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# INVESTIGATION OF TIME-VARYING SYSTEMS BY AVERAGING METHOD IN CASE OF DOMINANT ROOTS ABSENCE

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## ABSTRACT

In this paper an attempt was made to apply the averaging method to investigation of time-varying systems for which the characteristic equations have no dominant roots. The preliminary representation of the transfer function in the form of a continuous fraction, then reduction of the system's order with a subsequent application of averaging, provides good results for the given class of systems.

# Investigation of Time-Varying Systems by Averaging Method in Case of Dominant Roots Absence

In the work [1] it was shown that the approximate solution second-order differential equation

$$\ddot{X} - 2\sigma\dot{x} + (\sigma^2 + \omega^2)x = 0 \quad , \tag{1}$$

describing a linear time varying system can be found in the following form

$$X(t) = a_o \exp \left(\int_0^t \sigma(t)dt \sin \left(\int_0^t \omega(t)dt + \phi\right)\right. \quad , \tag{2}$$

Where $a_o, \phi_o$ are constants of integration and $\sigma(t)$ and $\omega(t)$ monotonous time functions.

The solution of high order linear differential equations with time-varying coefficients

$$\sum_{k=o}^{n} a_k s^k x = 0 \quad , \tag{3}$$

where

$$s = \frac{d}{dt} \, , \; a_k = a_k(t) , \qquad (k = 0,1,\dots,n)$$

may be also expressed in the form (2) presuming that the dominant roots of the characteristic equation

$$F(s) = \sum_{k=o}^{n} a_k s^k = 0 \quad , \tag{4}$$

correspond to the second order equation with monotonous coefficients:

$$s^2 - 2\sigma s + \sigma^2 + \omega^2 = 0 \quad , \tag{5}$$

where

$$\sigma = \sigma(t), \ \omega = \omega(t) \quad .$$

This method achieved good results as applied to high-order system with the evident dominant roots [2]. However, in practice, there exist a variety of systems where it is very difficult to distinguish the dominant roots. For example, in the case when several poles have equal or close real parts. In this case it is reasonable to take advantage of the methodology which allows us to represent the transfer function as a continuous fraction and to reduce the system of differential equations to the approximate second-order system [3]. When the behaviour of the control system is described by space-state equations, the order of the system can be reduced by dismembering the transfer matrix with the consequent throwing away of some of its elements.

Further we shall consider a typical feedback control system with a parallel corrective element (Figure 1). The transfer function of such a system in closed state can be written

$$\Phi(s) = \frac{G_1 + F_1}{1 + (G_1 + F_1)H} \quad , \tag{6}$$

Having divided the numerator and the denominator by $(G_1 + F_1)$ one can express (6) in the form of a fraction:

$$\Phi(s) = \frac{1}{H + \dfrac{1}{F_1 + G_1}} \quad . \tag{7}$$

If $G_1$ itself is a function at a higher order then the fraction (7) can be transferred into the continuous fraction of the following type:

$$\Phi(s) = \cfrac{1}{H_1 + \cfrac{1}{\frac{H_2}{s} + \cfrac{1}{H_3 + \cfrac{1}{\frac{H_4}{s} + \cfrac{1}{\cdots\cdots}}}}} \tag{8}$$

This expression corresponds to the structural scheme shown in Figure 2, which is a combination of feedforward and feedback relationships. It is very important that the character of the response function in the system will first be determined by the element $H_1$ and then $H_2$. The influence of the elements $H_i$ $(i = 1,2,\ldots,2n)$ on the response function is decreasing with the increase of the indexes i, e.g. the more the meaning of i, the less the influence of this element on the response function. This is the basic statement used in the reduction process of the order of systems. Thus, in order to receive a simplified model of the system of an order m one should neglect in (8) all the values except $H_1,H_2,\ldots,H_{2m}$.

Specifically, the simplified second-order model of expression (8) can be written as follows:

$$\Phi(s) = \cfrac{1}{H_1 + \cfrac{1}{\frac{H_2}{s} + \cfrac{1}{H_3 + \cfrac{1}{\frac{H_4}{s}}}}} \tag{9}$$

Thus, in order to receive a simplified model of the system one has to determine only elements $H_1,H_2,\ldots,H_{2m}$ of the continuous fraction (8).

The transfer function of a linear time-invariable system can be written in the form

$$\Phi(s) = \frac{P(s)}{Q(s)} = \frac{b_n + \cdots b_2 s^{n-2} + b_1 s^{n-1}}{a_n + \cdots + a_2 s^{n-2} + a_1 s^{n-1} + s^n}\,, \tag{10}$$

Here P(s), Q(s) are polynomials.

The first step in forming (8) includes the division of polynomial $Q(s)$ on polynomial $P(s)$ and the continuation of this process until the first member of the quotient. So we have,

$$\frac{Q(s)}{P(s)} = H_1 + \frac{Q_1(s)}{P(s)} \quad ,$$

where $Q_1(s)$ is the remainder from the division.

The next step is the similar division of the polynomial $P(s)$ on $Q_1(s)$.

$$\frac{P(s)}{Q_1(s)} = \frac{H_2}{s} + \frac{P_1(s)}{Q_1(s)} \quad .$$

Performing the similar procedure with the relationship $\frac{P_1(s)}{Q_1(s)}$ and repeating it as many times as it is necessary we receive the partial fraction (8).

Using the state-space approach one can write on the basis of Figure 1 the following state equation:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \cdot \\ \cdot \\ \cdot \\ \dot{z}_n \end{bmatrix} = - \begin{bmatrix} H_2H_1 & H_4H_1 & H_6H_1 & \cdots & H_{2n}H_1 \\ H_2H_1 & H_4(H_1+H_3) & H_6(H_1+H_3) & \cdots & H_{2n}(H_1+H_3) \\ H_2H_1 & H_4(H_1+H_3) & H_6(H_1+H_3+H_5) & \cdots & H_{2n}(H_1+H_3+H_5) \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ H_2H_1 & H_4(H_1+H_3) & H_6(H_1+H_3+H_5) & \cdots & H_{2n}(H_1+H_3+\cdots+H_{2n-1}) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \cdot \\ \cdot \\ \cdot \\ z_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} \text{X input} \qquad (11)$$

In order to receive the second order simplified matrix one has to choose four elements in the upper left corner, neglecting the rest.

In this case we have

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = - \begin{bmatrix} H_2H_1 & H_2H_1 \\ H_2H_1 & H_4(H_1+H_3) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{X input} \qquad (12)$$

As an example we consider the time-varying control system corresponding to the structured scheme shown in Figure 3.

Using the method of "fixed coefficients" we receive the transfer function of the system in the closed state:

$$\Phi(s) = \frac{1}{ms^3 + (13.5m - 1)s^2 + (88m - 13.5)s + 30k - 88}$$

Here $m = m(t) = \frac{5 - t}{2}$; $K = K(t) = 21 - 3.7t$; are functions changing in time. We accept that X input = 0. Having calculated elements $H_1$, $H_2$, $H_3$, $H_4$ as functions of parameters m(t) and k(t) we can write the system's characteristic equation in the following form:

$$s^2 + [H_1H_2 + H_4(H_1 + H_3)]s + H_1H_2H_3H_4 = 0 \quad . \tag{13}$$

Comparing (5) with (13) we see that

$$\left.\begin{aligned} -2\sigma &= H_1H_2 + H_4(H_1 + H_3) \\ \sigma^2 + \omega^2 &= H_1H_2H_3H_4 \end{aligned}\right\} \quad . \tag{14}$$

Using (14) one can determine the values of real and imaginary parts of complex conjugated roots of the equation (13) on the observed time period from t = 0 to t = 2 seconds.

The character of changing $\sigma(t)$ and $\omega(t)$ is shown in Figure 4. The discrete table meanings can be approximated by the third order polynomial and finally give the approximate expression for the response function related to the output at the initial condition

$$X \text{ output } (0) = 0.105 \frac{1}{\text{sec}};$$

$$\dot{X} \text{ output } (0) = 0$$

in the following form:

$$\begin{aligned} X(t) = 0{,}129 \exp\ &(0{,}0466t^4 - 0{,}3254t^3 + 0{,}7108t^2 \\ &- 5{,}1945t) \sin\ (0{,}0112t^4 - 0{,}0515t^3 \\ &+ 0{,}1444t^2 + 2{,}8406t + 0.5) \quad . \end{aligned}$$

The estimation of the accuracy of the solution was made by comparing the results received by this method with the computer simulation results. In Figure 5 the curves show the response functions of the system related to the output coordinate X output. Curve 1 was obtained as a result of the digital computer simulation of the third-order system, taking into account the varying character of the parameters m(t) and k(t). The solution of the same system, received by averaging is related to curve 2. Curve 3 was obtained as a result of preliminary reduction of the system up to the second order

From the comparison of these curves it is quite evident that the method of preliminary reduction of the system with the consequent application of averaging results in the considerable increase in accuracy of the approximate solution.

## REFERENCES

[1] Huston, R.L., and L.F. Doty, Note on the Krylov-Bogolinbov Method Applied to Linear Differential Equations, *SIAM Journal of Applied Mathematics*, 14, 3 (1966).

[2] Vorontsov, N., *The Averaging Method Applied to the Investigation of Substantial Time-Varying Systems of a Higher Order*, PP-77-8, International Institute for Applied Systems Analysis, Laxenburg, Austria, 1977.

[3] Chen, C.F., L.S. Chieh, A Novel Approach to Linear Model Simplification, *International Journal of Control*, 6, 6 (1968), 561-570.