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THE FORCES OF URBANIZATION UNDER VARYING NATURAL INCREASE AND MIGRATION RATES

Jacques Ledent

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Preface

Roughly 1.6 billion people, 40 percent of the world's population, live in urban areas today. At the beginning of the last century, the urban population of the world totaled only 25 million. According to recent United Nations estimates, about 3.1 billion people, twice today's urban population, will be living in urban areas by the year 2000.

Rapid rates of urban demographic and economic growth increase the difficulties of providing a population with adequate supplies of food, energy, employment, social services and infrastructure. The investment needed just to maintain present standards in many rapidly urbanizing countries calls for a doubling or tripling of institutional plant within the next 25 years.

Scholars and policy-makers often disagree when it comes to evaluating the desirability of current rapid rates of urban growth in many parts of the globe. Some see this trend as fostering national processes of socioeconomic development, particularly in the poorer and rapidly urbanizing countries of the Third World; whereas others believe the consequences to be largely undesirable and argue that such urban growth should be slowed down.

Professor Nathan Keyfitz of Harvard University spent the month of May this year collaborating with HSS scholars in their research on migration, urbanization and development. During his stay, he formulated a model of the urbanization process that stimulated a number of us. In particular, Jacques Ledent responded by writing a series of three papers dealing with extensions of the Keyfitz model. This paper, the third of the series, focuses on the dynamics of urbanization under varying regimes of natural increase and migration.

A list of related papers in the Population, Resources and Growth Series appears at the end of this publication.

Andrei Rogers
Chairman
Human Settlements
and Services Area

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Abstract

This paper is the third and last of a series seeking to shed some light on the question of whether a nation's urban population grows mostly by rural-urban migration or by natural increase. Again, the discussion evolves around an analytical study of the Keyfitz model of urbanization (Keyfitz, 1978) and the Rogers components-of-change model (Rogers, 1968) applied to a rural-urban system. Here, in contrast to the preceding papers in which rates of natural increase and migration were constant, the present paper allows these rates to vary.

A larger part of the analysis is based on the Keyfitz model, shown earlier to be less meaningful than the alternative model but lending itself to an easier tractability when rates are allowed to vary. In particular, the Keyfitz model is used in an attempt to connect the variations of rural-urban (net) migration rates to economic changes through a simple scheme of wage differentials, later supplemented by the Todaro hypothesis.

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INTRODUCTION

In a recent paper, intended to examine whether cities grow mostly by inmigration or natural increase, Keyfitz (1978) proposed a two-region continuous model of population growth and distribution in which migration between the rural and urban regions was viewed as a net flow out of the rural region.

This model was criticized by Ledent (1978a), who pointed out the existence of an asymmetry between the two regions capable, in some circumstances, of leading to some undesirable long-term evolution. As an alternative, he suggested the use of a continuous version of the Rogers multiregional components-of-change model (Rogers 1968), a model whose dynamics he showed to be more suitable for studying the sources of urban growth.

Ledent implemented his suggestion in a further paper (Ledent 1978b), in which he examined the evolution of urbanization in an initially entirely rural population system. He conducted his analysis in a manner closely following Keyfitz's original analysis, thus allowing a simple and immediate comparison of the results yielded by both models.

With regard to their ability to shed some light on the sources of urbanization, the two alternative analyses presented a common drawback stemming from their reliance on constant rates of natural increase and migration. Thus, following Rogers, who argues that "...one of the fundamental aspects of the 'mobility revolution' experienced by nations undergoing the structural transformation from agrarian to industrial societies is an increasing rate of migration" (Rogers, 1978, p. 1), we reexamine here both models, in which we allow migration (as well as natural increase) rates to vary over time. An attempt is even made to connect the variations of migration rates to economic changes through a simple scheme of wage differentials, later supplemented by the Todaro hypothesis.

However, the inclusion of varying rates increases the complexity of the alternative models. Their analytical tractability requires the assumption of an identical rate of natural increase in both the rural and urban regions.

The first part of the paper deals with the Keyfitz model, examined under various evolutive patterns of natural increase and rural-urban migration. The second part focuses on the two-region Rogers model whose analysis is, however, less developed due to additional considerations which seriously hamper the mathematical tractability of the model in the case of varying rates.

I. ANALYSIS BASED ON THE KEYFITZ MODEL

Let us consider a population system divided into two regions - urban and rural - which exhibit the same positive rate of natural increase, $r(t)$. In addition, suppose that internal migration can be viewed as a net migration flow from rural to urban defined as a positive fraction $m(t)$ of the rural population.

The evolution of this system is entirely described by

$$dP_T(t) = r(t)P_T(t) \quad . \tag{1}$$

$$dP_r(t) = [r(t) - m(t)]P_r(t) \quad , \tag{2}$$

where

$P_T(t)$ is the total population at time t ,

$P_r(t)$ is the rural population at time t .

Once $P_T(t)$ and $P_r(t)$ have been obtained by integrating (1) and (2) respectively, the urban population $P_u(t)$ is simply given by:

$$P_u(t) = P_T(t) - P_r(t) \quad . \tag{3}$$

Suppose now that the initial population is entirely rural. Then the integration of (1) leads to:

$$P_T(t) = P(0)e^{\int_0^t r(u)du} \quad , \tag{4}$$

and the integration of (2) to:

$$P_r(t) = P(0)e^{\int_0^t [r(u) - m(u)]du} \quad . \tag{5}$$

It thus follows from (3) that

$$P_u(t) = P(0)e^{\int_0^t r(u)du}\left(1 - e^{-\int_0^t m(u)du}\right) , \tag{6}$$

so that the ratio of urban to rural population is

$$S(t) = e^{\int_0^t m(u)du} - 1 \quad . \tag{7}$$

Note that the first derivative of S(t)

$$\frac{dS(t)}{dt} = m(t)e^{\int_0^t m(u)du} \quad , \tag{8}$$

is always positive (since we assumed m(t) to be positive). Consequently, whatever the specification of the migration function m(t), S(t) appears to be an increasing function of time.*

Now, let us define the ratio R(t) of urban net migration to natural increase:

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*Differentiating (8), we obtain

$$\frac{d^2S(t)}{dt^2} = \left[m^2(t) + \frac{dm(t)}{dt}\right]e^{\int_0^t m(u)du} \quad , \tag{9}$$

a relationship which shows that, if m(t) is an increasing function, the curvature of S(t) is directed upward.

$$R(t) = \frac{m(t)P_r(t)}{r(t)P_u(t)} , \tag{10a}$$

which can be rewritten as

$$R(t) = \frac{m(t)}{r(t)S(t)} . \tag{10b}$$

The variations of this ratio depend on the sign of

$$\frac{dR(t)}{R(t)} = \frac{dm(t)}{m(t)} - \frac{dr(t)}{r(t)} - \frac{dS(t)}{S(t)} . \tag{11}$$

Indeed, the variations of R(t) which depend on the values of r(t) and m(t) are not necessarily monotonic. However, for a large choice of the functions r(t) and m(t), R(t) can decrease monotonically. Let us suppose first that

$$\frac{dr(t)}{dt} \geq 0 . \tag{12}$$

Clearly, we have from (11),

$$\frac{dR(t)}{R(t)} < \frac{dm(t)}{m(t)} - \frac{dS(t)}{S(t)} . \tag{13}$$

The right-hand side of this inequality has the sign of

$$y(t) = dm(t)\left[e^{\int_0^t m(u)du} - 1\right] - (m(t))^2 e^{\int_0^t m(u)du} . \tag{14}$$

Differentiating y(t) with respect to time, we obtain

$$dy(t) = d^2m(t)\left[e^{\int_0^t m(u)du} - 1\right] - m(t)[dm(t) + (m(t))^2]e^{\int_0^t m(u)du} . \tag{15}$$

Consequently, if the migration function m(t) is such that

$$\frac{dm(t)}{dt} > 0 \quad , \qquad \text{and} \qquad \frac{d^2m(t)}{dt} < 0 \quad , \tag{16}$$

$\frac{dy(t)}{dt}$ decreases monotonically. Since $\frac{dy(0)}{dt} = 0$, it follows that $\frac{dy(t)}{dt}$ is always negative, i.e., y(t) decreases monotonically. Finally, since y(0) is negative, y(t) only takes negative values and $\frac{dR(t)}{dt}$ is always negative. To summarize, if the natural increase rate r(t) and the migration function m(t) are such that (12) and (16) respectively hold, R(t) monotonically decreases, which indicates the larger importance taken by natural increase vis-à-vis migration as the urban region grows.

In what follows, we attempt to study the evolution of the above system according to various schemes of variations for r(t) and m(t). Of major interest are the variations of R(t), which permit one to determine the time at which natural increase starts exceeding inmigration in accounting for urban growth.

Case of Constant Rates

We can assume that r(t) and m(t) remain constant and equal to r and m respectively: this is the hypothesis made by Keyfitz (1978).

Under these conditions, the integration of (1) and (2) leads to:

$$P_T(t) = P(0)e^{rt} \quad , \tag{17}$$

for the national population at time t, and

$$P_r(t) = P(0)e^{(r-m)t} \quad , \tag{18}$$

for the rural. Then, the urban population is

$$P_u(t) = P(0)e^{rt}(1 - e^{-mt}) \quad , \tag{19}$$

and the ratio of urban to rural population is

$$S(t) = e^{mt} - 1 \quad , \tag{20}$$

which shows that S(t) monotonically increases from zero (for t = 0) to + ∞ (as t → + ∞).*

Substituting (20) in (10) yields

$$R(t) = \frac{m}{r(e^{mt} - 1)} \quad , \tag{21}$$

so that R(t) monotonically decreases from + ∞ (for t = 0) to zero (for t → + ∞). The role of migration, initially preponderant in accounting for the growth of the urban region, diminishes as time passes by so that natural increase is eventually the unique source of urban growth.

Keyfitz (1978) refers to the point in time T, at which point natural increase is equal to migration, as the cross-over point. By definition,

$$R(T) = 1 \quad , \tag{22}$$

and thus we have from (10)

$$S(T) = \frac{m}{r} \quad . \tag{23}$$

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*Note that the two populations are monotonic. However, if $P_u(t)$ increases and becomes infinitely positive, $P_r(t)$ presents similar variations only if $r > m$; it decreases, tending toward zero, if $r < m$.

An expression of T is then obtained by substituting (20) in (23):

$$T = \frac{1}{m} \ln(1 + \frac{m}{r}) \quad . \tag{24a}$$

Keyfitz observes therefore that:

> The more rapidly the population as a whole increases the sooner the cross-over, and more surprisingly, the larger the value of m, the fraction of the country side migrating, the sooner comes the day when natural increase exceeds migration as a factor (Keyfitz, 1978, p. 5).

The problem just examined is visualized in Figure 1, whose schema (i) displays the straight line with ordinate $\frac{m}{r}$ and, by contrast, the curve describing the variations of S(t).

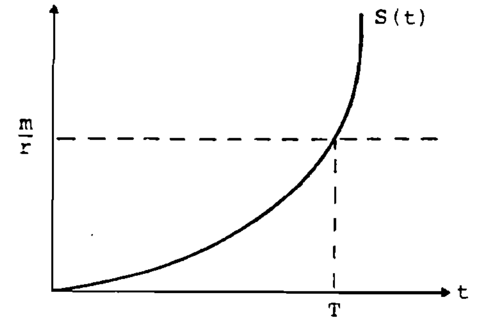
A particularly interesting observation is generated by a scenario involving the trajectory of the country that starts with an entirely rural population of 1 million, and is exposed to an unchanging rate of natural increase of $r = 0.03$ and a fixed fraction of migrating of $m = 0.02$. In this scenario, the role of natural increase, in accounting for urban growth, increases rapidly and exceeds that of migration after

$$T = \frac{1}{0.02} \ln 1.66 = 25.5 \text{ years} \quad .$$

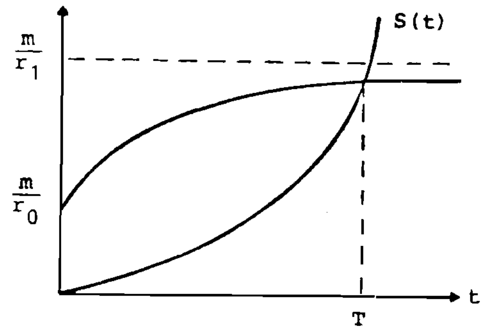
At this point, the ratio of urban to rural population is

$$S(T) = 0.66 \quad ,$$

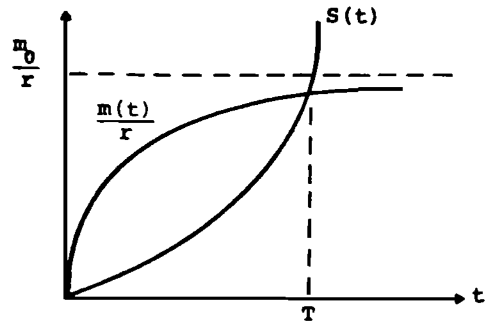
so that the part of the population which is urban is exactly 40 percent. (Table 1).



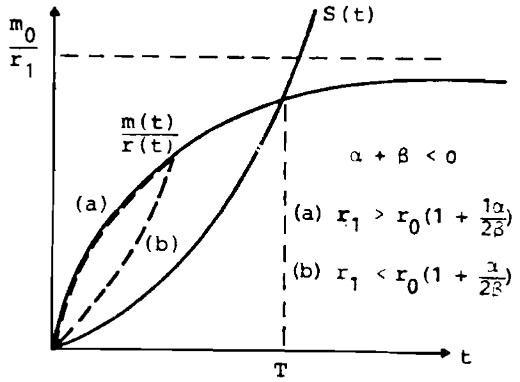
(i) $r(t) = r.$
$m(t) = m.$



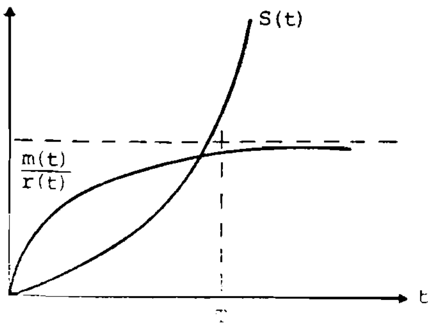
(ii) $r(t) = r_1 + (r_0 - r_1)e^{\beta t} (\beta < 0).$
$m(t) = m.$



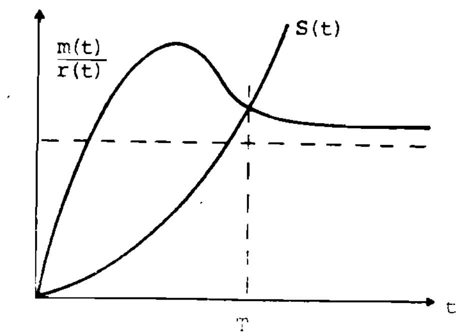
(iii) $r(t) = r.$
$m(t) = m_0(1 - e^{-\alpha t}) (\beta > 0).$



(iv) $r(t) = r_1 + (r_0 - r_1)e^{\beta t} (\beta < 0).$
$m(t) = m_0(1 - e^{-\alpha t}) (\alpha > 0).$



(v) $r(t) = r.$
$m(t)$ given by (68) $(g > \alpha)$.



(vi) $r(t) = r.$
$m(t)$ given by (68) $(g < \alpha)$.

Figure 1. The Keyfitz model: variations of $\frac{m(t)}{r(t)}$ and $S(t)$ contrasted according to various natural increase and migration patterns.

Table 1. Urbanization of an initially rural population of 1 million, with r = 0.03 and m = 0.02.

Year	Total	Rural	Urban	Percentage Urban	$\frac{m}{S(t)}$	R(t)
0	1	1	0	0	$+\infty$	$+\infty$
1	1.03	1.01	0.02	0.020	0.990	33.00
2	1.06	1.02	0.04	0.039	0.490	16.34
3	1.09	1.03	0.06	0.058	0.323	10.78
4	1.13	1.04	0.09	0.077	0.240	8.00
5	1.16	1.05	0.11	0.095	0.190	5.23
10	1.35	1.11	0.24	0.181	0.090	3.01
15	1.57	1.16	0.41	0.259	0.057	1.91
20	1.82	1.22	0.60	0.330	0.041	1.36
25	2.12	1.28	0.83	0.394	0.031	1.03
30	2.46	1.35	1.11	0.451	0.024	0.81
35	2.86	1.42	1.44	0.503	0.020	0.66
40	3.32	1.49	1.83	0.551	0.016	0.54
45	3.86	1.57	2.29	0.593	0.014	0.46
50	4.48	1.65	2.83	0.632	0.012	0.39
75	9.49	2.12	7.37	0.777	0.006	0.19
100	20.09	2.72	17.37	0.865	0.003	0.10

Suppose now that we observe an actual population system submitted to rates of natural increase and migration equal to r and m respectively, and presenting a ratio of urban to rural population equal to $\bar{s}$. From a result established by Ledent (1978a)*, it appears that this observed population system is

*He demonstrates that, when the rates of natural increase in the rural and urban areas are different (r and u respectively), a necessary condition for the observed population system to correspond to the subsequent state of a hypothetical population system, defined as above, is $r < u + m \frac{1 + \bar{s}}{\bar{s}}$. This condition reduces here to $0 < m \frac{1 + \bar{s}}{\bar{s}}$, a condition which always holds (independently of the specifications of r(t) and m(t)).

identical to a subsequent state of the above hypothetical population system.

The time t_D, at which this correspondence occurs, is simply observed as the root of $S(t) = \bar{s}$, which is unique due to the course of the evolution of $S(t)$. It is readily established that

$$t_D = \frac{1}{m} \ln(1 + \bar{s}) \quad . \tag{24b}$$

Consequently, if, around the observation period, the actual population exhibits the constant regimes of natural increase and migration defined by r and m, we can simply determine whether this system has already reached or will reach the cross-over point.

Letting T' denote the time span necessary to reach the cross-over point from the observation period, we have

$$T' = T - t_D \quad ,$$

and finally (Keyfitz, 1978),

$$T' = \frac{1}{m} \ln \frac{1 + \frac{m}{r}}{1 + \bar{s}} \quad . \tag{24c}$$

This relationship shows that the sign of T' depends on the relative values of $\frac{m}{r}$ and $\bar{s}$. In particular, if $\bar{s} > \frac{m}{r}$, the cross-over point appears to have been passed.

Suppose now that we observe an actual population in which $r = 0.03$ and $m = 0.02$, and the part of the population which is urban is 0.2 (i.e., $\bar{s} = 0.25$). Then, if this population is submitted to the constant regime of natural increase and migration defined by r and m respectively, it will reach the cross-over point at which natural increase and migration contribute equally to urban growth

$$T' = \frac{1}{0.02} \ln \frac{1.66}{1.25} = 14.4 \text{ years later} \quad .$$

Case of a Rate of Natural Increase Varying Exponentially

Let us suppose that

$$r(t) = r_1 + (r_0 - r_1)e^{\beta t} \quad , \tag{25}$$

in which r_0 and r_1 are both positive but such that, if r_0 is larger (smaller) than r_1, β is negative (positive).

The ensuing model can again be considered as evolving from an initial state in which the population is entirely rural. It is simple to establish that the total population is given by:

$$P_T(t) = P(0)e^{r_1 t + \frac{r_0 - r_1}{\beta}(e^{\beta t} - 1)} \quad , \tag{26}$$

and the rural population by

$$P_r(t) = P(0)e^{(r_1 - m)t + \frac{r_0 - r_1}{\beta}(e^{\beta t} - 1)} \quad . \tag{27}$$

It follows that the urban population is:

$$P_u(t) = P(0)e^{r_1 t + \frac{r_0 - r_1}{\beta}(e^{\beta t} - 1)}(1 - e^{-mt}) \quad , \tag{28}$$

so that the ratio of urban to rural population is again given by:

$$S(t) = e^{mt} - 1 \quad . \tag{29}$$*

As expected, since the rate of natural increase is the same in both regions, a change in r has no impact on the distribution of population, which depends solely on m.

Substituting (25) and (29) in (21) yields:

$$R(t) = \frac{m}{(r_1 + (r_0 - r_1)e^{\beta t})(e^{mt} - 1)} \quad . \tag{30}$$

Since m(t) is here a constant, (11) reduces to:

$$\frac{dR(t)}{R(t)} = - \frac{dr(t)}{r(t)} - \frac{dS(t)}{S(t)} \quad . \tag{31}$$

Clearly, if $\beta > 0$, $\frac{dr(t)}{r(t)}$ is positive and thus dR(t) is negative. By contrast, if $\beta < 0$, the sign of dR(t) cannot be derived without expliciting (30). In such circumstances dR(t) has the same sign as the following expression:

$$E = - me^{mt}\left[r_1 + (r_0 - r_1)e^{\beta t}\right] - \beta(r_0 - r_1)e^{\beta t}(e^{mt} - 1) \quad , \tag{32a}$$

in which $r_0 - r_1 > 0$.

Two subcases must be considered. Let us first suppose that $\beta + m > 0$ and let us rearrange E as

$$E = - mr_1e^{mt} - (\beta + m)(r_0 - r_1)e^{(\beta+m)t} - (-\beta)(r_0 - r_1)e^{\beta t} \quad . \tag{32b}$$

*The urban population always becomes infinitely positive. The rural population tends toward + ∞ (if $\beta > 0$ or if $\beta < 0$ and $r_1 > m$), and toward zero (if $\beta < 0$ and $r_1 < m$).

Then, it follows that E is negative since all the terms of (32b) which are positive have a negative sign.

Now, if we suppose $\beta + m < o$, we may rearrange E as

$$E = - mr_1 e^{mt} - (-\beta)(r_0 - r_1)(e^{\beta t} - e^{(\beta+m)t})$$

$$- m(r_0 - r_1)e^{(\beta+m)t} \quad . \qquad (32c)$$

Since m and β are such that $\beta < \beta + m < 0$, we clearly have $e^{\beta t} > e^{(\beta+m)t}$. Consequently, all the terms of (32c) which are positive have a negative sign and thus E is negative.

Consequently, whatever r(t) increases or decreases exponentially, R(t) monotonically decreases from $+ \infty$ (for $t = 0$) to zero (as $t \to + \infty$): again, the importance of migration in accounting for urban growth monotonically decreases to vanish in the long run. As in the case of a constant natural increase rate, there exists a cross-over point T characterized by equal natural increase and migration in the urban region, i.e., such that

$$S(t) = \frac{m}{r(t)} \quad , \qquad (33)$$

which, after substituting (25) and (29), defines T implicitly

$$e^{mT} - 1 = \frac{m}{r_1 + (r_0 - r_1)e^{\beta T}} \quad . \qquad (34)*$$

The above problem is visualized in Figure 1, whose schema (ii) shows the variations of $\frac{m}{r(t)}$ (for $\beta < 0$) and S(t).

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*It is readily established that, if β is negative (positive), this T-value is higher (smaller) than the T-value that would be obtained if r(t) would keep the constant value r_0.

In Table 2, we display the results of a scenario corresponding to the case of a country which has an $m = 0.02$ rural-urban net migration rate and exhibits an exponential decrease of $r(t)$ with parameters $r_0 = 0.045$, $r_1 = 0.01$, and $\beta = -0.05$.

Table 2. Urbanization of an initially rural population of 1 million with $m = 0.02$ and $r(t) = 0.01 + 0.035\, e^{-0.05t}$.

Year	Total	Rural	Urban	Percentage Urban	$\frac{m}{S(t)}$	r(t)	R(t)
0	1	1	0	0	$+\infty$	0.045	$+\infty$
1	1.05	1.02	0.02	0.020	0.990	0.043	22.90
2	1.09	1.05	0.04	0.039	0.490	0.042	11.78
3	1.14	1.07	0.07	0.058	0.323	0.040	8.07
4	1.18	1.09	0.09	0.077	0.240	0.039	6.22
5	1.23	1.11	0.12	0.095	0.190	0.037	5.11
10	1.45	1.19	0.26	0.181	0.090	0.031	2.90
15	1.68	1.24	0.40	0.259	0.057	0.027	2.16
20	1.90	1.27	0.63	0.330	0.041	0.023	1.72
25	2.11	1.28	0.83	0.394	0.031	0.020	1.54
30	2.32	1.27	1.05	0.451	0.024	0.018	1.37
35	2.53	1.26	1.27	0.503	0.020	0.016	1.23
40	2.73	1.23	1.50	0.551	0.016	0.015	1.11
45	2.93	1.19	1.74	0.593	0.014	0.014	1.00
50	3.13	1.15	1.98	0.632	0.012	0.013	0.89
75	4.19	0.93	3.25	0.777	0.006	0.011	0.53
100	5.44	0.74	4.71	0.865	0.003	0.010	0.31

Note that, in contrast to the scenario of Table 1, the rural population reaches a maximum at about the 25th year and then tends to vanish (since $m > r_1$).

Now, returning to our actual population system ($r = 0.03$, $m = 0.02$, $\bar{s} = 0.25$), we would like to know when it reached or will reach the cross-over point if, around the observation period, the natural increase rate follows the pattern embodied in (25) with $r_1 = 0.01$ and $\beta = -0.05$.

Indeed, the answer to this problem requires the knowledge of the value of r_0 which permits one to build the hypothetical population submitted to the natural increase pattern just described and which, at some point in time, is characterized by a natural increase equal to r and a regional distribution corresponding to $\bar{s}$. The time t_D at which the hypothetical population presents characteristics identical to those of the actual population is again given by $t_D = \frac{1}{m} \ln(1 + \bar{s})$. Since $r(t_D) = r_1 + (r_0 - r_1)e^{\beta t_D} = r$, we have that

$$r_0 = r_1 + (r - r_1)(1 + \bar{s})^{-\frac{\beta}{m}} \quad . \tag{35}$$

It follows that $r_0 = 0.045$, which is precisely the value we chose when generating the scenario corresponding to Table 2. Again, the hypothetical population presents the same characteristics as the observed population for $t_D = \frac{1}{0.02} \ln 1.25 = 11.2$ years. It appears that the cross-over is reached for T approximately equal to 45.1 years. Then the time span necessary to reach the cross-over is T' = 33.9 years from the observed period (against T' = 14.4 years in the case of r(t) remaining equal to r). Thus, the exponential decrease of r(t) delays the cross-over point by 19.5 years. Indeed, the delay in the occurrence of the cross-over point causes natural increase to take over in a more urbanized country. At the cross-over, the ratio S(t) of urban to rural population is equal to 1.46 (versus 0.66 in the case of a constant rate of natural increase): this corresponds to an increase in the part of the population which is urban from 40 percent to over 59 percent.

Case of a Rural-Urban Migration Rate Increasing Exponentially

We again assume r(t) to be constant but suppose that m(t) increases exponentially:

$$m(t) = m_0(1 - e^{-\alpha t}) \quad , \tag{36}$$

in which m_0 and α are positive.*

The ensuing model can again be considered as evolving from an initial state in which the population is entirely rural.

The total population is again given by (17), whereas the rural population is obtained by integrating

$$\frac{dP_r(t)}{P_r(t)} = r - m_0(1 - e^{-\alpha t}) \quad , \tag{37}$$

which leads to:

$$P_r(t) = P(0)e^{(r - m_0)t + \frac{m_0}{\alpha}(1 - e^{-\alpha t})} \quad . \tag{38}$$

It follows that the urban population is given by

$$P_u(t) = P(0)e^{rt}\left[1 - e^{-m_0 t + \frac{m_0}{\alpha}(1 - e^{-\alpha t})}\right] \quad , \tag{39}$$

and the ratio S(t) of urban to rural population is

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*Two remarks are in order here: first of all, the case of a migration rate decreasing exponentially could be treated in a similar way using

$$m(t) = m_0(1 + e^{-\alpha t}) \quad ,$$

in which m_0 and α are again positive.

Secondly, note that, unlike the varying migration rate considered by Rogers (1978) which becomes infinitely positive as t increases, the present rate tends toward a limit m_0.

$$S(t) = e^{m_0 t - \frac{m_0}{\alpha}(1 - e^{-\alpha t})} - 1 \quad . \tag{40}$$

Thus S(t) monotonically increases from zero (for t = 0) to + ∞ (as t → + ∞).*

Substituting (36) and (40) in (25) yields

$$R(t) = \frac{m_0 (1 - e^{-\alpha t})}{r\left[\left(e^{m_0 t - \frac{m_0}{\alpha}(1 - e^{-\alpha t})}\right) - 1\right]} \quad . \tag{41}$$

What are then the variations displayed by R(t)? As t is close to zero, the numerator and denominator of R(t) are equivalent to $m_0\alpha t$ and $\frac{rm_0\alpha t^2}{2}$ respectively. Consequently, R(t) is infinitely positive.

On the other hand, as t increases infinitely, the denominator of (41) also increases infinitely and R(t) tends toward zero.

Are the variations of R(t) monotonic between the above extreme values? Differentiating m(t) with respect to time, we obtain

$$\frac{dm(t)}{dt} = m_0 \alpha e^{-\alpha t} \quad , \tag{42}$$

whose first derivative is

$$\frac{d^2 m(t)}{dt^2} = - m_0 \alpha^2 e^{-\alpha t} \quad . \tag{43}$$

*$P_u(t)$ monotonically increases toward + ∞ while $P_r(t)$ can either become infinitely positive if $r > m_0$ or vanish if $r < m_0$.

The migration function $m(t)$ is then such that (16) holds. It immediately follows from one of the properties established when dealing with the generalities of the model, that $R(t)$ monotonically decreases from $+\infty$ (for $t = 0$) to zero (as $t \to +\infty$).

As in the preceding cases, there exists a cross-over point T characterized by equal natural increase and migration in the urban region, i.e., such that

$$S(t) = \frac{m(t)}{r} \quad , \tag{44}$$

which, after substituting (36) and (40), defines T implicitly:

$$e^{m_0 T - \frac{m_0}{\alpha}(1 - e^{-\alpha t})} - 1 = \frac{m_0}{r}(1 - e^{-\alpha t}) \quad . \tag{45)*$$

The occurrence of the cross-over is visualized in Figure 1, whose schema (iii) indicates the variations of $\frac{m(t)}{r}$ and $S(t)$.

In Table 3, we display the results of a scenario corresponding to the case of a country in which the rate of natural increase is $r = 0.03$ and the rural-urban net migration rate is given by (36) where $m_0 = 0.12$ and $\alpha = 0.0084$.

Observe again that the rural population reaches a maximum at about the 34th year before decreasing toward zero (since $m_0 > r$).

We now return to our actual population system ($r = 0.03$, $m = 0.02$, $\bar{s} = 0.25$) and ask ourselves when the cross-over point occurred or will occur if the rural-urban migration rate follows, around the observation period, the pattern embodied in (36) with $m_0 = 0.12$.

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*It can easily be established that this T-value is smaller than the T-value that would be obtained if $m(t)$ would have the constant value m_0.

Table 3. Urbanization of an initially rural population of 1 million with $r = 0.03$ and $m(t) = 0.12\ (1 - e^{-0.0084t})$.

Year	Total	Rural	Urban	Percentage Urban	m(t)	$\frac{(t)}{S(t)}$	R(t)
0	1	1	0	0	0	$+\infty$	$+\infty$
1	1.03	1.03	0.00	0.001	0.001	1.996	66.54
2	1.06	1.06	0.00	0.002	0.002	0.996	33.20
3	1.09	1.09	0.00	0.005	0.003	0.662	22.07
4	1.13	1.12	0.01	0.008	0.004	0.495	16.51
5	1.16	1.15	0.01	0.012	0.005	0.395	13.16
10	1.35	1.29	0.06	0.048	0.010	0.192	6.41
15	1.57	1.41	0.16	0.103	0.014	0.124	4.12
20	1.82	1.51	0.31	0.174	0.019	0.088	2.94
25	2.12	1.58	0.54	0.255	0.023	0.067	2.22
30	2.46	1.62	0.84	0.342	0.027	0.052	1.72
35	2.86	1.63	1.23	0.430	0.031	0.041	1.35
40	3.32	1.61	1.71	0.516	0.034	0.032	1.07
45	3.86	1.56	2.30	0.596	0.038	0.026	0.87
50	4.48	1.49	2.99	0.668	0.041	0.021	0.68
75	9.49	0.93	8.56	0.902	0.056	0.006	0.20
100	20.09	0.41	19.67	0.980	0.068	0.001	0.05

To answer this question, we must know the value of α that permits us to build the hypothetical population (a) submitted to a constant rate r of natural increase and to the migration scheme just described, and (b) presenting a state characterized by a rural-urban migration rate equal to m_0 and a regional distribution corresponding to $\bar{s}$.

The time t_D at which the hypothetical population presents characteristics identical to those of the actual population is such that

$$m_0(t_D) = m_0\left(1 - e^{-\alpha t_D}\right) = m \quad , \tag{46}$$

and

$$e^{m_0 t_D} - \frac{m_0}{\alpha}\left(1 - e^{-\alpha t_D}\right) - 1 = \bar{s} \quad . \tag{47}$$

Eliminating t_D between these two equations yields:

$$\alpha = - \frac{m_0 \ln\left(1 - \frac{m}{m_0}\right) + m}{\ln(1 + \bar{s})} \quad . \tag{48}$$

Consequently, $\alpha = 0.0084$, which is precisely the value we chose when generating the scenario displayed in Table 3. It follows that the hypothetical population presents the same characteristics as the observed population for

$$t_D = - \frac{1}{\alpha} \ln\left(1 - \frac{m}{m_0}\right) = 21.7 \text{ years} \quad .$$

It appears that the cross-over is reached for T approximately equal to 41.6 years. Then, the time span necessary to reach the cross-over is T' = 19.9 years from the observed period (against T' = 14.4 years in the case of r(t) and m(t) remaining constant). Thus, as expected, the exponential increase in m(t) delays the cross-over point by 5.5 years.

Again, the delay in the occurrence of the cross-over causes natural increase to take over migration in a more urbanized country. At the cross-over, the ratio S(t) of urban to rural population is equal to 1.18 (versus 0.66 in the case of constant rates): this corresponds to an increase in the part of the whole population which is urban from 40 percent to roughly 54 percent.

Case of Varying Natural Increase and Migration Rates

We may now combine the assumptions of the two preceding cases so as to have an exponentially decreasing rate of natural increase (given by (25)) and an exponentially increasing rate of rural-urban migration (given by (36)).

Again, starting from an entirely rural population, the total population at time t is given by (26). The rural population is now obtained by integrating

$$\frac{dP_r(t)}{P_r(t)} = r_1 + (r_0 - r_1)e^{\beta t} - m_0(1 - e^{-\alpha t}) \quad , \tag{49}$$

in which $m_0 > 0$, $0 < r_1 < r_0$, $\alpha > 0$ and $\beta < 0$, which leads to:

$$P_r(t) = P(0)e^{(r_1 - m_0)t + \frac{r_0 - r_1}{\beta}(e^{\beta t} - 1) + \frac{m_0}{\alpha}(1 - e^{-\beta t})} \quad . \tag{50}$$

The urban population is now obtained from

$$P_u(t) = P(0)e^{r_1 t + \frac{r_0 - r_1}{\beta}(e^{\beta t} - 1)}\left[1 - e^{-m_0 t + \frac{m_0}{\alpha}(1 - e^{-\alpha t})}\right] . \tag{51}$$

Consequently, as expected, since r(t) does not affect it, the ratio S(t) of urban to rural population is again given by (40).*

Thus, the ratio R(t) of urban migration to natural increase can be expressed as

$$R(t) = \frac{m_0(1 - e^{-\alpha t})}{\left[r_1 + (r_0 - r_1)e^{\beta t}\right]\left[e^{m_0 t - \frac{m_0}{\alpha}(1 - e^{-\alpha t})} - 1\right]} . \tag{52}$$

---

*$P_u(t)$ monotonically increases toward $+\infty$ while $P_r(t)$ can either become infinitely positive if $r_1 > m_0$ or vanish if $r_1 < m_0$. But, S(t) monotonically increases from zero (for t = 0) to $+\infty$ (for $t \to +\infty$).

What are then the variations of R(t)? Since t is close to zero, the numerator and denominator of R(t) are equivalent to $m_0\alpha t$ and $\frac{r_0 m_0 \alpha t^2}{2}$ respectively: R(0) is then infinitely positive. By contrast, as t increases infinitely, the denominator of (52) also increases infinitely and R(t) tends toward zero. Does then R(t) decrease monotonically from $+\infty$ to zero as $t \to +\infty$?

Unfortunately, the complexity of (52) does not permit us to establish such a property. Nevertheless, we can establish the variations of $\frac{m(t)}{r(t)}$ and S(t) which are pictured on schema (iv) of Figure 1: both functions monotonically increase since their first derivatives are positive. As we have shown, S(t) has a negative first derivative and its curvature is directed upward. The curvature of $\frac{m(t)}{r(t)}$, however, depends on the parameter values. Let us assume that $\alpha + \beta < 0$. Then we can show in Appendix 1 that if $r_1 > r_0(1 + \frac{\alpha}{2\beta})$, the second derivative of $\frac{m(t)}{r(t)}$ is always negative and therefore its curvature is directed downward. In the alternative case, $r_1 < r_0(1 + \frac{\alpha}{2\beta})$, the curvature of $\frac{m(t)}{r(t)}$ is first directed upward and then downward.

In any case, if $\alpha + \beta < 0$, as suggested by schema (iv) of Figure 1, the curve $\frac{m(t)}{r(t)}$ lies above S(t) for small values of t (since $R(t) = \frac{m(t)}{r(t)}/S(t)$ is infinite for t = 0) and therefore, whatever the parameter values, the two curves $\frac{m(t)}{r(t)}$ and S(t) can and do intersect only once.

The resulting cross-over point T is defined by

$$\left[r_1 + (r_0 - r_1)e^{\beta T}\right]\left[e^{m_0 T - \frac{m_0}{\alpha}(1 - e^{-\alpha T})} - 1\right] = m_0(1 - e^{-\alpha T}) \quad . \tag{53}$$

In Table 4, we display the figures of a scenario corresponding to the case of a country submitted to

$$r(t) = 0.01 + 0.059e^{-0.05t} \quad , \tag{54}$$

and

$$m(t) = 0.12(1 - e^{-0.0084t}) \quad . \tag{55}$$

Returning to our actual population system (r = 0.03, m = 0.02, $\bar{s}$ = 0.25), we ask ourselves when the cross-over would occur if r(t) would decrease exponentially toward r_1 (with β = -0.05), and m(t) increase exponentially toward m_0.

Table 4. Urbanization of an initially rural population of 1 million submitted to (54) and (55).

Year	Total	Rural	Urban	Percentage Urban	m(t)	$\frac{m(t)}{S(t)}$	r(t)	R(t)
0	1	1	0	0	0	0	0.069	+ ∞
1	1.07	1.07	0.00	0.001	0.001	0.996	0.066	30.16
2	1.14	1.14	0.00	0.002	0.002	0.996	0.063	15.70
3	1.21	1.21	0.01	0.005	0.003	0.662	0.061	10.89
4	1.29	1.28	0.01	0.008	0.004	0.495	0.058	8.49
5	1.37	1.35	0.02	0.012	0.005	0.395	0.056	7.05
10	1.76	1.67	0.08	0.048	0.010	0.192	0.046	4.20
15	2.17	1.94	0.22	0.103	0.014	0.124	0.038	3.26
20	2.58	2.13	0.45	0.174	0.019	0.088	0.032	2.78
25	2.98	2.22	0.76	0.255	0.023	0.067	0.027	2.47
30	3.38	2.22	1.16	0.342	0.027	0.052	0.023	2.26
35	3.77	2.15	1.62	0.430	0.031	0.041	0.020	2.00
40	4.14	2.01	2.14	0.516	0.034	0.032	0.018	1.79
45	4.51	1.83	2.69	0.600	0.038	0.026	0.016	1.58
50	4.88	1.62	3.26	0.668	0.041	0.021	0.015	1.38
60	5.60	1.19	3.83	0.787	0.048	0.020	0.013	0.99
65	5.96	0.99	4.97	0.834	0.051	0.010	0.012	0.82
70	6.33	0.81	5.52	0.872	0.053	0.008	0.012	0.67
75	6.71	0.65	6.05	0.902	0.056	0.006	0.011	0.53
100	8.79	0.18	8.61	0.980	0.068	0.001	0.010	0.14

One can establish that the parameter α of m(t) should be identical to that of the preceding case and that r_0 should be taken as

$$r_0 = r_1 + (r - r_1)\left(1 - \frac{m}{m_0}\right)^{\frac{\beta}{\alpha}} . \quad (56)$$

Consequently, $r_0 = 0.069$, which is precisely the value we chose when generating the scenario displayed in Table 4.

Again, the hypothetical population presents the same characteristics as the observed population for $t_D = 21.7$ years. It appears that the cross-over is reached for T approximately equal to 59.8 years. Then, the time span necessary to reach the cross-over is T' = 38.1 years from the observed period (against T' = 14.4 years in the case of r(t) and m(t) remaining constant). In other words, the exponential decrease of the rate of natural increase and the exponential increase of the rural-urban migration rate delay the cross-over point by as many as 23.7 years. Indeed, this delay causes natural increase to take over migration in a more urbanized country: S(t) = 3.66 versus 0.66, which corresponds to an increase of the part of the population which is urban from 40 percent to roughly 78 percent.

Exploration of the Todaro Hypothesis

The migration function (36) has been put down above without any justification. However, can it be given any economic interpretation?

For example, let us consider that the rural per capita income increases exponentially:

$$w_r(t) = w_r(0)e^{\alpha t} . \quad (57)$$

If, in addition, we assume that the urban per capita income increases faster than the rural per capita income but in such a way that the ratio of the urban to rural growth rates in per capita income decreases to tend ultimately toward one, we have

$$w_u(t) = w_u(0)(e^{\alpha t} - 1) \quad . \tag{58}$$

Assume further that the rural-urban migration rate varies in direct proportion to the ratio of the per capita incomes in the two regions. Then we have:

$$m(t) = \gamma \frac{w_u(t)}{w_r(t)} = \gamma \frac{w_r(0)}{w_u(0)} \frac{e^{\beta t} - 1}{e^{\beta t}} \quad , \tag{59}$$

or, by substituting m_0 for $\frac{\gamma w_r(0)}{w_u(0)}$,

$$m(t) = m_0(1 - e^{-\beta t}) \quad ,$$

which is precisely the migration function (36).

Following Todaro (1969), Rogers (1978) argues that the notion of rural-urban migration incorporated in (59) is insufficient and should include the probability of getting a job at the region of destination. Then, we could substitute for (59) the following

$$m(t) = m_0\, p(t) \quad , \tag{60}$$

in which the probability $p(t)$ that a rural-urban migrant will find a job in an urban center is defined (Todaro 1976) by

$$p(t) = g(t) \frac{1 - u(t)}{u(t)} \quad , \tag{61}$$

where g(t) is the net rate of growth of modern sector employment opportunities in the urban region and u(t) is the rate of unemployment that prevails in this region. Here we assume g(t) to be constant, i.e.,

$$g(t) = g \quad . \tag{62}$$

Further, we assume that there is a bias in the growth of population, $P_u(t)$, and employment opportunities, $E_u(t)$ in the urban region, i.e.,

$$\frac{E_u(t)}{P_u(t)} = \frac{xe^{\delta t}}{-1 + ye^{\delta t}} \quad ; \tag{63)*$$

Then, if the labor force participation is constant, the unemployment rate

$$u = 1 - \frac{E_u(t)}{\rho P_u(t)} \quad . \tag{64}$$

can be expressed as

$$u(t) = \frac{(\rho y - x)e^{\delta t} - 1}{\rho(ye^{\delta t} - 1)} \quad . \tag{65}$$

Differentiating (65) with respect to time, we obtain:

$$\frac{du(t)}{u(t)} = \frac{1}{\rho} \frac{[y\delta(1 - e^{\delta t}) + x\delta]e^{\delta t}}{(ye^{\delta t} - 1)^2} \quad . \tag{66}$$

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*The ratio of employment opportunities to population is thus hypothesized to decrease monotonically from $\frac{x}{-1 + y}$ (y > 1) (for t = 0), to $\frac{x}{y}$.

Thus, u(t) monotonically increases from

$$u_0 = \frac{\rho y - (1 + x)}{\rho y - 1} , \qquad (67a)$$

to:

$$u_1 = \frac{\rho y - x}{\rho y} . \qquad (67b)$$

Substituting (62) and (65) into (61) leads to:

$$p(t) = g \frac{\dfrac{xe^{\delta t}}{\rho(ye^{\delta t} - 1)}}{\dfrac{(\rho y - x)e^{\delta t} - 1}{\rho(ye^{\delta t} - 1)}} = \frac{gxe^{\delta t}}{(\rho y - x)e^{\delta t} - 1} . \qquad (68)$$

Using (67b), this can be rewritten as

$$p(t) = \frac{g\rho y(1 - u_1)e^{\delta t}}{u_1 \rho y e^{\delta t} - 1} . \qquad (69)$$

To eliminate the unknown ρ, we further assume that $p(0) = 1$. We obtain

$$p(t) = \frac{g(1 - u_1)e^{\delta t}}{(1 - u_1)(1 + g) - 1 + u_1 e^{\delta t}} , \qquad (70)$$

and we thus have, after substituting (70) into (60)

$$m(t) = m_0(1 - u_1)(1 - e^{-\alpha t}) \frac{e^{\delta t}}{(1 - u_1)(1 + g) - 1 + u_1 e^{\delta t}} .$$

(71)

What are the variations of this migration function over time? Differentiating m(t) with respect to time, we obtain

$$\frac{dm(t)}{m(t)} = \frac{\alpha e^{-\alpha t}}{1 - e^{-\alpha t}} + \delta \frac{u_1 \delta e^{\delta t}}{(1 - u_1)(1 + g) - 1 + u_1 e^{\delta t}} \quad . \tag{72}$$

This expression has the sign of

$$z(t) = \alpha u_1 + (\alpha - \delta)Ae^{-\delta t} + \delta Ae^{(\alpha-\delta)t} \quad , \tag{73}$$

in which $A = (1 - u_1)(1 + g) - 1$ is negative.

Differentiating z(t) with respect to time, we have that

$$\frac{dz(t)}{dt} = \delta(\alpha - \delta)Ae^{-\delta t}(e^{\alpha t} - 1) \quad . \tag{74}$$

It follows that $\frac{dz(t)}{dt}$ has the sign $(\delta - \alpha)$. Two subcases must then be examined:

(a) $\delta > \alpha$

$\frac{dz(t)}{dt}$ is positive and thus z monotonically increases from $\alpha(u_1 + A) = \alpha g(1 - u_1)$, a positive value. Consequently, z(t) is always positive and m(t) monotonically increases from zero (for $t = 0$) to $m_0 \frac{1 - u_1}{u_1}$.

(b) $\delta < \alpha$

$\frac{dz(t)}{dt}$ is negative and thus z monotonically decreases from $\alpha g(1 - u_1)$, which is positive, $\beta - \infty$. In other words, there exists a value t_0 of t such that z(t) is positive for $t < t_0$ and negative for $t > t_0$. Consequently, m(t) increases for t varying from $t = 0$ to $t = t_0$ and decreases thereafter to reach the limit $m_0 \frac{1 - u_1}{u_1}$.

The variations of the migration function in both these cases are visualized in Figure 2.

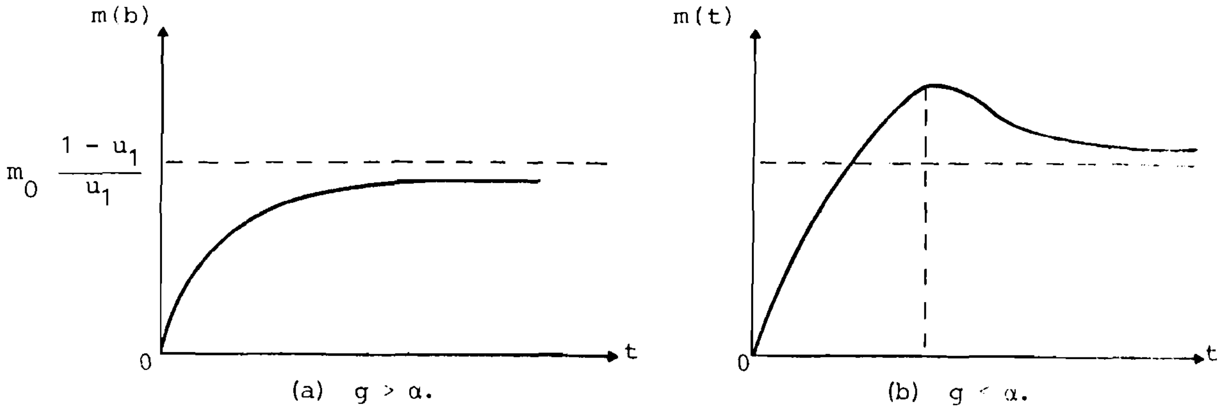

Figure 2. Exploration of the Todaro hypothesis: the variations of m(t).

Now, let us consider a population system, initially entirely rural, in which both regions are submitted to the same rate of natural increase r. In addition, the economic conditions are supposed to induce a rural-urban net migration rate m(t) given by (71).

Then, the natural population at time t is given by (17). Similar expressions relating to the rural and urban populations cannot be obtained here due to the difficulty of integrating equation (2).

In any case, we know from (8) that S(t) monotonically increases from zero (for t = 0); moreover, since m(t) tends toward a limit $m_0 \frac{1 - u_1}{u_1}$, S(t) becomes infinitely positive as $t \to +\infty$. Then, what about the variations of the ratio R(t) of migration to natural increase in the urban region?

Let us recall that the ratio R(t) of urban migration to natural increase is equal to the ratio of $\frac{m(t)}{r}$ to S(t). Again, it is simple to show that this ratio is initially infinite and tends toward zero as $t \to \infty$, so that there exists at least one point at which $R(t) = 1$, i.e., natural increase is equal to in-migration in the urban region (see in Figure 1, schemata (v) and (vi), corresponding to the two subcases distinguished earlier).

In contrast to the previous cases of varying natural increase and migration rates, the present evolution of $m(t)$ does not lend itself to an easy derivation of the sign of its second derivative and therefore does not allow one to conclude whether $R(t)$ monotonically decreases or not. It is expected that, in reality, $R(t)$ follows such a pattern and thus that there exists a unique cross-over point after which the growth of the urban region is more and more the fact of natural increase.

Because the analytical integration of $m(t)$ is not so straightforward, we cannot generate here a simple illustration of the model as in the previous cases. Fortunately, we can resort to using the discrete equivalent of the above model which, in fact, leads to very similar results. (Compare the results of Table 3 (stemming from the continuous formulation) with those of the table in Appendix 2 (relating to the discrete formulation) in the case of a constant rate of natural increase ($r = 0.03$) and a migration rate increasing exponentially).

Indeed the inclusion of the Todaro hypothesis leads to a reduction in the rural-urban net migration rate whose effect, for a given rate of natural increase, is to hasten the occurrence of the cross-over point.

However, the reduction in the pace of the migration rate increase does not seem to affect so much the trajectory of $R(t)$ in the useful period of the model (see Appendix 2), even in case the parameters of the model lead to a turning point in the variations of $m(t)$ (i.e., when $\delta < \alpha$). Therefore, the occurrence of the cross-over is only slightly hastened.

It follows that the introduction of the Todaro hypothesis does not radically modify the results of the case in which the rural-urban migration rate is a simple function of the wage differentials between the two regions.

II. ANALYSIS BASED ON THE ROGERS TWO-REGION MODEL

As an alternative to the model examined above, we can use a continuous two-region version of the interregional components-of-change model developed by Rogers (1968). In this model, a more symmetric treatment of the migration flows between the rural and urban regions is posited: gross migration flows out of the two regions rather than the consolidated net flow are considered.

Again let r(t) denote the positive rate of natural increase common to the two regions and let $o_u(t)$ and $o_r(t)$ denote the migration rates out of the urban and rural regions respectively.

Equation (1) remains valid so that the total population is still given by

$$P_T(t) = P(0)e^{\int_0^t r(u)du} \quad . \qquad (4)$$

However, the equation describing the growth of the rural population becomes

$$dP_r(t) = [r(t) - o_r(t)]P_r(t) + o_u(t)P_u(t) \quad , \qquad (75)$$

and, after substituting (3),

$$dP_r(t) = o_u(t)P_T(t) + [r(t) - o_r(t) - o_u(t)]P_r(t) \quad . \qquad (76)$$

Letting

$$P_r(t) = P(0)e^{\int_0^t [r(u) - o_r(u) - o_u(u)]du} y(t) \quad , \qquad (77)$$

(76) becomes

$$e^{\int_0^t [r(u) - o_r(u) - o_u(u)]du} dy(t) = o_u(t)e^{\int_0^t r(u)du} , \qquad (78)$$

or

$$dy(t) = o_u(t)e^{\int_0^t [o_r(u) + o_u(u)]du} . \qquad (79)$$

Integrating (79) leads to

$$y(t) = \int_0^t o_u(v)e^{\int_0^v [o_r(u) + o_u(u)]du} dv + K , \qquad (80)$$

in which, if the initial population is entirely rural, K = 1. Therefore, the rural population at time t is given by

$$P_r(t) = P(0)e^{\int_0^t [r(u) - o_r(u) - o_u(u)]du}\left(1 + \int_0^t o_u(v)e^{\int_0^v [o_r(u) + o_u(u)]du} dv\right). \qquad (81)$$

It thus follows from (3) that

$$P_u(t) = P(0)e^{\int_0^t r(u)du}$$

$$\left[1 - e^{-\int_0^t [o_r(u) + o_u(u)]du}\left(1 + \int_0^t o_u(v)e^{\int_0^v [o_r(u) + o_u(u)]du} dv\right)\right] , \qquad (82)$$

so that the ratio of urban to rural population is

$$S(t) = \frac{e^{\int_0^t [o_r(u) + o_u(u)]du}}{1 + \int_0^t o_u(v) e^{\int_0^v [o_r(u) + o_u(u)]du} dv} - 1 \quad ,(83)$$

or

$$S(t) = \frac{\int_0^t o_r(v) e^{\int_0^v [o_r(u) + o_u(u)]du} dv}{1 + \int_0^t o_u(v) e^{\int_0^v [o_r(u) + o_u(u)]du} dv} \quad . \qquad (84)$$

Note that the first derivative of S(t) has the sign of

$$F = o_r + \left[o_r \int_0^t o_u(v) e^{\int_0^v [o_r(u) + o_u(u)]du} dv - o_u \int_0^t o_r(v) e^{\int_0^v [o_r(u) + o_u(u)]du} dv \right] . \qquad (85)$$

Unlike the model of the preceding section, the present model is not necessarily characterized by a monotonic increase of S(t). Moreover, since the ratio R(t) of urban net migration to natural increase is

$$R(t) = \frac{o_r(t) S_r(t) - o_u(t) P_u(t)}{o_r(t) P_u(t)} = \frac{o_r(t)}{r(t) S(t)} - \frac{o_u(t)}{r(t)} \quad , \qquad (86)$$

there is, in general, no possibility of studying the sources of urban growth in an analytical way: recourse to a simulation analysis is then necessary.

A Tractable Case: The Case of Proportional Gross Migration Rates

However, if the ratio of $o_r(t)$ to $o_u(t)$ remains constant, i.e.,

$$\frac{o_r(t)}{o_u(t)} = k \quad , \qquad \text{for all } t \quad , \tag{87}$$

(this assumption is assumed to hold in the rest of this paper), the analysis is still tractable analytically.

When substituting (87) into (81), the quantity between brackets becomes

$$1 + \int_0^t o_u(v) e^{\int_0^v (1+k) o_u(u) du} dv \quad , \quad \text{i.e.,} \qquad 1 + \frac{e^{\int_0^t (1+k) o_u(u) du} - 1}{1 + k} \quad ,$$

so then the rural population at time t is

$$P_r(t) = P(0) \frac{e^{\int_0^t r(u) du}}{1 + k} \left[1 + k e^{-\int_0^t (1+k) o_u(u) du} \right] \quad . \tag{88}$$

Subtracting (88) from (4) yields

$$P_u(t) = P(0) e^{\int_0^t r(u) du} \frac{k}{1 + k} \left(1 - e^{-\int_0^t (1+k) o_u(u) du} \right) \quad . \tag{89}$$

Then, the ratio S(t) of urban to rural population is

$$S(t) = \frac{k\left(1 - e^{-\int_0^t (1+k)o_u(u)du}\right)}{1 + ke^{-\int_0^t (1+k)o_u(u)du}} \quad . \tag{90}$$

Since substituting (87) in (85) yields $F = o_r$, it is clear that the variations of S(t) are monotonic: it increases from zero (for t = 0) to k (for $t = \infty$).

The ratio R(t) of migration to natural increase in the urban region is obtained by substituting (90) in (86). We have

$$R(t) = \frac{(1+k)o_u(t)}{r(t)\left[e^{(1+k)\int_0^t o_u(u)du} - 1\right]} \quad . \tag{91}$$

It is clear that the function R(t) is similar to the one obtained earlier when using the Keyfitz model: $o_r(t) + o_u(t) = (1 + k) o_u(t)$ is substituted for m(t). Therefore, if r(t) monotonically increases, and/or $o_u(t)$ (and therefore $o_r(t)$) is such that its first derivative is positive and its second derivative negative, R(t) monotonically decreases, which again indicates the greater importance of natural increase vis-à-vis migration as the urban region grows.

Case of Constant Rates

We begin with the assumption that r(t), $o_r(t)$ and $o_u(t)$ remain constant, equal to r, o_r and o_u respectively: this is the hypothesis made by Ledent (1978a), with the further assumption that the urban rate of natural increase is identical to that of the rural region.

Under these conditions, the total population at time t is

$$P_T(t) = P(0)e^{rt} \quad , \tag{17}$$

and the rural population is obtained as

$$P_r(t) = \frac{P(0)e^{rt}}{o_r + o_u} \left[o_u + o_r e^{-(o_r + o_u)t} \right] \quad . \tag{92}$$

Then, the urban population is

$$P_u(t) = \frac{P(0)e^{rt}}{o_r + o_u} o_r \left[1 - e^{-(o_r + o_u)t} \right] \quad , \tag{93}$$

so that the ratio S(t) of urban to rural population is

$$S(t) = \frac{o_r\left(1 - e^{-(o_r + o_u)t}\right)}{o_u + o_r e^{-(o_r + o_u)t}} \quad . \tag{94}$$

It can be seen that, as expected, $\frac{dS(t)}{dt} > 0$ and thus S(t) increases monotonically: from zero (for t = 0) to $\frac{o_r}{o_u}$ (for $\rightarrow + \infty$).* Moreover, one can demonstrate that $\frac{d^2S(t)}{dt^2} < 0$ so that the direction of the curvature of S(t) is in opposition to the direction it had in the Keyfitz model with constant rates.

Substituting (94) into (91), we have that

*Note that both the urban and rural populations become infinitely positive as $t \rightarrow + \infty$.

$$R(t) = \frac{(o_r + o_u)e^{-(o_r + o_u)t}}{r\left(1 - e^{-(o_r + o_u)t}\right)} \quad . \tag{95}$$

Differentiating (95) leads to the result that $\frac{dR(t)}{dt}$ is negative, i.e., R(t) monotonically decreases: from + ∞ (for t = 0) to zero (for t → + ∞). Again, the role of migration, initially preponderant in accounting for the growth of the urban region, diminishes as time passes by so that natural increase is eventually the unique source of urban growth.

Consequently, there exists a cross-over point T at which natural increase equals migration in the urban region. Note that in contrast to the general case, for which rural and urban rates of natural increase are different (Ledent 1978b), this cross-over point always exists. At that point, we have, by substituting (22) into (86)

$$S(T) = \frac{o_r}{o_u + r} \quad . \tag{96}$$

An expression of T is then obtained by substituting (94) in (96)

$$T = \frac{1}{o_r + o_u} \ln\left(1 + \frac{o_r + o_u}{r}\right) \quad , \tag{97}$$

from which we draw conclusions similar to those drawn by Keyfitz from (24). The higher the common rate of natural increase, the sooner the cross-over; and the larger the values of both o_u and o_r, the sooner comes the day when natural increase exceeds migration.

In Table 5 we present a scenario involving the trajectory of a country that starts with an entirely rural population of 1 million. Its population is submitted to a rate of natural increase r = 0.03 and the gross outmigration rates are respectively

$o_r = 0.025$ and $o_u = 0.02$. In this scenario, the role of natural increase in accounting for urban growth increases rapidly and exceeds that of migration after

$$T = \frac{1}{0.03} \ln 2 = 20.4 \text{ years} \quad .$$

At this point, the ratio of urban to rural population is S(T) = 0.5 so that the part of the population which is urban is equal to one-third.

Table 5. Urbanization of an initially rural population of 1 million, with r = 0.03, $o_r = 0.025$ and $o_u = 0.02$.

Year	Total	Rural	Urban	Percentage Urban	m(t)	$\frac{o_r}{S(t)} - o_u$	R(t)
0	1	1	0	0	0.025	+ ∞	+ ∞
1	1.03	1.01	0.03	0.02	0.025	0.988	32.59
2	1.06	1.01	0.05	0.05	0.024	0.478	15.93
3	1.09	1.02	0.08	0.07	0.024	0.311	10.38
4	1.13	1.02	0.10	0.09	0.023	0.278	7.61
5	1.16	1.03	0.13	0.11	0.023	0.178	5.94
10	1.35	1.08	0.37	0.20	0.020	0.079	2.64
15	1.57	1.14	0.43	0.27	0.018	0.047	1.56
20	1.82	1.22	0.60	0.33	0.015	0.031	1.03
25	2.12	1.32	0.79	0.38	0.013	0.022	0.72
30	2.46	1.45	1.01	0.41	0.011	0.016	0.52
35	2.86	1.60	1.26	0.44	0.009	0.012	0.39
40	3.32	1.78	1.54	0.46	0.008	0.009	0.30
45	3.86	2.00	1.86	0.48	0.006	0.007	0.23
50	4.48	2.25	2.23	0.50	0.005	0.005	0.18
75	9.49	4.40	5.09	0.54	0.002	0.002	0.05
100	20.09	9.05	11.03	0.55	0.001	0.001	0.02

Now suppose that the actual population system, considered in the first section, exhibits a gross migration pattern such that $o_r = 0.025$ and $o_u = 0.02$. When will this system reach the point at which the urban region grows equally from natural increase and migration, if the parameters of the system remain constant?

Let us recall that Ledent (1978a) shows that any population system whose urban (rural) regions exhibit natural increase and outmigration rates equal u and o_u (r and o_r) respectively, is identical to the subsequent state of a hypothetical population, initially entirely rural and characterized by the same parameters only if the observed ratio $\bar{s}$ urban to rural population is such that

$$\bar{s} < \frac{u - o_u - (r - o_r) + \sqrt{(u - o_u - r + o_r)^2 + 4o_r o_u}}{2o_u} , \tag{98a}$$

i.e., if u = r, as in the present case,

$$\bar{s} < \frac{o_r}{o_u} . \tag{98b}$$ *

Substituting the parameters of our population system indicates that (98b) holds.

The time t_D, at which this correspondence occurs, is simply obtained as the root of $S(t) = \bar{s}$, which is unique since S(t) monotonically increases. It is easy to establish that

$$t_D = \frac{1}{o_r + o_u} \ln \frac{o_r(1 + \bar{s})}{o_r - \bar{s}o_u} . \tag{99}$$

*Note that the evolution is hardly surprising since the highest value of S(t) is $S(\infty) = \frac{o_r}{o_u}$.

Consequently, if around the observation period the actual population exhibits the constant regimes of natural increase and migration defined by r, o_u and o_r, the time span necessary to reach the point at which natural increase and migration are equal is

$$T' = T - t_D = \frac{1}{o_r + o_u} \ln \frac{\left(1 + \frac{o_r + o_u}{r}\right)\left(1 - \bar{s}\,\frac{o_u}{o_r}\right)}{1 + \bar{s}} \quad . \tag{100}$$

This relationship shows that the sign of T' depends on the relative position of $\frac{o_r}{r + o_u}$ and $\bar{s}$. In particular, if $\bar{s} > \frac{o_r}{r + o_u}$, urban population growth is primarily due to natural increase since the cross-over point appears to have been passed.

Substituting the parameter values of our actual population system into (100) indicates that the cross-over at which natural increase and migration contribute equally to urban growth will be reached in

$$T' = 10.4 \text{ years} \quad .$$

Case of a Rate of Natural Increase Varying Exponentially

Let us suppose now that

$$r(t) = r_1 + (r_0 - r_1)e^{\beta t} \quad , \tag{25}$$

in which r_0 and r_1 are both positive but such that, if r_0 is larger (smaller) that r_1, β is negative (positive); and let the gross migration rates remain constant.

Substituting (25) into (4), we obtain the total population at time t:

$$P_T(t) = P(0)e^{r_1 t + \frac{r_0 - r_1}{\beta}(e^{\beta t} - 1)} . \quad (26)$$

The rural population is given by

$$P_r(t) = \frac{P(0)e^{r_1 t + \frac{r_0 - r_1}{\beta}(e^{\beta t} - 1)}}{o_r + o_u}\left[o_u + o_r e^{-(o_r + o_u)t}\right] , \quad (101)$$

and the urban population by

$$P_u(t) = \frac{P(0)e^{r_1 t + \frac{r_0 - r_1}{\beta}(e^{\beta t} - 1)}}{o_r + o_u} o_r\left[1 - e^{-(o_r + o_u)t}\right] , \quad (102)$$

so that the ratio S(t) of the urban to rural population is as above

$$S(t) = \frac{o_r\left(1 - e^{-(o_r + o_u)t}\right)}{o_u + o_r e^{-(o_r + o_u)t}} , \quad (94)$$

which was expected, since natural increase has no impact on population distribution. The variations of this function have been described above.*

Substituting (25) and (94) in (86), we have that

*Again, note that both $P_r(t)$ and $P_u(t)$ become infinitely positive as $t \to +\infty$.

$$R(t) = \frac{(o_r + o_u)e^{-(o_r + o_u)t}}{\left[r_1 + (r_0 - r_1)e^{+\beta t}\right]\left[1 - e^{-(o_r + o_u)t}\right]} \quad , \quad (103)$$

or

$$R(t) = \frac{o_r + o_u}{\left[r_1 + (r_0 - r_1)e^{\beta t}\right]\left[e^{(o_r + o_u)t} - 1\right]} \quad . \quad (104)$$

Note that (104) is identical to (30) in which $(o_r + o_u)$ is substituted for m. Then, without further calculations, we can state that R(t) decreases monotonically from + ∞ (for t = 0) to vanish in the long-run. There exists a cross-over point T characterized by equal natural increase and migration in the urban region, at which

$$S(t) = \frac{o_r}{r(t) + o_u} \quad . \quad (105)$$

Pursuing further the analogy of the present model with the Keyfitz model, T is implicitly defined by an equation similar to (34):

$$e^{(o_r + o_u)T} - 1 = \frac{o_r}{r_1 + (r_0 - r_1)e^{\beta t} + o_u} \quad . \quad (106)$$

Table 6 shows the evolution of a hypothetical country of 1 million which is characterized by the same gross migration rates as in Table 5, and having a rate of natural increase decreasing from $r_0 = 0.043$ to $r_1 = 0.01$ with parameter $\beta = -0.05$.

Table 6. Urbanization of an initially rural population of 1 million with $o_u = 0.02$, $o_r = 0.025$ and $r(t) = 0.01 + 0.033\ e^{-0.05t}$.

Year	Total	Rural	Urban	Percentage Urban	$\frac{o_r}{S(t)} - o_u$	r(t)	R(t)
0	1	1	0	0	+ ∞	0.043	+ ∞
1	1.04	1.02	0.03	0.02	0.98	0.041	23.71
2	1.09	1.03	0.05	0.05	0.48	0.040	12.03
3	1.13	1.05	0.08	0.07	0.31	0.038	8.14
4	1.17	1.07	0.11	0.09	0.23	0.037	6.19
5	1.22	1.08	0.14	0.11	0.18	0.036	5.01
10	1.43	1.14	0.29	0.20	0.08	0.030	2.65
15	1.64	1.20	0.45	0.27	0.067	0.026	1.83
20	1.85	1.24	0.61	0.33	0.031	0.021	1.40
25	2.05	1.28	0.77	0.38	0.021	0.019	1.11
30	2.25	1.32	0.93	0.41	0.016	0.017	0.91
35	2.44	1.37	1.08	0.44	0.012	0.016	0.75
40	2.63	1.41	1.22	0.46	0.009	0.014	0.62
45	2.82	1.46	1.36	0.48	0.007	0.014	0.51
50	3.01	1.52	1.50	0.50	0.005	0.013	0.42
75	4.02	1.86	2.16	0.54	0.002	0.011	0.15
100	5.22	2.35	2.87	0.55	0.000	0.010	0.05

Note that in this scenario, the rural population keeps increasing as time goes by instead of reaching a maximum and then vanishing as in the similar scenario based on the Keyfitz model.

Indeed, the value of r_0 has been chosen so that one state of the system in this scenario will be identical to our actual population system.* As in the case of a constant rate of natural increase, this state is reached for $t_D = 9.9$ years.

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*Once β and r_1 are chosen, r_0 is obtained from (35) in which $o_r + o_u$ is substituted for m.

Moreover, the cross-over appears to be reached for T approximately equal to 27.6 years. Then, the time span necessary to reach the cross-over is T' = 17.7 years from the observed period (against T' = 10.4 years when r(t) remains equal to r). Thus, the exponential decrease of r(t) delays the cross-over point by 7.3 years.*

Case of Gross Migration Rates Increasing Exponentially

Again, we assume r(t) = r (for all t), but we now allow $o_u(t)$ and $o_r(t)$ to vary such that (87) holds. We posit:

$$o_u(t) = m_0 (1 - e^{-\alpha t}) \quad . \tag{107}$$

The total population is again given by (17) whereas the rural population is obtained by substituting (87) and (107) into (81) and then integrating:

$$P_r(t) = P(0)e^{rt} \frac{1 + ke^{-\left[m_0(1 + k)\left(t + \frac{e^{-\alpha t} - 1}{\alpha}\right)\right]}}{1 + k} \quad . \tag{108}$$

Thus,

$$P_u(t) = P(0)e^{rt} \frac{k\left(1 - e^{-\left[m_0(1 + k)\left(t + \frac{e^{-\alpha t} - 1}{\alpha}\right)\right]}\right)}{1 + k} \quad , \tag{109}$$

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*Indeed, the cross-over is now reached in a more urbanized nation: almost 40 percent of the population appears to be urban if r(t) is allowed to decrease (versus one third if r(t) remains constant).

and

$$S(t) = \frac{k\left(1 - e^{-\left[m_0(1 + k)\left(t + \frac{e^{-\alpha t} - 1}{\alpha}\right)\right]}\right)}{1 + ke^{-\left[m_0(1 + k)\left(t + \frac{e^{-\alpha t} - 1}{\alpha}\right)\right]}} \quad . \qquad (110)$$

It is readily established that both $P_r(t)$ and $P_u(t)$ increase monotonically and that $S(t)$ increases monotonically as well: from zero (for $t = 0$) to k (for $t = +\infty$). In the long run, the ratio of the urban to rural population thus tends to become equal to the constant ratio of gross migration rates.

Now, what about the variations of $R(t)$? Substituting (107) in (91), we have

$$R(t) = \frac{m_0(1 + k)(1 - e^{-\alpha t})}{r\left[e^{m_0(1 + k)\left(t + \frac{e^{-\alpha t} - 1}{\alpha}\right)} - 1\right]} \quad . \qquad (111)$$

Note that $R(t)$ is identical to (41) in which m_0 has been replaced by $m_0(1 + k)$. Thus $R(t)$ monotonically decreases: from $+\infty$ (for $t = 0$) to zero (as $t \to +\infty$). There exists a cross-over point T characterized by equal natural increase and migration in the urban region, i.e., such that

$$S(t) = \frac{km_0(1 - e^{-\alpha t})}{r + km_0(1 - e^{-\alpha t})} \quad . \qquad (112)$$

Equating (112) with (110) finally defines T implicitly.

In Table 7, we display the results of a scenario corresponding to the case of a country in which the rate of natural increase is $r = 0.02$ and the ratio of the rural and urban migration is equal to $\frac{0.025}{0.02} = 1.25$. In addition, the maximal value of the urban gross migration rate was taken as m_0 = 0.12 and we chose the parameter α to be 0.0122.

It appears that the cross-over point takes place at time t = 27.8 years, when the ratio S(t) of urban to rural population appears to be equal to about one half (i.e., about one third of the population is urban at the cross-over point).

Table 7. Urbanization of an initially rural population of 1 million with a constant rate of natural increase and gross migrations increasing exponentially.

Year	Total	Rural	Urban	Percentage Urban	$\frac{o_r}{S(t)} - o_u$	R(t)
0	1	1	0	0	+ ∞	+ ∞
1	1.03	1.03	0.00	0.001	34.164	1138.82
2	1.06	1.06	0.00	0.003	8.560	285.33
3	1.09	1.09	0.01	0.007	3.807	126.92
4	1.13	1.11	0.01	0.011	2.144	71.36
5	1.16	1.14	0.02	0.018	1.368	45.59
10	1.35	1.26	0.09	0.066	0.333	11.08
15	1.57	1.36	0.21	0.136	0.139	4.64
20	1.82	1.43	0.39	0.215	0.072	2.38
25	2.12	1.50	0.62	0.293	0.041	1.34
30	2.46	1.57	0.89	0.362	0.024	0.80
35	2.86	1.66	1.20	0.420	0.013	0.43
40	3.32	1.78	1.54	0.464	0.009	0.29
45	3.86	1.94	1.92	0.497	0.005	0.18
50	4.48	2.16	2.33	0.519	0.003	0.10
75	9.49	4.24	5.25	0.554	0.000	0.00
100	20.09	8.93	11.16	0.555	0.000	0.00

The values of k and α were chosen so that one state of the system in this scenario presents the same characteristics as our actual population system. Indeed, the parameter k was taken equal to the ratio of the gross outmigration rates in the observed population. Moreover, once m_0 was chosen, α was obtained by eliminating t between (107) and $S(t) = \bar{s}$ in the same way that we eliminated t between (46) and (47).

$$\alpha = -\frac{1+k}{k}\,\frac{m_0 \ln\left(1 - \frac{o_r}{m_0}\right) + o_r}{\ln\left(\frac{1+\bar{s}}{1-\frac{\bar{s}}{k}}\right)} \quad . \tag{113}$$

Note that the hypothetical population presents the same characteristics as the observed population for

$$t_D = -\frac{1}{\alpha}\ln\left(1 - \frac{o_r}{m_0}\right) = 19.1 \text{ years} \quad .$$

Recalling that the cross-over occurs for T = 27.8 years, it follows that the time span necessary to reach it is T' = 8.7 years from the observed period (against 10.4 years in the case of all rates being constant). Then, rather surprisingly, a proportional increase in the gross migration rates hastens the cross-over point.

CONCLUSION

In this paper, we examined the relative importance of inmigration and natural increase in the growth of urban areas with the help of two alternative urbanization models, in which rates of natural increase and migration were allowed to vary. Table 8 displays a comparison of the time spans necessary to reach the cross-over as obtained from both models under alternative patterns of natural increase and migration. Indeed, the difference between the numerical values offered by both models is quite considerable.

Table 8. Numerical comparison of the time spans necessary to reach the cross-over.

THE KEYFITZ MODEL			THE TWO-REGION ROGERS MODEL	
T'	S(T)		T'	S(T)
14.4	0.66	Constant Rates	10.4	0.50
33.9	1.46	Varying Rate of Natural Increase	17.7	0.65
19.9	1.18	Varying Migration Rates	8.7	0.50

The problem is then one of knowing which of the two models provides better insights into the urbanization process. On the one hand, the Rogers model appears to be more appropriate because its symmetrical consideration of gross migration flows in both regions prevents the rural population from vanishing in the long run, as in some applications of the Keyfitz model. On the other hand, in contrast to the Keyfitz model, the Rogers model does not lend itself to an analytical use if the model parameters vary over time. As seen in this paper, the analytical tractability of this model requires making the additional assumption that the urban and rural outmigration rates are in constant proportions, which is a rather restrictive assumption.

In conclusion, the Rogers model is in theory more desirable than the alternative model (see Ledent 1978a, b for a longer discussion of this statement), but its slightly more complicated specification prevents, in the case of varying rates, the development of an analytical study similar to the one carried out with the Keyfitz model and shown in the first part of this paper. Consequently, further insights into the process of urbanization seem to require a simulation analysis with the help of the Rogers model.

REFERENCES

Keyfitz, N. (1978), *Do Cities Grow by Natural Increase or by Migration?* (Unpublished paper).

Ledent, J. (1978a), *The Dynamics of Two Demographic Models of Urbanization*, RM-78-56, International Institute for Applied Systems Analysis, Laxenburg, Austria.

Ledent, J. (1978b), *The Factors and Magnitude of Urbanization Under Unchanged Fertility and Mobility Patterns*, RM-78-57, International Institute for Applied Systems Analysis, Laxenburg, Austria.

Rogers, A. (1968), *Matrix Analysis of Interregional Population Growth and Distribution*, University of California Press, Berkeley.

Rogers, A. (1978), *A Comment on Keyfitz's "Do Cities Grow by Natural Increase or by Migration?"*, (Unpublished paper).

Todaro, M.P. (1969), A Model of Labor Migration and Urban Employment in Less Developed Countries, *American Economic Review*, 59, 138-148.

Todaro, M.P. (1976), Urban Job Expansion, Induced Migration and Rising Unemployment, *Journal of Development Economics*, 3, 211-225.

APPENDIX 1. The Keyfitz Model with Varying Rates of Natural Increase and Migration: Derivation of the Sign of the Second Derivative of $\frac{m(t)}{r(t)}$

In this case, the ratio $\frac{m(t)}{r(t)}$ is given by

$$\frac{m(t)}{r(t)} = \frac{m_0(1 - e^{-\alpha t})}{r_1 + (r_0 - r_1)e^{\alpha t}} \quad , \tag{A1}$$

in which $\alpha > 0$ and $\beta < 0$.

Differentiating this expression with respect to time leads to

$$\frac{\frac{d\left(\frac{m(t)}{r(t)}\right)}{dt}}{\frac{m(t)}{r(t)}} = \frac{\alpha e^{-\alpha t}}{(1 - e^{-\alpha t})} - \frac{(r_0 - r_1)\beta e^{\beta t}}{r_1 + (r_0 - r_1)e^{\beta t}}$$

$$= \frac{\alpha}{(e^{\alpha t} - 1)} - \frac{(r_0 - r_1)\beta}{r_1 e^{-\beta t} + (r_0 - r_1)} \quad . \tag{A2}$$

Differentiating again with respect to time gives

$$\frac{\frac{d^2\left(\frac{m(t)}{r(t)}\right)}{dt^2}}{\frac{m(t)}{r(t)}} - \left[\frac{\frac{d\left(\frac{m(t)}{r(t)}\right)}{dt}}{\frac{m(t)}{r(t)}}\right]^2 = - \frac{\alpha^2 e^{\alpha t}}{(e^{\alpha t} - 1)^2} - \frac{(r_0 - r_1)r_1\beta^2 e^{-\beta t}}{\left(r_1 e^{-\beta t} + r_0 - r_1\right)^2} \quad . \tag{A3}$$

When substituting (A2) into (A3), we have

$$\frac{\dfrac{d^2\left(\dfrac{m(t)}{r(t)}\right)}{dt^2}}{\dfrac{m(t)}{r(t)}} = -\frac{\alpha^2}{e^{\alpha t} - 1} + \frac{(r_0 - r_1)\beta^2\left[r_0 - r_1 - r_1e^{-\beta t}\right]}{\left(r_1e^{-\beta t} + r_0 - r_1\right)^2}$$

$$- \frac{2\alpha\beta(r_0 - r_1)}{(e^{\alpha t} - 1)\left(r_1e^{-\beta t} + r_0 - r_1\right)} \quad . \qquad \text{(A4)}$$

Multiplying both sides of (A4) by $\frac{m(t)}{r(t)}$ and substituting (A1) gives

$$\frac{d^2\left(\dfrac{m(t)}{r(t)}\right)}{dt^2} = \frac{m_0e^{-(\alpha + \beta)t}}{\left(r_1e^{-\beta t} + r_0 - r_1\right)^3} \; F(t) \quad , \qquad \text{(A5)}$$

in which

$$F(t) = \alpha^2\left(r_0 - r_1 + r_1e^{-\beta t}\right)^2 + \beta(r_0 - r_1)\left[2\alpha\left(r_0 - r_1 + r_1e^{-\beta t}\right)\right.$$

$$\left. - \beta(e^{\alpha t} - 1)\left(r_0 - r_1 - r_1e^{-\beta t}\right)\right] \quad . \qquad \text{(A6)}$$

Differentiating (A6) yields

$$dF(t) = e^{(\alpha - \beta)t}\left[\beta^2r_1(r_0 - r_1)(\alpha - \beta) + \beta r_1(r_0 - r_1)\right.$$

$$(\beta^2 - 2\alpha(\alpha + \beta))e^{-\alpha t} - \alpha\beta^2(r_0 - r_1)^2e^{\beta t}$$

$$\left. - 2\alpha^2\beta r_1^2e^{-(\alpha + \beta)t}\right] \quad , \qquad \text{(A7)}$$

whose variations have the sign of

$$G(t) = -\alpha\beta r_1(r_0 - r_1)(\beta^2 - 2\alpha(\alpha + \beta))e^{-\alpha t} - \alpha\beta^3(r_0 - r_1)^2 e^{\beta t}$$

$$+ 2\alpha^2\beta(\alpha + \beta)r_1^2 e^{-(\alpha + \beta)t} \quad . \tag{A8}$$

If we suppose that $\alpha + \beta < 0$, then the three terms of $G(t)$ in (A8) have a positive coefficient and thus $F(t)$ monotonically increases. Its smallest value, $F(0) = \alpha r_0[\alpha r_0 + 2\beta(r_0 - r_1)]$, can be either positive or negative. In the first case, i.e., $r_1 > r_0\left(1 + \frac{\alpha}{2\beta}\right)$, $F(t)$ is always positive and thus the second derivative of $\frac{m(t)}{r(t)}$ is negative for all t. In the second case, i.e., $r_1 < r_0\left(1 + \frac{\alpha}{2\beta}\right)$, $F(t)$ is negative for $t <$ same value t_r and positive thereafter. It follows that the second derivative of $\frac{m(t)}{r(t)}$ is positive for $t < t_r$ and negative thereafter.

APPENDIX 2. An Illustration of the Todaro Hypothesis

Year	Total	Rural	Urban	S(t)	m(t)	$\frac{m(t)}{S(t)}$	R(t)
0	1	1	0	0			
1	1.03	1.03	0.00	0.001	0.001	1.039	34.32
2	1.06	1.06	0.00	0.002	0.002	0.770	25.68
3	1.09	1.09	0.01	0.005	0.003	0.569	18.97
4	1.13	1.12	0.01	0.008	0.004	0.446	14.89
5	1.16	1.15	0.01	0.012	0.005	0.366	12.20
10	1.34	1.28	0.06	0.050	0.009	0.198	6.27
15	1.56	1.40	0.16	0.115	0.014	0.123	4.09
20	1.81	1.49	0.31	0.210	0.018	0.088	2.94
21	1.86	1.51	0.35	0.233	0.0194	0.083	2.77
22	1.92	1.52	0.39	0.258	0.0203	0.078	2.62
25	2.09	1.56	0.53	0.342	0.023	0.067	2.22
30	2.43	1.60	0.83	0.519	0.027	0.052	1.73
35	2.81	1.60	1.21	0.755	0.031	0.041	1.36
40	3.26	1.58	1.68	1.066	0.035	0.032	1.08
45	3.78	1.53	2.25	1.477	0.038	0.026	0.86
50	4.38	1.45	2.93	2.023	0.042	0.021	0.69
75	9.18	0.88	8.30	9.428	0.068	0.006	0.20
100	19.22	0.38	18.84	49.988	0.069	0.001	0.04
25	2.09	1.56	0.53	0.341	0.022	0.065	2.18
30	2.43	1.61	0.82	0.509	0.025	0.050	1.65
35	2.81	1.63	1.18	0.721	0.028	0.039	1.29
40	3.26	1.64	1.62	0.990	0.030	0.031	1.02
45	3.78	1.63	2.15	1.325	0.032	0.025	0.82
50	4.38	1.60	2.78	1.743	0.034	0.020	0.66
75	9.18	1.30	7.88	6.071	0.041	0.007	0.23
100	19.22	0.91	18.30	20.008	0.046	0.002	0.08

The upper part of the above table displays the evolution of an initially rural population system submitted to a rate of natural increase $r(t) = 0.02$ and to a migration rate increasing exponentially as in the scenario of Table 3. The difference with the scenario shown in Table 3 is that the calculations have been made using the discrete equivalent of the Keyfitz model.

Note that, by about the 22nd year (as in the continuous case), the system reaches a state whose characteristics are similar to those of our actual population system ($m = 0.02$ and $\bar{s} = 0.25$).

The implementation of the Todaro hypothesis after this 22nd year (with $s = 0.007$, $g = 0.05$ and $u = 0.12$), leads to an evolution of the population system shown in the bottom part of the above table. As expected, since the inclusion of the Todaro hypothesis contributes to diminish $m(t)$, the cross-over point is reached faster than when the Todaro hypothesis is not considered: it occurs 18.4 years after the observation period against 19.7 years in the alternative case. This difference is rather small when one considers the more important consequences that the implementation of the Todaro hypothesis has on $P_r(t)$ and especially on $m(t)$.

Note that, in spite of the favorable choice of δ, $m(t)$ continually increases in the above table. The decline in $m(t)$ occurs for high values of t, long after natural increase has taken over migration in the urban region.

<u>Papers of the Population, Resources and Growth Study</u>

1. Nathan Keyfitz, *Understanding World Models*, RM-77-18.

2. Andrei Rogers, *Migration, Urbanization, Resources, and Development*, RR-77-14, forthcoming in *Alternatives for Growth: The Engineering and Economics of Natural Resources Development*, H. McMains, and L. Wilcox, eds., J. Wiley, New York.

3. Roman Kulikowski, *Optimization of Rural-Urban Development and Migration*, RM-77-41, published in Migration and Settlement: Selected Essays, *Environment and Planning A*, <u>10</u>, 5, 1978.

4. Frans Willekens, *Spatial Population Growth in Developing Countries: With a Special Emphasis on the Impact of Agriculture*, internal working papers, 1977, forthcoming as a Research Report.

5. Henry Rempel, *The Role of Rural-Urban Migration in the Urbanization and Economic Development Occurring in Kenya*, RM-78-12.

6. Allen Kelley, and C. Swartz, *The Impact of Family Structure on Household Decision Making in Developing Countries: A Case Study in Urban Kenya*, internal working paper, published in the Proceedings of the IUSSP Conference on *Economic and Demographic Change: Issues for the 1980's*, 1978.

7. Donaldo Colosio, Luis J. Castro, and Andrei Rogers, *Migration, Urbanization, and Development: A Case Study of Mexico*, internal working paper, 1978, published in abridged form in *Memoria Cuarto Congreso Academia Nacional de Ingenieria*, A.C., October, 1978.

8. Clark W. Reynolds, *A Shift-Share Analysis of Regional and Sectoral Productivity Growth in Mexico Since 1940: The Narrowing Dimensions of Policy Space*, forthcoming.

9. Mahendra M. Shah and Frans Willekens, *Rural-Urban Population Projections for Kenya and Implications for Development*, RM-78-55.

10. Jacques Ledent, *The Dynamics of Two Demographic Models of Urbanization*, RM-78-56.

11. Jacques Ledent, *The Factors and Magnitude of Urbanization Under Unchanged Natural Increase and Migration Patterns*, RM-78-57.