (22) is a deterministic two-level programming problem, we can introduce the extended concepts of Stackelberg solution for the original fuzzy random two-level linear programming problem (1).

Definition 3 (AD-N-Stackelberg solution) *A feasible solution $(\boldsymbol{x}_1^*,\boldsymbol{x}_2^*)\in X$ is called an AD-N-Stackelberg solution, meaning a Stackelberg solution through absolute deviation minimization using necessity, if $(\boldsymbol{x}_1^*,\boldsymbol{x}_2^*)$ is an optimal solution to the following two-level linear fractional programming problem:*

$$\left.\begin{array}{ll}
\underset{\boldsymbol{x}_1}{\textit{minimize}} & Z_1^{N,AD}(\boldsymbol{x}_1,\boldsymbol{x}_2) \\
\textit{where } \boldsymbol{x}_2 \textit{ solves} & \\
\underset{\boldsymbol{x}_2}{\textit{minimize}} & Z_2^{N,AD}(\boldsymbol{x}_1,\boldsymbol{x}_2) \\
\textit{subject to} & A_1\boldsymbol{x}_1 + A_2\boldsymbol{x}_2 \leq \boldsymbol{b} \\
& \displaystyle\sum_{j=1}^{2}\sum_{k=1}^{n_j}\left\{\sum_{s_l=1}^{S_l} p_{ls_l}d_{ljk} - d_{ljks_l}\right\}x_{jk} - r^+_{ls_l} + r^-_{ls_l} = -z_l^0, \\
& \qquad\qquad\qquad\qquad l=1,2,\ s_l=1,2,\ldots,S_l \\
& r^+_{ls_l}\cdot r^-_{ls_l} = 0,\ l=1,2,\ s_l=1,2,\ldots,S_l \\
& \boldsymbol{x}_1 \geq \mathbf{0},\ \boldsymbol{x}_2 \geq \mathbf{0},\ r^+_{ls_l}\geq 0,\ r^-_{ls_l}\geq 0,\ l=1,2,\ s_l=1,2,\ldots,S_l
\end{array}\right\}. \qquad (23)$$

Similarly to the possibility case, for any feasible decision $\hat{\boldsymbol{x}}_1$ given by DM1, the rational reaction $\boldsymbol{x}_2(\hat{\boldsymbol{x}}_1)$ of DM2 can be obtained by solving the following problem:

$$\left.\begin{array}{ll}\text{minimize} & \dfrac{\sum_{s_2=1}^{S_2} p_{2s_2}\left(r_{2s_2}^{+}+r_{2s_2}^{-}\right)}{\sum_{k=1}^{n_2}\gamma_{22k}x_{2k}-z_2^1+z_2^0+\sum_{k=1}^{n_1}\gamma_{21k}\hat{x}_{1k}} \\ \text{subject to} & A_2\boldsymbol{x}_2 \le \boldsymbol{b}-A_1\hat{\boldsymbol{x}}_1 \\ & \sum_{k=1}^{n_2}\left(\sum_{s_l=1}^{S_l} p_{ls_l}d_{l2k}-d_{l2ks_l}\right)x_{2k}-r_{ls_l}^{+}+r_{ls_l}^{-} \\ & =-z_l^0-\sum_{k=1}^{n_1}\left(\sum_{s_l=1}^{S_l} p_{ls_l}d_{l1k}-d_{l1ks_l}\right)\hat{x}_{1k},\ l=1,2,\ s_l=1,2,\ldots,S_l \\ & r_{ls_l}^{+}\cdot r_{ls_l}^{-}=0,\ l=1,2,\ s_l=1,2,\ldots,S_l \\ & \boldsymbol{x}_2\ge\boldsymbol{0},\ r_{ls_l}^{+}\ge 0,\ r_{ls_l}^{-}\ge 0,\ l=1,2,\ s_l=1,2,\ldots,S_l\end{array}\right\}, \tag{24}$$

and DM1 should select an optimal solution $(\boldsymbol{x}_1, \boldsymbol{x}_2)$ to the following problem:

$$\left.\begin{array}{ll}\text{minimize} & Z_1^{N,AD}(\boldsymbol{x}_1,\boldsymbol{x}_2) \\ \text{subject to} & A_1\boldsymbol{x}_1+A_2\boldsymbol{x}_2\le\boldsymbol{b} \\ & \sum_{j=1}^{2}\sum_{k=1}^{n_j}\left\{\sum_{s_l=1}^{S_l} p_{ls_l}d_{ljk}-d_{ljks_l}\right\}x_{jk}-r_{ls_l}^{+}+r_{ls_l}^{-}=-z_l^0, \\ & \qquad l=1,2,\ s_l=1,2,\ldots,S_l \\ & r_{ls_l}^{+}\cdot r_{ls_l}^{-}=0,\ l=1,2,\ s_l=1,2,\ldots,S_l \\ & \boldsymbol{x}_1\ge\boldsymbol{0},\ \boldsymbol{x}_2\in RR(\boldsymbol{x}_1),\ r_{ls_l}^{+}\ge 0,\ r_{ls_l}^{-}\ge 0,\ l=1,2,\ s_l=1,2,\ldots,S_l\end{array}\right\}. \tag{25}$$

Observe that the optimal solution to (25) is an AD-N-Stackelberg solution.

As discussed in the possibility case, we can consider the following relaxed problem:

$$\left.\begin{array}{l}\underset{\boldsymbol{x}_1}{\text{minimize}}\ Z_1^{N,AD}(\boldsymbol{x}_1,\boldsymbol{x}_2)=\dfrac{\sum_{s_1=1}^{S_1} p_{1s_1}\left(r_{1s_1}^{+}+r_{1s_1}^{-}\right)}{\sum_{j=1}^{2}\sum_{k=1}^{n_j}\gamma_{1jk}x_{jk}-z_1^1+z_1^0} \\ \text{where } \boldsymbol{x}_2 \text{ solves} \\ \underset{\boldsymbol{x}_2}{\text{minimize}}\ Z_2^{N,AD}(\boldsymbol{x}_1,\boldsymbol{x}_2)=\dfrac{\sum_{s_2=1}^{S_2} p_{2s_2}\left(r_{2s_2}^{+}+r_{2s_2}^{-}\right)}{\sum_{j=1}^{2}\sum_{k=1}^{n_j}\gamma_{2jk}x_{jk}-z_2^1+z_2^0} \\ \text{subject to}\ A_1\boldsymbol{x}_1+A_2\boldsymbol{x}_2\le\boldsymbol{b} \\ \qquad \sum_{j=1}^{2}\sum_{k=1}^{n_j}\left\{\sum_{s_l=1}^{S_l} p_{ls_l}d_{ljk}-d_{ljks_l}\right\}x_{jk}-r_{ls_l}^{+}+r_{ls_l}^{-}=-z_l^0, \\ \qquad\qquad l=1,2,\ s_l=1,2,\ldots,S_l \\ \qquad \boldsymbol{x}_1\ge\boldsymbol{0},\ \boldsymbol{x}_2\ge\boldsymbol{0},\ r_{ls_l}^{+}\ge 0,\ r_{ls_l}^{-}\ge 0,\ l=1,2,\ s_l=1,2,\ldots,S_l\end{array}\right\}. \tag{26}$$

The computational method for obtaining AD-N-Stackelberg solutions follows very much in the same fashion as that for obtaining AD-P-Stackelberg solutions and will, therefore, be omitted.

4 Numerical example

In order to demonstrate the feasibility and efficiency of the proposed computational methods, consider the following two-level linear programming problem involving fuzzy random variable coefficients:

$$\left.\begin{array}{ll}\underset{\text{DM1}}{\text{minimize}} & z_1(\boldsymbol{x}_1,\boldsymbol{x}_2)=\tilde{\bar{\boldsymbol{C}}}_{11}\boldsymbol{x}_1+\tilde{\bar{\boldsymbol{C}}}_{12}\boldsymbol{x}_2\\ \underset{\text{DM2}}{\text{minimize}} & z_2(\boldsymbol{x}_1,\boldsymbol{x}_2)=\tilde{\bar{\boldsymbol{C}}}_{21}\boldsymbol{x}_1+\tilde{\bar{\boldsymbol{C}}}_{22}\boldsymbol{x}_2\\ \text{subject to} & \boldsymbol{a}_{11}\boldsymbol{x}_1 + \boldsymbol{a}_{12}\boldsymbol{x}_2 \le b_1\\ & \boldsymbol{a}_{21}\boldsymbol{x}_1 + \boldsymbol{a}_{22}\boldsymbol{x}_2 \le b_2\\ & \boldsymbol{a}_{31}\boldsymbol{x}_1 + \boldsymbol{a}_{32}\boldsymbol{x}_2 \le b_3\\ & \boldsymbol{a}_{41}\boldsymbol{x}_1 + \boldsymbol{a}_{42}\boldsymbol{x}_2 \le b_4\\ & \boldsymbol{x}_1=(x_{11},x_{12},x_{13})^T\ge\boldsymbol{0}\\ & \boldsymbol{x}_2=(x_{21},x_{22},x_{23})^T\ge\boldsymbol{0}\end{array}\right\}\quad(27)$$

where $\tilde{\bar{\boldsymbol{C}}}_{lj}$, $l=1,2$, $j=1,2$ are vectors whose elements $\tilde{\bar{C}}_{ljk}$, $k=1,2,\ldots,n_j$ are fuzzy random variables.

Values of coefficients in constraints, values of d_{ljks_l} for each s_l, β_{ljk} and γ_{ljk}, $l=1,2$ are shown in Tables 1, 2 and 3, respectively.

Table 1: Values of coefficients in constraints

	x_{11}	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}	$\boldsymbol{b}$
$\boldsymbol{a}_1$	2	3	1	2	3	3	65
$\boldsymbol{a}_2$	4	4	2	3	2	1	80
$\boldsymbol{a}_3$	2	4	3	3	2	2	105
$\boldsymbol{a}_4$	−3	−2	−2	−4	−1	−4	−70

Table 2: Values of d_{1jks_1} for each $s_1\in\{1,2,3\}$, β_{1jk} and γ_{1jk}.

	x_{11}	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}
Scenario $s_1=1$ ($p_{11}=0.25$)	2.3	−1.0	1.3	−1.3	−1.8	2.0
Scenario $s_1=2$ ($p_{12}=0.40$)	2.0	−1.3	2.0	1.1	−2.1	2.4
Scenario $s_1=3$ ($p_{13}=0.35$)	1.9	−2.4	2.7	−1.5	−1.2	3.8
β_{1jk}	0.8	1.2	0.7	0.9	1.3	0.6
γ_{1jk}	0.8	1.1	0.5	0.6	0.9	1.0

Calculating z_l^0 and z_l^1 from (7) and (8) yields $z_1^0=121$, $z_1^1=-48$, $z_2^0=76$ and $z_2^1=-81$. Considering these values, membership functions of fuzzy goals for objective functions are determined as shown in Figure 6.

For illustrative purposes, we first derive an AD-P-Stackelberg solution to (27). For this numerical example, in step 1, after transforming (17) into (18) by the variable transformation method, (18) is solved by the simplex method. For the obtained value of

Table 3: Values of d_{2jks_2} for each $s_2 \in \{1,2,3\}$, β_{2jk} and γ_{2jk}.

	x_{11}	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}
Scenario $s_2 = 1$ ($p_{21} = 0.45$)	3.0	1.7	-1.6	-1.4	-1.6	1.7
Scenario $s_2 = 2$ ($p_{22} = 0.15$)	1.7	1.3	-2.3	-0.8	-1.9	2.6
Scenario $s_2 = 3$ ($p_{23} = 0.40$)	2.3	0.9	-1.0	-2.0	-1.2	3.5
β_{2jk}	0.7	1.2	0.8	0.5	0.9	1.1
γ_{2jk}	0.7	0.9	0.6	1.0	0.8	0.9

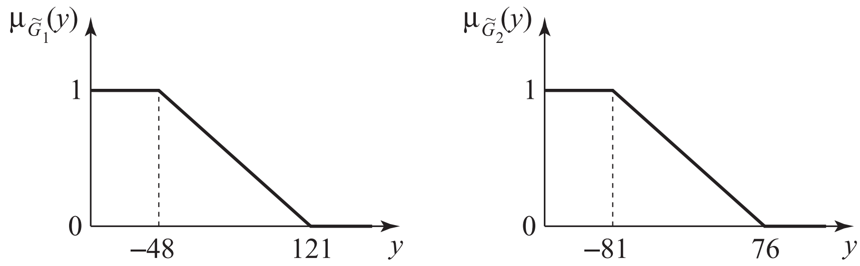

Figure 6: Membership functions $\mu_{\tilde{G}_l}(\cdot)$, $l = 1,2$.

$(\boldsymbol{x}_{1[1]}^T, \boldsymbol{x}_{2[1]}^T)^T = (14.58, 0.00, 0.83, 3.88, 9.08, 0.00)^T$, let $W := \{(\boldsymbol{x}_{1[1]}^T, \boldsymbol{x}_{2[1]}^T)^T\}$, $U := \emptyset$. In step 2, after transforming (19) into (20) by the variable transformation method, we solve (20) by the simplex method in order to obtain the rational reaction for $\boldsymbol{x}_{1[1]}$. Since the optimal solution to (20) $\tilde{\boldsymbol{w}}_{2[1]}/u_{[1]} = (3.23, 11.70, 0.00)^T$ is not equal to $\boldsymbol{x}_{2[1]} = (3.88, 9.08, 0.00)^T$, the current extreme point $(\boldsymbol{x}_{1[1]}^T, \boldsymbol{x}_{2[1]}^T)^T$ is not an AD-P-Stackelberg solution. In step 3, we enumerate feasible extreme points $(\boldsymbol{x}_1^T, \boldsymbol{x}_2^T)^T$ which are adjacent to $(\boldsymbol{x}_{1[1]}^T, \boldsymbol{x}_{2[1]}^T)^T$ and satisfy $Z_1^{\Pi,AD}(\boldsymbol{x}_1, \boldsymbol{x}_2) \geq Z_1^{\Pi,AD}(\boldsymbol{x}_{1[1]}, \boldsymbol{x}_{2[1]})$, and make $W_{[1]}$. Then, let $U := U \cup (\boldsymbol{x}_{1[1]}^T, \boldsymbol{x}_{2[1]}^T)^T$ and $W := (W \cup W_{[1]}) \backslash U$. In step 4, we find a feasible extreme point $(\boldsymbol{x}_1^T, \boldsymbol{x}_2^T)^T$ in W whose $Z_1^{\Pi,AD}(\boldsymbol{x}_1, \boldsymbol{x}_2)$ is the least and let it be the next extreme point $(\boldsymbol{x}_{1[i+1]}^T, \boldsymbol{x}_{2[i+1]}^T)^T$. Then, let $i := i+1$ and return to step 2. By repeating the procedures, we can obtain an AD-P-Stackelberg solution

$$(\boldsymbol{x}_{1,ADP}^T, \boldsymbol{x}_{2,ADP}^T)^T = (0.00, 12.67, 10.30, 1.36, 0.00, 4.66)^T$$

where

$$Z_1^{\Pi,AD}(\boldsymbol{x}_{1,ADP}, \boldsymbol{x}_{2,ADP}) = 0.0163, \quad Z_2^{\Pi,AD}(\boldsymbol{x}_{1,ADP}, \boldsymbol{x}_{2,ADP}) = 0.0745.$$

On the other hand, using the computational method quite similar to that for obtaining an AD-P-Stackelberg solution, we can obtain an AD-N-Stackelberg solution

$$(\boldsymbol{x}_{1,ADN}^T, \boldsymbol{x}_{2,ADN}^T)^T = (4.92, 11.34, 0.47, 3.05, 0.00, 4.85)^T$$

where

$$Z_1^{N,AD}(\boldsymbol{x}_{1,ADN}, \boldsymbol{x}_{2,ADN}) = 0.0287, \quad Z_2^{N,AD}(\boldsymbol{x}_{1,ADN}, \boldsymbol{x}_{2,ADN}) = 0.0338.$$

5 Conclusions

In this paper, assuming noncooperative behavior of the decision makers, computational methods for obtaining Stackelberg solutions to two-level linear programming problems involving fuzzy random variable coefficients have been presented. Considering vague natures of decision makers' judgments, fuzzy goals were introduced into the formulated fuzzy random noncooperative two-level linear programming problems. On the basis of the possibility and necessity measure that each objective function fulfills the corresponding fuzzy goal, the fuzzy random two-level linear programming problems to minimize each objective function with fuzzy random variables were transformed into stochastic two-level programming problems to maximize the degree of possibility and necessity that each fuzzy goal is fulfilled. Through the use of absolute deviation minimization in stochastic programming, the transformed stochastic two-level programming problems were reduced to deterministic two-level programming problems. For the transformed problems, AD-P- and AD-N-Stackelberg solutions were introduced and computational methods were also presented. It is significant to note here that AD-P- and AD-N-Stackelberg solutions can be obtained through the combined use of the variable transformation method and the Kth best algorithm for two-level linear programming problems. To illustrate the proposed computational methods, a numerical example for obtaining AD-P- and AD-N-Stackelberg solutions was provided. Extensions to other stochastic programming models will be considered elsewhere. Further considerations from the view point of fuzzy random cooperative two-level programming will be required in the near future.

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