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Risk, Security and Robust Solutions

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Foreword

Standard risk management deals with threats generated by exogenous events. Repetitive observations are used to characterize risk by a probability distribution that can be used for risk-decision support. Statistical decision theory, expected utility theory and the more general stochastic optimization (STO) theory provide common approaches for this purpose. In contrast to standard risk management, security management addresses threats generated (intentionally or unintentionally) in multi-agent environment by intelligent agents, which may affect large territories and communities. An obvious example is terrorism. Less evident examples are floods which are often triggered by rains, hurricanes and earthquakes in combination with inappropriate land use planning, maintenance of flood protection systems and behavior of various agents. Other examples include: civil, social, energy, food and water security issues. Threats associated with such systems are usually affected by decisions of different agents, say an increase of biofuels production may change market prices, induce threats of environmental degradation, destabilize supplies of food and water and disturb natural environments. In contrast to classical situations such threats cannot be characterized by a uniquely defined probability distribution. Inherent uncertainties of complex interdependent systems with the lack and even absence of repetitive observations restrict exact evaluations and predictions. Future paths of these systems may be dramatically affected by old and new policies. The main issue in this case is the design of robust solutions. Although exact evaluations are impossible, the preference structure among feasible alternatives of policies, regulations, structures etc, provide a stable basis for comparative analysis. This is used in order to find solutions, which ensure robustness in the sense of maintaining functioning of systems, in the face of a vast variety of uncertainties.

The main purpose of this paper is to develop a decision-theoretic approach to security management. It shows that robustness of solutions in security management can be achieved by developing new stochastic optimization tools for models with uncertain multi-dimensional probability distributions, extreme events and multiple criteria. One approach, using the common Stackelberg game is built on strong assumptions of perfect information about all agents and leading to unstable solutions and discontinuous models with respect to slight variations of initial data. Our proposed decision-theoretic approach does not destroy convexities but still preserves the two-stage structure of the Stackelberg “leader-follower” decisions. The paper analyzes problems of homeland security, electricity networks and other areas of systemic security and risk management. It provides an overview of existing relevant computational methods to be further developed and analyses promising new methods based on specific representations of uncertain probabilities.

Abstract

The aim of this paper is to develop a decision-theoretic approach to security management of uncertain multi-agent systems. Security is defined as the ability to deal with intentional and unintentional threats generated by agents. The main concern of the paper is the protection of public goods from these threats allowing explicit treatment of inherent uncertainties and robust security management solutions. The paper shows that robust solutions can be properly designed by new stochastic optimization tools applicable for multicriteria problems with uncertain probability distributions and multivariate extreme events.

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About the Authors

Yuri Ermoliev, from the Glushkov Institute of Cybernetics, Kiev, Ukraine, came to IIASA in October 1991 to work on advanced methods for proper treatment of endogenous uncertainty and catastrophic risks in complex decision problems; continues to be the Institute Scholar but contributes his research to the Integrated Modeling Environment Project. Prof. Ermoliev was the recipient of the State Award in Science of the Ukraine and the State Award in Science of the USSR. He is a Member of the National Ukrainian Academy of Sciences. He has been Head of the Department of Mathematical Methods of Operations Research at the Institute of Cybernetics of the Ukrainian Academy of Sciences, Kiev. His scientific interests are modeling and application of decision processes in the presence of risks and uncertainties, non-smooth stochastic and dynamic systems optimization, optimization on networks, and nonlinear dynamics. His major publications include books on Stochastic Programming Methods, Stochastic Models in Economics, Methods of Optimal Control using finite difference approximations, Extremal Problems on Graphs, Identification Problems and Techniques for Stochastic Optimization. Other publications concern the study of path-dependent adaptation processes, pollution control problems, energy and agriculture modeling, reliability theory, optimization of discontinuous systems including discrete event systems, studies of robust solutions under multivariate extreme events and risks.

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Risk, Security and Robust Decisions

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1. Introduction:

Standard risk management deals with threats generated by exogenous events. Typically, such situations allow to separate risk assessment from risk management. Repetitive observations are used to characterize risk by a probability distribution that can be used in risk management. Statistical decision theory, expected utility theory and more general stochastic optimization (STO) theory provide common approaches for this purpose.

Security management includes threats generated (intentionally or unintentionally) by intelligent agents. Obvious examples are threats to public goods and homeland security from terrorists [16]. Less evident examples are floods which are often triggered by rains, hurricanes, and earthquakes in combination with inappropriate land use planning, maintenance of flood protection systems and behavior of various agents. The construction of levees, dikes, and dams which may break on average, say, once in 100 years, create an illusion of safety and in the absence of proper regulations developments close to these constructions can create potential catastrophic events of high consequences.

Other examples include social, financial, economic, energy, food and water security issues. Water and food security deals with the robust functioning of complex multi-agent water and food supply networks. Threats associated with such systems depend on decisions of different agents. For example, an increase of bio-fuel production may change market prices, induce threats of environmental degradation, destabilize supplies of food and water, and disturb rural developments.

These examples illustrate threats that cannot be characterized by a single probability distribution. Inherent uncertainties of related decision problems with the lack and even absence of repetitive observations restrict exact evaluations and predictions. The main issue in this case is the design of robust solutions. Although exact evaluations are impossible, the preference structure among feasible alternatives provides a stable basis for relative ranking of them in order to find solutions robust with respect to all potential scenarios of uncertainties. As we know, the heavier parcel can be easily found without exact measuring of the weight.

The main purpose of this paper is to analyze specifics of decision problems arising in the security management. It shows that robustness of solutions can be achieved by using STO tools applicable for models with uncertain probability distributions, multivariate extreme events, and multiple criteria. Since the focus of CwU workshop is on broad audience, this paper avoids mathematical technicalities. In particular, it pays specific attention to motivations and clarifications.

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In order to develop robust decision-theoretic approaches, sections 2, 3, 4 analyze similarities and fundamental differences between frequent standard risks, multivariate multi-agent catastrophic risks with the lack and even absence of repetitive observations, and risks generated by intelligent agents.

In the case of standard risks, the term “robust” was introduced in statistics [22] in connections with irrelevant “bad” observations (outliers) which ruin the standard mean values, least square analysis, regression and variance/covariance analysis. The mean is not robust to outlier, whereas the median is robust. Section 2 shows, that switching from quadratic (least square) smooth optimization principles in statistics to non-smooth stochastic minimax optimization principles leads to robust statistical decisions. This idea is generalized in the following sections.

In general decision problems (section 3) under inherent uncertainty the robustness of decisions is achieved first of all by a proper representation of uncertainty, adequate sets of decisions and performance indicators characterizing socio-economic, technological, environmental, safety, security, equity, etc. perspectives. This leads to specific STO problems. In particular, a key issue is the sensitivity (singularity) of robust solutions with respect to low-probability extreme events. Section 3 introduces similar to section 2 multicriteria versions of risk measures and new robust STO models applicable for managing systemic catastrophic risks involving multivariate extreme events.

Section 4.1 analyses security management problems with several agents formulated as principal-agent or agency problems where a principal agent (PA) regulates performance of other agents in order to secure overall performance of a system [29], [34], [35]. These problems have features of a two-stage Stackelberg games, in which a ”leader” chooses first and a ”follower” chooses next, with full knowledge of the leaders’ decision. The traditional formulation of the Stackelberg game is problematic because of the assumption about exactly known responses of agents and commitments of agents to these responses. Section 5 shows that this easily leads to degenerated responses of agents inducing instabilities and discontinuities even for linear objective and constraints functions of agents. Implicitly, such assumptions are also used in bi-level mathematical programs with equilibrium constraints [6], [25], [27]. Section 4.1 discusses also serious limitations of Bayesian games. The use of Nash games (sections 6) destroys essential two-stage structure of principal-agent problems. Important stochastic bi-level mathematical programs are analyzed in [20].

Sections 4.2, 5 introduce concepts of robust decision-theoretic versions of the principle-agent problem using the PA’s perceptions of agents’ behavioral scenarios and general stochastic and probabilistic maximin principles. Section 5 analyses systemic security management problems, in particular, preventive robust solutions in randomized strategies, defensive allocation of resources, and modeling of systemic failures and damages. Section 6 discusses security of electricity networks. Section 7 analyses computational methods for problems with extreme events and uncertain probability distributions. Applications of these methods to security management can be found in [4], [39]. Section 8 concludes.

2. Standard risks

Standard risk analysis relies on observations from an assumed true model specified by a probability distribution P. Repetitive observations allow deriving the probability distribution P and its characteristics required for related decision support models. A key issue in this case is concerned with “bad” observations or “outliers”, which may easily ruin standard mean values, variance, least-square analysis, regressions and covariances [11]. [22], [26]. Therefore, traditional deterministic models using mean values may produce wrong results. The main approach in such cases is to use robust models which are not sensitive to irrelevant bad observations and at the same time, which are able to cope with relevant rare extreme events of high consequences.

The term “robust” was introduced into statistics in 1953 by Box and received recognition after the path-breaking publication by Huber [22], although the discussion about rejection of bad observations is at least as old as the 1777 publication of Daniel Bernoulli. The straightforward rejection of outliers is practically impossible in the case of massive data sets, because it may also delete important and relevant observations. Huber introduced rigorous notions of robustness based on probabilistic minimax approach. Its main idea can be developed for general decision problems emerging in security management (section 4). By using appropriate neighborhoods of probability distributions (e.g. ε – contaminated probabilities, neighborhoods of imprecise probabilities) Huber derived robust estimates optimizing the worst that can happen in a specific probabilistic sense over the neighborhood of the model. In other words, robust statistical analysis is equivalent to switching from smooth least square optimization principles to non-smooth minimax STO principles. The mean is not robust to outliers, whereas the median is robust. The mean value of a random variable θ minimizes the quadratic function

$$M(x) = E(x-\theta)^2 = \int (x-\theta)^2 P(d\theta)\,, \tag{1}$$

whereas the median and more generally a quantile minimizes function

$$Q(x) = E\max\{\alpha(x-\theta), \beta(\theta-x)\} = \int \max\{\alpha(x-\theta), \beta(\theta-x)\} P(d\theta)\,, \tag{2}$$

with non-smooth random function $\max\{\alpha(x-\theta), \beta(\theta-x)\}$, where P is a probability distribution function, and $\alpha, \beta > 0$. This follows from convexity of functions $M(x)$, $Q(x)$. For example assume that P has a continuous density, i.e. M(x), Q(x) are continuously differentiable functions. Then intuitively we have

$$Q'(x) = \alpha \Pr ob[\theta < x] - \beta \Pr ob[\theta \geq x] = 0$$

i.e., a solution x of stochastic minimax problem (2) satisfies the equation [12], [15], page 95, [26], [36]:

$$\Pr ob[\theta \geq x] = q\,,\; q = \frac{\alpha}{\alpha+\beta}\,. \tag{3}$$

Remark 1 (Uniqueness of quantile). If $Q(x)$ is not a continuously differentiable function, then optimality conditions satisfy analogue of (3) equations using subgradients [12] of function (2). In this case, equation (3) has a set of solutions. Quantile x_q is defined as minimal *x* satisfying equation $\Pr ob[\theta \geq x] \leq q$. A slight contamination of θ in (2), say by normal random variable, $(1-\varepsilon)\theta + \varepsilon N(0,1)$, makes $Q(x)$ strongly convex and continuously differentiable function [13]. The convergence of resulting quantile x_q^{ε} to x_q follows from the monotonicity of x_q^{ε}, that is $x_q^{\varepsilon_2} < x_q^{\varepsilon_1}$ for $\varepsilon_2 < \varepsilon_1$. Therefore, in the following we avoid using subgradients by assuming that equation (3) has a unique solution. For $\alpha = \beta$ equation (3) defines the median.

Remark 2 (Equivalent calculations of quantiles). It is easy to see that $Q(x) = \alpha x + (\alpha + \beta) E \max\{0, \theta - x\} - \alpha E\theta$. Therefore, x_q minimizes also function

$$x + (1/q) E \max\{0, \theta - x\}, \ q = \frac{\alpha}{\alpha + \beta}. \tag{4}$$

This simple rearrangement is used in section 3 to formulate robust STO decision support models applicable for security management. Formula (3) connects quantiles with a simple convex STO model (2). This became a key approach in risk management because direct use of quantiles destroys continuity of even linear performance indicators [14], page 9.

Problems (1), (2) are simplest examples of STO models. Model (2) is an example of important stochastic minimax problems arising in the security analysis (section 4). Equation (3) shows that even the simplest case of such problems generates robust solutions characterized by quantiles. In general decision models under uncertainty, any relevant decision x results in multiple outcomes dependent on *x* and uncertainty characterized by a scenario (event, random vector) $\omega \in \Omega$, where Ω denotes a set of admissible states ω. For complex systems it is natural that different performance indicators should be used (see, e.g., [9], [11], [22]) to evaluate robustness of x similar to the use of different indicators of health (e.g., temperature and blood pressure) for humans. This leads to STO models formulated as optimization (maximization or minimization) of an expectation function

$$F_0(x) = Ef_0(x,\omega) = \int_{\Omega} f_0(x,\omega) P(d\omega) \tag{5}$$

subject to constraints

$$F_i(x) = Ef_i(x,\omega) = \int_{\Omega} f_i(x,\omega) P(d\omega) \geq 0, \ i = 1,...,m, \tag{6}$$

where vector $x \in X \subseteq R^n$ and ω in general represent decisions and uncertainties in time $t = 0,1,...$, i.e., $x = (x(0), x(1),...)$, $\omega = (\omega(0), \omega(1),...)$. Models with ex-ante and ex-post time dependent decisions can be always formulated [14], page 16, in terms of the first stage solutions x as in (5), (6). Therefore, model of type (5), (6) allows to assess multi-stage dynamic trade-offs between anticipative ex-ante and adaptive ex-post decisions and the learning (section 7.2) arising in security management (section 6). Random performance indicators $f_i(x,\omega)$, $i = \overline{0,m}$, are often non-smooth functions as in (2). In the case of discontinuous functions $f_i(x,\omega)$, expected values $F_i(x)$ of constraints (6) characterize often risks of different parts $\overline{1,m}$ of the system [3], [13], [14], [28], [37] in the form of the chance constraints: $\Pr ob[f_i(x,\omega) \geq 0] \geq p_i$, $i = \overline{1,m}$, where p_i is a desirable level of safety. Say, an insolvency of insurers is regulated with $1 - p_i \approx 8*10^{-2}$, meaning that balances (risk reserves) may be violated only once in 800 years. In models presented in [9] these type constraints characterize a dynamic systemic risk of systems composed of individuals, insurers, governments, and investors.

Remark 3 (Scenario analysis). It is often used as a straightforward attempt to find a decision x that is robust with respect to all scenarios ω by maximizing $f_0(x,\omega)$, s.t. $f_i(x,\omega) \geq 0$, $i = 1,...,m$, for each possible scenarios ω. Unfortunately, a given decision x for different scenarios ω may have rather contradictory outcomes, which do not really tell us which decision is reasonable good (robust) for all of them. For example, models (1), (2) show that for any scenario ω the optimal solution is $x(\omega) = \omega$, i.e., the scenario-by-scenario analysis will not suggest solutions in the form of quantile (3). This straightforward scenario analysis faces computational limits even for very small number of examined decisions and scenarios, e.g., analysis of all combinations of 10 scenarios and 10 different decisions may easy require 10^{10} sec. > 100 years.

Models (1), (2) illustrate the main specifics of STO problems of the following sections. Objective functions (1), (2) are analytically intractable because in statistics the probability distribution P is unknown. Instead only observations of ω are available. Analytical intractability of functions $F_i(x)$ is a common feature of STO models. For example, even a sum of two random variables commonly has analytically intractable probability distribution although distributions of both variables are given analytically. Therefore, the main issue of this paper is the development of effective “distribution-free” methods applicable for different type of distributions [3], [14], [28], [37] and large number of decision variables and uncertainties (section 7).

Remark 4 (Uncertain probabilities, Bayesian and non-Bayesian models). The standard stochastic optimization model (5), (6) is characterized by a single probability distribution P, therefore can be defined as Bayesian STO model. When observations are extremely sparse or not available distribution P is elicited from experts [23], [33]. Yet, often it is difficult to identify uniquely probability P. Most people cannot clearly distinguish between probability ranging roughly from 0.3 to 0.5. Decision analysis then has to rely on imprecise statements, for example, that event e_1 is more probable than event e_2 or that the probability of event e_1 or of event e_2 is greater than 50% and less than 90%. Therefore only feasible sets of probabilities are identified by inequalities such as $p_1 > p_2$, $0.5 \le p_1 + p_2 \le 0.9$. It is typical for models arising in security management (sections 4, 5). In such cases we may speak of non-Bayesian STO models., i.e. STO models which are not defined by a single probability distribution, but by a family of distributions with uncertain parameters or, more generally, by an uncertain distribution. Probability distributions depending on decisions are discussed in subsection 3.3.

3. Catastrophic and systemic risks

Standard "known" risks are characterized by a single probability distribution that can be derived from repetitive observations of ω. The essential new feature of catastrophic risks is the lack and even absence of real repetitive observations. Experiments may be expensive, dangerous, or impossible. The same catastrophe never strikes twice the same place. In addition, catastrophes affect different location and agents generating multivariate risks and needs for developing new STO models integrating risk reductions, risk transfers and risk sharing [9].

As a substitute of real observations, so-called catastrophe modeling (catastrophe generators) is becoming increasingly important for estimating spatio-temporal hazard exposures and potential catastrophic impacts. The designing of a catastrophe model is a multidisciplinary task. To characterize "unknown" catastrophic risks, that is, risks with the lack of repetitive real observations we should at least characterize the random patterns of possible disasters, their geographical locations, and their timing. We should also design a map of values and characterize the vulnerabilities of buildings, constructions, infrastructure, and activities. The resulting catastrophe model allows deriving histograms of mutually dependent losses for a single location, a particular zone, a country, or worldwide from fast Monte-Carlo simulations rather than real observations [9], [38]

3.1. Applicability of mean values, systemic risk. The use of different sources of information, including often rather contradictory expert opinions usually leads to multimodal distributions of ω and random indicators $f_i(x,\omega)$. The mean value of such indicator can be even outside the set of admissible values requiring the use of quantile, e.g., the median of $f_i(x,\omega)$. Unfortunately, the straightforward use of quantiles destroys the additive structure and concavity (convexity) of model (5), (6), even for linear functions $f_i(\cdot,\omega)$ because, in contrast to the mean value $quantile \sum_i f_i \ne \sum_i quantile(f_i)$. This lack of additivity makes it practically impossible to use many computational methods relying on additive structure of models, e.g., dynamic programming equations and Pontryagin's maximum principle.

Equations (3), (4) allow the following promising approach for using quantiles. Let us denote a quantile of $f_i(x,\omega)$ by $Q_i(x)$, $i = 0,1,...,m$. Then we can formulate the following robust version of STO model (5)-(6): maximize

$$Q_0(x) + \mu_0 E \min\{0, f_0(x,\omega) - Q_0(x)\}$$

subject to

$$Q_i(x) + \mu_i E \min\{0, f_i(x,\omega) - Q_i(x)\} \geq 0, \; i = 1,...,m,$$

where $\mu_i > 1$ are risk parameters regulating potential variability of $f_i(x,\omega)$ below $Q_i(x)$, $i = 0,1,...,m$. Unfortunately the direct use of $Q_i(x)$ destroys concavity of functions $F_i(x)$. This can be avoided by the following reformulation of the problem. According to model (2), equation (3) and Remark 2, the formulated above robust version of STO model (5), (6) can be equivalently rewritten in a similar to (4) form: maximize w.r.t. (z, x) function

$$z_0 + \mu_0 E \min\{0, f_0(x,\omega) - z_0\}, \tag{7}$$

subject to

$$z_i + \mu_i E \min\{0, f_i(x,\omega) - z_i\} \geq 0, \; i = 1,...,m. \tag{8}$$

For concave functions $f_i(\cdot,\omega)$ this is a concave STO model. The following Proposition 1 shows, that components $z_i^*(x)$, $i = 0,1,...,m$, solving (7), (8) w.r.t. *z* = (z_0, z_1, …, z_m) are quantiles $Q_i(x)$. Therefore, (7), (8) is a robust version of model (5), (6) where mean values Ef_i are substituted by quantiles of indicators f_i with a safety levels μ_i controlling their variability. In a sense, the model (7), (8) can also be viewed as a concave version of STO models with probabilistic safety constraints [3], [11], [14], [28] outlined in section 2. Equation (9) shows that model (7), (8) is defined by multicriteria versions of VaR and CVaR risk measures [36] controlling safety/security of overall system, i.e., a systemic risk. An alternative formulation of quantile optimization problems (subject to quantile constraints) and a corresponding mixed-integer programming solution technique is considered in [32].

Proposition 1 (Quantiles of $f_i(x,\omega)$): Assume $f_i(x,\cdot)$, $i = 0,1,...,m$, have continuous densities (Remark 1); $\mu_i > 1$, (z^*, x^*) is a solution of model (7), (8) and $\lambda^* = (\lambda_1^*,...,\lambda_m^*) \geq 0$ is a dual solution. Then for *i* = 0 and active constraints $i = \overline{1,m}$,

$$\mathrm{Prob}[f_i(x^*,\omega) \leq z_i^*] = 1/\mu_i, \; i = 0,1,...,m. \tag{9}$$

Proof: Let $\varphi_i(z_i, x, \omega) := z_i + \mu_i \min\{0, f_i(x,\omega) - z_i\}$. From the duality theory follows that z_i^* maximizes

$$E\varphi_0(z_0, x^*, \omega) + \sum_{i=1}^{m} \lambda_i^* E\varphi_i(z_i, x^*, \omega) .$$

Thus, if $\lambda_i^* > 0$, $i = 1,...,m$ then z_i^* maximizes $E\varphi_i(z_i, x^*, \omega)$. Therefore, from Remark 2 follows equation (9) for $i = 1,...,m$. Equation (9) for $i = 0$ follows from the complementary condition $\sum_{i=1}^{m} \lambda_i^* E\varphi_i(z_i^*, x, \omega) = 0$ and formula (3).

Let us also not that the variability of outcomes $f_i(x, \omega)$ can be controlled by using a vector of quantiles $z^i = (z_{i0}, z_{i1}, ..., z_{il})$ generated as in (7)-(8) by performance indicators $\sum_l (z_{il} + \mu_{il} \min\{0, f_i(x, \omega) - z_{il}\})$, $i = \overline{0, m}$, where $1 < \mu_{i1} < \mu_{i2} < \dots$.

3.2. Distributional heterogeneities. Extreme events. The following simple examples illustrate critical importance of quantiles to represent distributional characteristics of performance indicators.

Example 1 (Annualization, temporal heterogeneity). Extreme events are usually characterized by their expected arrival time say as a 200-year flood, that is, an event that occurs on average once in 200 years. Methodologically, this view is supported by so-called annualization, i.e., by spreading losses from a potential, say, 30-year crash of airplane, equally over 30 years. In this case, roughly speaking, the crash risk is evaluated as a sequence of independent annual crashes: one wheel in the first year, another wheel in the second year, and so on, until the final crash of the navigation system in the 30th year. The main conclusion from this type of deterministic mean value analysis is that catastrophes are not a matter although they occur as random “explosions” in time and space that may destabilize a system for a long time.

Example 2 (Collective losses). A key issue is the use of proper indicators for collective losses. In a sense, we often have to show that $100 >> \overbrace{1+1+...+1}^{100}$. Assume that each of 100 locations has an asset of the same type. An extreme event destroys all of them at once with probability 1/100. Consider also a situation without the extreme event, but with each asset still being destroyed independently with the same probability 1/100. From an individual point of view, these two situations are identical: an asset is destroyed with probability 1/100, i.e., individual losses are the same. Collective (social) losses are dramatically different. In the first case 100 assets are destroyed with probability 1/100, whereas in the second case 100 assets are destroyed with probability 100^{-100}, which is practically 0. This example also bears on individual versus systemic (collective) risk, risk sharing and the possibility to establish a mutuality.

Model (7), (8) allows to analyze properly risk sharing portfolios involving both type of situations. In Example 2 the standard worst case scenario is identical for both situations, that is losses of

100 assets. Stochastic worst case scenario as in stochastic maximin problems (16) of section 4.2 is determined only by extreme events, i.e., losses of 100 assets with probability 1/100.

3.3. Unknown risks. A fundamental methodological challenge in dealing with catastrophic risks is the endogenous character of catastrophes. Catastrophic losses occur often due to inappropriate land use planning and maintenance of engineering standards. In these cases functions $F_i(x)$ in (5) – (6) have the following structure:

$$F_i(x) = \int f_i(x,\omega)P(x,d\theta),\ i = 0,1,...,m .$$

In other words, there is no single probability distribution defining the structure of functions $F_i(x)$ for all x. Instead, there are probability distributions $P(x,d\theta)$, which are different for different decisions x. Therefore, this is a non-Bayesian STO model (Remark 4). Usually probability distribution $P(x,d\theta)$ is given implicitly by a Monte Carlo type simulations, which allow to observe in general only values of random functions $f_i(x,\omega)$ for a given x (section 7.1). The decision dependent measure $P(x,d\omega)$ may easily overthrow convexity. Fortunately, this is not the case with decision dependent measure defined as in (15) of section 4.2.

4. Security management: principal agent problem

Security management has essentially a multi-agent domain. The main source of uncertainty and risks is associated with behavioral patterns of agents motivated often and shaped by other uncertainties. In contrast to “unknown” risks of section 3 which can be characterized by catastrophe models, security management deals in a sense with “unknowable” risks dependent on decisions of agents. This section analyzes two ways to represent behavioral uncertainties: game theoretic and decision theoretic approaches.

4.1. Game theoretic approach. The search for proper regulations protecting public goods is often formulated as the principal-agent problem [2], [20], [29] or Stackelberg game [34], [35]. Important issues concerns nonmarket institutions [1]. In rather general terms the problem is summarized as the following. The principal agent (PA) introduces a regulatory policy characterized by a vector of decision variables $x = (x_1,...,x_n)$. Other agents, which are often called adversaries, know x and they commit to a unique response characterized by a vector function $y(x)$. The PA knows $y(x)$ and he knows that agents commit to $y(x)$. Therefore his main problem is formulated as to find a decision x^* maximizing an objective function

$$R(x, y(x)) \tag{10}$$

subject to some constraints given by a vector-function $r(x, y)$,

$$r(x, y(x)) \geq 0. \tag{11}$$

The game theoretic approach assumes that components of the vector-function $y(x)$ maximize individual objective functions of agents

$$A(x, y) \tag{12}$$

subject to their individual feasibility constraints

$$a(x, y) \geq 0, \tag{13}$$

where A, a are in general vector-functions, i.e., in general, there may be many principals and agents. For the sake of notational simplicity, we will view them as single-valued functions. Since PA knows functions A, a, he can derive responses $y(x)$ by solving agents individual optimization problems. Since $y(x)$ is assumed to be a unique solution, then agents have strong incentive to choose $y(x^*)$ afterwards, i.e., x^* is the Stackelberg equilibrium.

This approach relies on the strong assumptions of perfect information that the PA has about the preference structure of agents and their commitments to a unique response $y(x)$. Section 5 shows that $R(x, y(x))$, $r(x, y(x))$ are non-convex and discontinuous functions even for linear functions $R(x, y)$, $r(x, y)$, $A(x, y)$, $a(x, y)$. This leads to degenerated solutions and sensitivity of solutions to small variations of data.

Remark 5 (bi-level mathematical programming). A solution procedure for PA can be defined by solving bi-level mathematical programs [6]:

$$\text{maximize } R(x, y) \tag{14}$$

subject to constraints $r(x, y) \geq 0$ and optimality conditions (for a given x) for all individual models (12), (13).

Example 3 (Bayesian games: Cournot duopoly). These games deal with situations in which some agents have private information. Therefore, agents make decisions relying on their beliefs about each other under certain consistency assumptions. The following example illustrates these assumptions restricting the applicability of Bayesian games for PA models.

The profit function of two firms are given as

$$\pi_i = (x_i + x_j - \omega_i)x_i \,,\ i \neq j \,,\ i, j = 1,2 \,.$$

Firm 1 has $\omega_1 = 1$, but firm 2 has private information about ω_2. Firm 1 believes that $\omega_2 = \alpha$ with probability p and $\omega_2 = \beta$ with probability $1 - p$. Decision problem of firm 2 is to

$$\max_{x_2} (x_2 + x_1 - \omega_2) x_2 \,,$$

which has solution $x_2^*(x_1, \omega_2) = \frac{1}{2}(\omega_2 - x_1)$. Assume that firm 1 knows response function $x_2^*(x_1, \omega_2)$, then its decision problem is to

$$\max_{x_1} [\, p(x_1 + x_2^*(x_1, \alpha) - 1)x_1 + (1 - p)(x_1 + x_2^*(x_1, \beta) - 1)x_2 \,,$$

which has solution $x_1^*(\alpha, \beta, p)$ dependent on α, β, p. Assume that the private information of firm 2 is consistent with the believe of firm 1: firm 2 is type $\omega_2 = \alpha$ (observes $\omega_2 = \alpha$ before making decisions) with probability p and $\omega_2 = \beta$ with probability $1 - p$. Only then firm 2 (agent) has incentives to use decisions $x_2^*(x_1^*, \alpha)$, $x_2^*(x_1^*, \beta)$. Therefore, the Bayesian games are applicable in the cases when firm 1 (PA) exactly knows the unique response function $x_2^*(x_1, \omega_2)$ of firm 2 (agent) and the exact distribution of agent's uncertainties ω_2. For general model (10)-(13) Bayesian games require exact information about dependencies of functions A, a on uncertainties ω (say, functions $A(x, y, \omega)$, $a(x, y, \omega)$) and probability distribution of ω, assuming also a unique response function $y(x, \omega)$ solving problem (12), (13).

4.2. Decision-theoretic approach. The game theoretic approach introduces behavioral scenarios of agents by uniquely defined known response functions $y(x)$. This raises a key issue regarding actual outcomes of derived solutions in the presence of uncertainty. The decision-theoretic approach explicitly addresses uncertainty based on PA's perceptions of agents behavioral scenarios. These scenarios can be represented (see examples in sections 5, 6) either by a set Π of mixed strategies $\pi \in \Pi$ defined on a set of pure strategies Y, or by a set Y of pure strategies $y \in Y$. This leads then to two classes of STO models.

Probabilistic maximin models associate robust solutions with distributions characterizing desirable indicators (say, social welfare function) over the worst that may happen from $\pi \in \Pi$, i.e., of the form:

$$F(x) = \min_{\pi \in \Pi} E \int f(x, y, \omega) \pi(dy), \qquad (15)$$

for some random function $f(x, y, \omega)$, where ω is an exogenous uncertainty.

Stochastic maximin models of the type (2) associate the robustness with respect to the worst-case random events generated by $y \in Y$:

$$F(x) = E \min_{y \in Y} f(x, y, \omega). \qquad (16)$$

where *Y* may depend on x, ω.

Remark 6 (Extreme events and robust statistics). Extreme values (events) theory analyses distributions of minimax (maximin) $M_n = \min(\xi_1, ..., \xi_n)$, where $\xi_1, ..., \xi_n$ is a sequence of identically distributed independent random variables [7]. The model (16) has connections with this theory: it focuses on random events generated by extreme values $\min_{y \in Y} f(x, y, \omega)$ with respect to scenarios $y \in Y$. In other words, (16) can be viewed as a decision oriented analogue of the extreme events models with mutually dependent multivariate endogenous (dependent on decision variables x) extreme events. The use of expected values in (16) may not be appropriate, i.e., (16) has to be modified as (7)-(8). Probabilistic maximin model (15) corresponds to minimax approaches introduced by Huber in robust statistics. The integral (15) with respect to an extreme measure $\Pi(x, dy)$ indicates links to Choquet integrals used also by Huber for simple sets Π of imprecise probabilities. The key issue is a proper representation of Π, that is discussed in section 7.

Decision theoretic approaches aim to address uncertainties of agents responses $y(x)$. Namely, assumptions of game theoretic approach:

- agents commit to a unique $y(x)$,
- PA knows $y(x)$ and the commitments of agents and, hence, chooses x maximizing function (10)

are substituted by assumptions about the PA perception of agents scenarios. For example, the PA may use his perceptions $A(x, y, \omega)$, $a(x, y, \omega)$ of real functions $A(x, y)$, $a(x, y)$ "contaminated" by uncertain parameters ω. In this case random sets of agents scenarios $Y(x, \omega)$ can be defined as

$Y(x,\omega)=\{y: a(x,y,\omega)\geq 0\}$.

In other cases [39] these sets can be characterized by experts opinions combined with probabilistic inversions. The overall decision problem is formulated as multicriteria (multi-objective) STO problem with random functions $R(x,y,\omega)$, $A(x,y,\omega)$, $r(x,y,\omega)$, $a(x,y,\omega)$. For example, it can be formulated as the maximization of function $F(x)=E\min_{y\in Y(x,\omega)} R(x,y,\omega)/A(x,y,\omega)$ or $F(x)=E\min_{y\in Y(x,\omega)} [R(x,y,\omega)-A(x,y,\omega)]$ under constraints defined by functions $r(x,y,\omega)$. This leads to stochastic maximin models (16). In general, function $F(x)$ may have the form $F(x)=E\min_{y\in Y(x,\omega)} \varphi(A,R,x,y,\omega)$ for some function φ , e.g., a welfare function $\varphi=\delta A+(1-\delta)R$, $0<\delta<1$ with economic perspectives of welfare analysis regarding possible transferable utilities, side payments, contracts, contingent claims. Definitely, in these cases insurance and finance supplement the safety measures and may mitigate many related problems besides prevention[1].

5. Systemic security

Under increasing interdependencies of globalization processes the protection of public goods is becoming a critical topic, especially against uncertain threats generated by agents. In rather general terms such problems can be formulated by using “defender-attacker” terminology. The agents can be intentional attackers such as terrorists, or agents generating extreme events such as electricity outage, oil spills, or floods by the lack of proper regulations, e.g., land use planning. The main issues in these cases concern coping with extreme events generated by agents directly and indirectly through cascading systemic failures. As a result, the security of the whole system can be achieved only by coordinated security management of all its interconnected subsystems, i.e., the systemic security management. In general, arising complex interdependent problems require developing new specific models and methods. This section and section 6 discuss some related issues.

5.1. Preventive randomized solutions. This section analyzes situations requiring solutions in randomized strategies as in probabilistic maximin model (15). The simplicity of selected model allows easy to illustrate specifics of both game theoretic and decision theoretic approaches.

The following model is a simplified version of the model analyzed in [34]. Consider a PA (defender) providing civil security say to houses $i=\overline{1,n}$ to prevent an attack (robbery). A pure strategy i is to visit a house i , whereas x_i is portion of times the pure strategy i is used in overall security control policy $x=(x_1,...,x_n)$, $\sum_i x_i=1$, $x_i\geq 0$. It is assumed that the agent (attacker) knows randomized strategy x and commits to a randomized strategy $y(x)=(y_1(x),...,y_n(x))$ maximizing his expected rewards:

[1] We thank our anonymous reviewer for pointing on these issues.

$$A(x,y)=\sum_{i,j} r_{ij}x_i y_j\ ,\ \sum_j y_j=1\ ,\ y_j\geq 0\ ,\ j=\overline{1,n}\ , \tag{17}$$

assuming that the response $y(x)$ is a unique vector-function. Since PA knows the agent's commitment to $y(x)$, the PA maximizes his expected rewards

$$R(x,y(x))=\sum_{i,j} R_{ij}x_i y_j(x)\ ,\ \sum_i x_i=1\ ,\ x_i\geq 0\ ,\ i=\overline{1,n}\ . \tag{18}$$

The randomized strategy x definitely increases the security of the PA. At the same time, the randomized strategy y increases uncertainty about the agent.

A discontinuity of $R(x)$ can be easily seen for $n=2$, $i=1,2$. The response function $y(x)=(y_1(x),y_2(x))$ maximizes $(r_{11}x_1+r_{21}x_2)y_1+(r_{12}x_1+r_{22}x_2)y_2$, $y_1+y_2=1$, $y_1,y_2\geq 0$, and it has the following simple structure. Let $\alpha=(r_{22}-r_{21})/(r_{11}-r_{12})$, then

$$\left.\begin{aligned} &y_1(x)=1, y_2(x)=0,\ for\ \ x_1<\alpha x_2\\ &y_1(x)=0, y_2(x)=1,\ \ otherwise.\end{aligned}\right\}, \tag{19}$$

i.e., $R(x,y(x))$ is a discontinuous function on the line $x_1=\alpha x_2$:

$$R(x,y(x))=\begin{cases} R_{11}x_1+R_{21}x_2, & for\ \ x_1>\alpha x_2,\\ R_{12}x_1+R_{22}x_2, & for\ \ x_1<\alpha x_2.\end{cases}$$

The deterministic game theoretic model (17), (18) relies strongly on perfect information about randomized strategies x, y. As a result $y(x)$ attains degenerated 0-1 values. It is natural to expect that formulations which take into account uncertainties will lead to more reasonable solutions. Consider first a straightforward generalization of model (17), (18). Instead of deterministic r_{ij}, let us assume that the PA perceives agent's rewards as random variables $r_{ij}(\omega)$ defined on a set Ω of admissible probabilistic scenarios ω. In general, $\{r_{ij}(\omega)\}$ is a random matrix of interdependent variables. The PA uses now his perception of the agent model and can derive agent's random response function $y(x,\omega)$ by maximizing with respect to y

$$A(x,y,\omega)=\sum_{i,j} r_{ij}(\omega)x_i y_j\ ,\ \sum_j y_j=1\ ,\ y_j\geq 0\ ,\ j=\overline{1,n}\ . \tag{20}$$

Assuming that the PA still follows exactly the logic of model (10), (11), i.e. PA maximizes now the expected value

$$\overline{R}(x) = E \sum_{i,j} R_{ij} x_i y_j (x, \omega) , \tag{21}$$

where for the simplicity of illustration we assume that $\{R_{ij}\}$ is a deterministic matrix. It is easy to see that this formal introduction of uncertainty into the game-theoretic model already smoothes function $R(x, y(x))$. Consider random variable $\alpha(\omega) = (r_{22}(\omega) - r_{21}(\omega))/(r_{11}(\omega) - r_{12}(\omega))$, then similar to (19):

$$y_1(x,\omega) = 1,\ y_2(x,\omega) = 0 \text{ with } \Pr ob[\alpha(\omega) > x_1 / x_2],$$

$$y_1(x,\omega) = 0,\ y_2(x,\omega) = 1 \text{ with } \Pr ob[\alpha(\omega) \le x_1 / x_2].$$

Therefore,

$$\overline{R}(x) = (R_{11}x_1 + R_{21}x_2) \Pr ob[\alpha(\omega) > x_1 / x_2] + (R_{12}x_1 + R_{22}x_2) \Pr ob[\alpha(\omega) \le x_1 / x_2].$$

Remark 7 (Non-concave and discontinuous models). If distribution of $\alpha(\omega)$ has a continuous density, then $\overline{R}(x)$ is a continuous but, in general, non-concave function. Otherwise, $\overline{R}(x)$ is again a discontinuous function purely due to the structure of the Stackelberg models, that is, in fact, meaningful only under perfect information about commitments of agents to $y(x,\omega)$.

Thus, the game theoretic approach orients PA decisions on unique best-case scenarios $y(x)$ or $y(x,\omega)$ from agents' perspectives, whereas the decision theoretic approach orients decisions on extreme random scenarios of agents from PA perspectives. In particular, the PA can take position to oppose the agent's interests, i.e., to view perceived rewards $A(x, y, \omega)$ as his losses. Therefore, the PA decision model can be formulated as the following stochastic maximin model: maximize

$$F(x) = E \min_{y \in Y} f(x, y, \omega) ,\ x \in X ,$$

where $f(x,y,\omega)=R(x,y,\omega)-A(x,y,\omega)$, $X=\left\{x\geq 0:\sum_i x_i=1\right\}$, $Y=\left\{y\geq 0:\sum_i y_i=1\right\}$.

In general cases X and Y may reflect various additional feasibility constraints of agents. For example, Y may represent prior information in the form of such comparative statements as the following: the agent plans to visit i more probably then j, $y_i \geq y_j$; or the probability to visit objects i, k, l is higher then objects k, m, n, s, t, i.e., $y_i+y_k+y_l\geq y_m+y_n+y_s+y_t$, etc. Sets X, Y may include also budget constraints. In particular, if c_i is the cost per visit of location i, then the total costs should not exceed a given budget C, $\sum_i c_i x_i \leq C$.

Example 4 (uncertain distributions). It is essential that decision theoretic models can be formulated in a different case-dependent manner. Consider an important situation. Practically, the PA observes results of random trials i, j from randomized strategies x, y and he can see whether $i=j$, or not. If the information about rewards is not available, then the PA problem can be formulated as finding randomized strategy $x=(x_1,...,x_n)$ that "matches" feasible randomized strategy $y=(y_1,...,y_n)$ of the agent as much as it is possible. In this case, a rather natural way to derive optimal randomized strategy x is by minimizing the function

$$\max_{y\in Y}\sum_i x_i \ln\frac{x_i}{y_i},\ x\in X,$$

where $\sum_i x_i \ln\frac{x_i}{y_i}$ defines the Kullback-Leibler distance between distributions x and y. This distance is a concave in x and a convex in y function. A simple effective solution procedures similar in spirit to sequential downscaling methods [17] can be developed in the case of sets X, Y defined by linear constraints.

5.2. Defensive resource allocation. A problem of resource allocation for protecting public goods against attackers is demonstrated in [39] as an application of the stochastic minimax model (16). A typical setting is that the PA (defender) wants to minimize the perceived payoffs to the agents (attackers). In the following we shortly summarize this study advanced during 2010 IIASA's Young Scientists Summer Program.

Suppose the defender is faced with potential attacks on a collection of targets (e.g., cities, critical infrastructures, public transportation systems, financial systems, energy or food supply systems, and etc.). The defender's objective is to minimize the consequences from attacker choices. A Stackelberg game is usually used to model this situation when there is no uncertainty about the attacker preferences. In reality, the attacker's preferences are not fully known to the defender. In the face of such uncertainty, the defender cannot predict the attacker's best response for sure; therefore, a STO model is needed to minimize the perceived total consequences.

For simplicity, suppose the defender is faced with one attacker, whose decision is to choose a target i among n targets with the highest payoff to attack. The defender objective is to minimize

$$E \max_i g_i(x,\omega)$$

where $x \in X$ is the defensive resource allocation decision among targets under a budget constraint

$$X = \{x \in R^n \mid \sum_{i=1}^{n} x_i \le B, x_i \ge 0, i = 1,..., n\}$$

for some $B > 0$, $g_i(x,\omega)$ is the perception of attacker utility function on each target. Therefore, this model focuses on extreme attacks (events) maximizing perceived utility of attackers (see also Remark 6). In general, this model also considers the interdependencies between multiple targets and agents if the agent's utility functions depend on all components of x, ω. In particular, $g_i(x,\omega) = p(x_i)u_i(\omega)$ is a product of target vulnerability (success probability)

$$p(x_i) = e^{-\lambda_i x_i}$$

and the attack consequence

$$u_i(\omega) = \sum_{j=1}^{m-1} w_j A_{ij} + w_m \varepsilon_i .$$

Note that in this model $\omega = (w_1,..., w_m, \varepsilon_1,..., \varepsilon_n)$ is a random vector representing all uncertain parameters in the attacker's utility function, λ_i is the cost effectiveness of defensive investment on target i. For example, at the cost effectiveness level of 0.02, if the investment is measured in millions of dollars, then every million dollars of defensive investment will reduce the success probability of an attack by about 2%.

It is assumed that consequences are valued by the attacker according to a multi-attribute utility function with m attributes (of which $m-1$ are assumed to be observable by the defender). A_{ij} is attacker utility of target i on the j th attribute, where A_{ij} takes values in [0,1], with 1 representing the best possible value and 0 the worst, ε_i is utility of the unknown (by the defender) m th attribute of target i, $(w_1,..., w_m)$ are weights on the m attributes, where $\sum_{j=1}^{m} w_j = 1$ and $w_j \ge 0, j = 1,..., m$.

The inherent and deep uncertainty about agent behaviors is critical to models of protecting public goods. Solutions obtained in a deterministic model are usually unstable to even a subtle change in the agent parameters. The STO models are developed for robust solutions against such uncertainties. Therefore, quantifying uncertainty becomes an important task to provide input for the STO models. When direct judgments on the uncertain parameters ω are available, the uncertainties can be quantified directly through probability distributions or simulated scenarios.

However, in some cases direct judgments are not available. For example, in the case study of defensive resource allocations against intentional attacks [39] available are only expert opinions about attacker's ranking of cities (targets i). Therefore, so-called probabilistic inversion is used to simulate scenarios about attribute weights and unobserved attributes $\omega = (w_1,...,w_m, \varepsilon_1,...,\varepsilon_n)$. In other words, if there are expert opinions on attacker rankings of potential targets, it is possible to probabilistically invert subjective distributions (as simulated scenarios) on the relative importance of targets attributes (e.g., expected loss or profits from attacks, population, national icon, difficulty of launching an attack, and etc), and even the characteristics of unknown attributes.

5.3. Systemic failures and damages. The model of this section can be used as a module of systemic security management model. The main issues concerns the following. Such an "attack" involving different agents as catastrophic flood, financial melt-down, oil spill, or terrorists strikes may have direct and indirect long-term consequences with cascading failures and damages. Example 2 illustrates a vital importance of systemic damages distributions of which may significantly exceed the sum of isolated damages of related subsystems.

The development of an appropriate model reflecting dependencies among failures, damages, and decisions of different subsystems requires special attention. An attack may produce a chain of indirect damages. For example, a rain affects simultaneously different locations of a region and may cause landslides and formation of damps, lakes; overfilling and breakdowns of dams may further cause floods, fires, and destruction of buildings, communication networks, and transportation systems. Fires may affect computer networks and destroy important information, etc. A failure in a peripheral power grid and financial organization may trigger cascading failures with catastrophic systemic outages and global financial crisis. The indirect losses can even significantly exceed direct impacts. Therefore it is important to develop a model capable of analyzing the propagation of failures through the system and their total direct and indirect impacts. In the following a simple model is described, which is related to notions such as random fields and Bayesian nets. Versions of this model have been used in studies of catastrophic risks at IIASA. The model distinguishes N subsystems or elements (buildings, infrastructures, locations, agents, etc.) $l = 1,...,N$ of a system (region). Possible damage at each l is characterized by random variable ς_l assuming M levels: for sake of simplicity $1,2,...,M$. Hence damages of the system are described by the random vector $\varsigma = (\varsigma_1,...,\varsigma_N)$. A fixed value of this vector is denoted by z and the set of all possible damages by Z. Let us denote by p_k^{lt} the probability that the damage at l is equal *k* at time t, $\sum_{k=1}^{M} p_k^{lt} = 1$, $p_k^{lt} \geq 0$. Dependencies between subsystems are represented as a graph, where elements $i = 1,...,N$ are nodes of the graph and links between them are represented by arrows between nodes. The dependency graph $G = (V,U)$ is characterized then by the set of nodes $V = \{1,2,...,N\}$ and the set of arrows (directed arcs) U. If nodes l,s belong to V, $l,s \in V$, and there is an arrow from l to s, then l is an adjacent to s node. Define as V_s the set of all adjacent to s nodes and z_{V_l} is sub-vector of the vector of damages indexed by V_l. For example, $z_{V_l} = (z_2, z_5)$ for $V_l = (2,5)$.

Damages z_l are described by a conditional probability $H^l(z_l \mid z_{V_l}, x)$, i.e., damages at l depend on current values of damages at *l* and adjacent nodes as a function of available mitigation measures x. Let this function is known for each l. This is a common assumption of catastrophe modeling (section 3). Say, probability of a dam break is conditional on probabilities of potential discharge curves; the probability of inundation is conditional on a dike break; damages of buildings and other constructions are conditional on inundation patterns, and so on. In the same manner we can model, say, financial crisis spreading through regions. Functions H^l define the propagation of indirect events and related damages through the system according to the following relation

$$p_k^{l,t+1} = \sum_{z_{V_l} \in Z} H^l(\varsigma_l^t = k \big| \varsigma_{V_l}^{t-1} = z_{V_l}, x) P(\varsigma_{V_l}^{t-1} = z_{V_l}),$$

where $p_k^{l,t} = P(\varsigma_l^t = k)$. To define completely the propagation of failures and damages it is necessary to fix an initial distribution of ς_l^t for $t = 0$, i.e., at the moment when the attack occurred. This equation together with initial distribution allow the exact calculation (under certain assumptions on the structure of graph *G*) of $p_k^{l,t}$ for any $t \geq 0$. Of course, for complex graphs it is practically impossible to derive analytical formulas for $p_k^{l,t}$ as functions of decision variables x. Hence the damages may have rather complex dynamic implicit dependencies on decision vector x requiring developments of specific decision support tools. The most important approaches have to rely on STO in combination with fast Monte Carlo simulation as in section 7.1. Paper [39] reports on computational effectiveness of these methods for realistic problems of security management with very large number of simulated scenarios and two-stage decision variables required for coping with extreme events. The values $p_k^{l,t}$ reflect the dynamic of propagation of initial (direct) damages through the system after the occurrence of an attack. Scenarios of damages can be simulated at any $t \geq 0$. For example, $t = 0$ corresponds to the distribution of direct damages.

6. Security of electricity networks

This section presents a decision theoretic model for regulating electricity markets (networks). The California energy crises in 2001 and the collapse of ENRON raised serious concerns about proper regulation of the market power, that is, the ability of electricity suppliers to raise prices above competitive levels for a significant period of time. This is considered as a major obstacle to successful reforms of centralized electricity sectors to competitive markets [5], [40]. Leader-follower type models are being used to support policy decisions on design of electricity markets and various regulatory tasks. Unfortunately, these models are usually inherently non-convex and sensitive to assumptions on their parameters. It is recognized [5] that no modeling approach can predict prices in oligopolistic markets, therefore the value of models is considered in their ability to provide robust results on relative differences of feasible market structure and regulations. Let us consider a model [40] where the independent system operator (ISO) controls the transmission system and generator outputs so as to maximize social welfare of consumers while meeting all the network and security constraints.

An electricity market can be represented by a set N of nodes and a set of transmission lines. The strategic decision variables of the ISO are import/export quantities r_i, $i \in N$, which must be balanced

$$\sum_i r_i = 0, \tag{22}$$

and such that the resulting power flows don't exceed secure thermal limits of the transmission lines in both directions

$$-K_l \leq \sum_i D_{il} r_i \leq K_l, \; l \in L, \tag{23}$$

where D_{il} are distribution factors (parameters) which specify the flow on a line l from a unite of flow increase at a node i.

Given the ISO's (leader) decisions r_i, each producer (follower) i maximizes the profit function

$$P_i(q_i + r_i)q_i - C_i(q_i), \; 0 \leq q_i \leq \bar{q}_i, \; i \in N, \tag{24}$$

where $P_i(\cdot)$, $C_i(\cdot)$ are the inverse demand function (wiliness-to-pay) and generation cost function; $\bar{q}_i$ are upper capacity bounds.

The leader-follower models rely on the following perfect information assumptions: the ISO knows the response functions $q_i(r_i)$ and chooses r_i, $i \in N$, maximizing the welfare function of consumers

$$\sum_i \left[\int_0^{r_i + q_i(r_i)} P_i(v)dv - C_i(q_i(r_i)) \right]. \tag{25}$$

The resulting model is one-leader multi-follower Stackelberg game. This type of model may have none or multiple degenerated solutions. There might exist also no-equilibrium in pure strategies due to non-convexity and even discontinuities in the welfare function (25). Slight deviations in $q_i(r_i)$, say due to volatility of price/demand functions $P_i(\cdot)$ may have significant consequences [5] on the market power mitigation and equilibrium.

Another approach is to assume that the ISO is a Nash player that acts simultaneously with producers. This unilateral approach for regulation of network interdependencies ignores dependence of agents' decisions q_i on regulations r_i, that removes the non-convexity from

ISO's optimization problem (22), (23), (25). Yet again, this approach requires perfect information about all producers, demand and system contingencies. It ignores uncertainties of fluctuations that stem from unforeseen events, such as demand uncertainty and transmission and generation outranges. The critical shortcoming of the Nash equilibrium is that it ignores the two-stage character of the ISO and producers decisions. In particular, it excludes proper modeling of forward markets allowing participants to secure more stable prices reducing opportunities to manipulate the market.

We shall now describe a two-stage STO model (see also section 7.2) in which the ISO determines its forward decisions under uncertainties at stage 1 (comprising possibly many random time interval), and producers act at stage 2 after a scenario of uncertainty is revealed. Let us consider a network affected by a set of random events (shocks) $\omega \in \Omega$ which are assumed to be elements of a probability space. These events lead to variability of functions $P_i(\cdot,\omega)$, bounds $K_i(\omega)$, $\bar{q}_i(\omega)$, and cost functions $C_i(\cdot,\omega)$. In general, these random functions and parameters can be viewed as ISO's perception of producer's model.

In the presence of uncertainties the best ISO strategy would be a collective risk sharing maximizing the social welfare function of consumers and producers:

$$F(r) = \sum_i Ef_i(r_i,\omega), \qquad (26)$$

$$f_i(r_i,\omega) = \max_{q_i} \int_0^{q_i+r_i} P_i(\upsilon,\omega)d\upsilon + P_i(q_i+r_i,\omega)q_i - C_i(q_i,\omega), \; 0 \le q_i \le \bar{q}_i(\omega),$$

under constraints (22), (23). This is a two-stage STO problem as in section 7.2. Function $F(r)$ orients regulatory decisions on achieving best possible outcomes with respect to all potential behavioral scenarios of agents (see also Remark 6).

7. Computational methods

A discussion of computational methods and applications of Stackelberg games can be found in [20], [25], [27], [34], [35]. Concept of Nash equilibrium smoothes the problem but it ignores essential two-stage structure of leader-follower decisions (section 6). Explicit treatment of uncertainty in PA models with bi-level structure is considered in [2], [20]. Paper [39] advances the decision-theoretic approach to homeland security models.

The development of effective decision-theoretic computational methods essentially depends on specifics of arising STO models. The main issue is analytical intractability of performance indicators $F_i(x) = Ef_i(x,\omega) = \int f_i(x,\omega)P(x,d\omega)$, where $P(x.d\omega)$ may implicitly depend on x as in problem (15) and Remark 4. If functions $F_i(x)$ are analytically tractable, then the problem can be solved by using standard deterministic methods. Unfortunately, this is rarely the case. In fact, standard deterministic models are formulated usually by switching from $F_i(x) = Ef_i(x,\omega)$ to deterministic functions $f_i(x, E\omega)$. Simple examples show that this substitution may result in wrong conclusions as in section 3.2. More specifically, outcomes $\exp(\omega x)$ for $\omega = \pm 100$ with probability $1/2$ have considerable variability, whereas $\exp(E\omega x) = 1$.

Instead straightforward evaluations of integrals $F_i(x)$, STO methods [3], [4], [14], [28], [37] use only random values $f_i(x,\omega)$ available from Monte Carlo simulations. The following section outlines the main idea of these powerful methods which avoid the "curse of dimensionality" (Remark 3) and allow to solve problems which cannot be solved by other existing methods [8], [9], [19]. An important application of these methods for security management can be found in [39].

7.1. Adaptive Monte Carlo optimization. For the simplicity of illustration, let us consider the minimization of a function $F(x) = \int f(x,\omega)P(x,d\omega)$ without constraints.

Computations evolve from an initial solution x^0. Instead of computing values of integral $F(x)$, what is practically impossible, the procedure uses only observable (simulated) random values $f(x,\omega)$.

For a solution x^k calculated after k-th step, simulate two independent observations $\omega^{k,1}$, $\omega^{k,2}$ of ω correspondingly from $P(x^k + \gamma_k \eta^k, d\omega^{k,1})$, $P(x^k, d\omega^{k,2})$ and calculate values $f(x^k + \gamma_k \eta^k, \omega^{k,1})$, $f(x^k, \omega^{k,2})$, where γ_k is a positive number, $\eta^k = (\eta_1^k, ..., \eta_n^k)$ is a random vector with, say, independent identically uniformly distributed in interval $[-1,1]$ components. New approximate solution x^{k+1} is computed by moving from x^k in direction of so-called stochastic quasigradient [8], [19]:

$$\xi^k = \frac{f(x^k + \gamma_k \eta^k, \omega^{k,1}) - f(x^k, \omega^{k,2})}{\gamma_k} \eta^k$$

with a step size $\rho_k > 0$. The convergence of x^k to the set of optimal solutions with probability 1 follows from the fact that random vector ξ^k is a stochastic quasigradient (SQG) of $F(x)$, i.e., $E[\xi^k \mid x^k] \approx F_x(x^k)$. In other words, ξ^k is an estimate of the gradient $F_x(x^k)$ or its analogs for nondifferentiable and discontinuous functions [8], [13], [19]. Step-sizes ρ_k, γ_k have to satisfy some simple requirements, e.g., $\rho_k = const/k$, $\sum_k \rho_k \gamma_k < \infty$.

This method simulates realistic adaptive processes. Only two random observations of function $F(x)$ are used at each step to identify the direction of transition from x^k to x^{k+1} and its size, whereas values of $F(x)$ and its derivatives remain unknown. In fact, changes of $F(x^k)$ can be relatively tracked, in a sense, by $F^k = \frac{1}{k}\sum_{s=1}^{k} f(x^s, \omega^s)$ due to the convergence of $F(x^k)$, $k \to \infty$. This allows designing adaptive regulations of ρ_k to speed up the convergence. Applications of fast Monte Carlo simulations usually require nontrivial analytical analysis [13] of involved stochastic processes. The next section outlines a version of the method that utilizes the analytical structure of $f(x, \omega)$ to achieve faster simulations.

7.2. Two-stage STO models. The robustness is achieved by two-stage structure of decisions combining both fundamental for coping with uncertainty mechanisms of anticipation and adaptation. Forward-looking anticipative decisions are made before new information about uncertainty become available, whereas other options are created and remain open for adaptive adjustments to potential new information when it becomes available. The two-stage STO problem [3], [14], [28], [37] as an attempt to incorporate both fundamental mechanisms for coping with uncertainty seems to be the most suitable for framing principle-agent decision models under uncertainty. An example of two-stage model is given in section 6, equation (26). Problem (2) has also two-stage formulation important for modeling the climate change dilemma [31]. A rather general two-stage STO model is formulated as follows. A long-term decision x must be made at stage 1 before the observation of uncertainty ω is available. At stage 2, for given $x \in X$ and observed ω, the adaptive short-term decision $y(x, \omega)$ is chosen so as to solve the problem: find $y \in Y$, such that

$$g_i(x, y, \omega) \geq 0,\ i = \overline{1, l}, \text{ and} \tag{27}$$

$$g_0(x, y, \omega), \tag{28}$$

is maximized (minimized) for some functions g_i, $i = \overline{1, l}$. Then the main problem is to find decision x, such that

$$F(x) = Ef(x, \omega),\ f(x, \omega) = g_0(x, y(x, \omega), \omega),\ x \in X \tag{29}$$

is maximized.

We can see that equations (27), (28) correspond to equations (12), (13) of the agents' models, and (29) to the goal (10) of the principle-agent model. Computational methods for this general model are discussed in [8]. If $y(x, \omega)$ maximizes (28), this problem corresponds to the (two-stage) recourse STO model; otherwise – stochastic maximin problem (16). In general, (29) may be (section 2) a dynamic two-stage STO model. Multistage STO models with more then two stages arise in cases when ω remains unknown after new information become available.

In general, two-stage STO models (29) are solved by using adaptive Monte Carlo optimization [8] methods. The main idea can be easy illustrated by using the simplest stochastic minimax model (2). As in section 7.1, instead of integral $F(x) = Ef(x,\omega)$, $f(x,\omega) = \max\{\alpha(x-\omega), \beta(\omega - x)\}$, the method sequentially updates an initial solution x^0 by using only available on-line independent observations or simulations ω_0, ω_1, ... of random variable ω. Let x^k is an approximate solution computed at step $k = 0,1,...$. Observe (simulate) ω_k and change x^k by the rule

$$x^{k+1} = x^k - \rho_k \xi^k,\ k = 0,1,...,\ \ \xi^k = \begin{cases} \alpha,\ if\ \ x^k \geq \omega_k, \\ \beta,\ otherwise. \end{cases}$$

Again, ξ^k is a SQG of non-smooth function $F(x)$ at $x = x^k$. The step-size multiplier ρ_k satisfies the same type conditions as in section 7.1.

7.3. Uncertain distributions. In this section we assume that ω is characterized by a vector $v = (v_1,...,v_m)$ of random parameters $v \in V \subset R^m$. Analyzed in previous sections STO models can be formulated as maximization of the function

$$F(x) = Ef(x,v) = \int_V f(x,v)dH(v),\ x \in X \subset R^n, \tag{30}$$

where $v \in V \subset R^m$ is a vector of random parameters, $H(y)$ is a cumulative distribution function, i.e., $P(dv) = dH(v)$, and $f(x,\cdot)$ is a random function possessing all the properties necessary for expression (30) to be meaningful.

As previous sections show, we often do not have full information on $H(y)$. For new decision problems, in particular, arising in security analysis, we often have large a number of unknown interdependent variables v, x and only very restricted samples of real observations which don't allow to derive the distribution P.

Experiments to generate new real observations may be extremely expensive, dangerous or simply impossible. Instead, the natural approach for dealing with new problems can be based on using all additional information on P to derive a set of feasible distributions.

Let us denote by K the set of distributions consistent with available information on P. The robust solution can be defined as $x \in X$ maximizing

$$F(x) = \min_{P \in K} \int f(x,v)P(dv) = \int f(x,v)P(x,dv), \tag{31}$$

where $P(x,dv)$ denotes the extreme distribution as in section 4. Thus, we have a general case of STO models with probability measure affected by decision x as in sections 3.3, 7.1.

Assume that in accordance with available sample and our beliefs we can split the set V into disjoint subsets $\{C_s, s=1,...,S\}$. Some of them may correspond to clusters of available observations whereas others may reflect expert opinions on the degree of uncertainty and its heterogeneity across the admissible set V. For instance, we can distinguish some critical zones ("catalogues of earthquakes") which may cause significant reductions of performance indicators. Accordingly, the additional beliefs can be given in terms of a "quantile" class

$$K = \left\{ P: \int_{C_s} P(dv) = \alpha_s, s = 1,...,S \right\}, \tag{32}$$

where $\sum_{s=1}^{S} \alpha_s = 1$; more generally – in terms of ranges of probabilities

$$K = \left\{ P: \alpha_s \le \int_{C_s} P(dv) \le \beta_s, s = 1,...,S \right\}, \tag{33}$$

where α_s, β_s are given numbers such that $\sum_{s=1}^{S} \alpha_s \le 1 \le \sum_{s=1}^{S} \beta_s$. This class is considered as the most natural elicitation mechanism.

Let us denote $\gamma_s = \int_{C_s} P(dv)$, $\gamma = (\gamma_1,...,\gamma_S)$. In general, additional beliefs can be represented in a form of various inequalities among components of vector γ (see Remark 4), of the type

$$K = \{P: A\gamma \le b, \gamma \ge 0\} \tag{34}$$

for some matrix A and vector b.

Proposition 2: For any function $f(x,v)$ assumed to be integrable w.r.t. all P in K defined by (34)

$$\min_{P \in K} \int f(x,v) P(dv) = \min_{\gamma} \left\{ \sum_{s=1}^{S} \gamma_s \min_{v \in C_s} f(x,v) \mid A\gamma \le b \right\}.$$

In the case of K defined by (32), (33)

$$\min_{P\in K}\int f(x,v)P(dv) = \sum_{s=1}^{S}\alpha_S \min_{v\in C_S} f(x,v),$$

$$\min_{P\in K}\int f(x,v)P(dv) = \min_{\{\gamma_s\}}\left\{\sum_{s=1}^{S}\gamma_s \min_{v\in C_s} f(x,v) \mid \alpha_s \le \gamma_s \le \beta_s, \sum_{s=1}^{S}\gamma_s = 1\right\}.$$

Proof: We can choose a distribution P concentrated at any collection of points $v_S \in C_S$, $s = \overline{1,S}$, therefore

$$\min_{P\in K}\int f(x,v)P(dv) \le \min_{\gamma}\left\{\sum_{s=1}^{S}\gamma_S \min_{v\in C_S} f(x,v) \mid A\gamma \le b\right\}.$$

On the other hand,

$$\int f(x,v)P(dv) = \sum_S \int_{C_s} f(x,v)P(dv) \ge \min_{\gamma}\left\{\sum_{s=1}^{S}\gamma_S \min_{v\in C_S} f(x,v) \mid A\gamma \le b\right\}.$$

Proposition 2 reduces maximization problem (31) to deterministic maximin problems which can be solved by linear or nonlinear programming methods. There are important connections between dual solutions of these problems and CVaR measures discussed in sections 2, 3.

7.4. Generalized moment problem. Often we know bounds for the mean value or other moments of H in (30). Such information can often be written in terms of constraints

$$Q^k(H) = Eq^k(v) = \int_V q^k(v)dH(v) \le 0, \ k = \overline{1,l},$$

$$\int_V dH(v) = 1,$$

where the $q^k(v)$, $k=\overline{1,l}$, are known functions. Let K is the set of functions H satisfying these constraints.

Consider again the problem (31). Methods of maximizing $F(x)$ in (31) depend on solution procedures for the following “inner” minimization problem: find a distribution function H that minimizes

$$Q^0(H) = Eq^0(v) = \int_V q^0(v)dH(v)$$

subject to $H \in K$ for some function q^0. This is a generalization of the known moments problem (see, e.g., [10]). It can also be regarded as a generalization of the nonlinear programming problem

$$\min\left\{q^0(v) : q^k(v) \le 0, v \in V, k=\overline{1,l}\right\}$$

to an optimization problem involving randomized strategies as in section 5.

There are two main approaches [10] for minimizing $Q^0(H)$ in K: generalized linear programming (GLP) methods and dual maximin approach.

7.5. GLP methods. Minimization of $Q^0(H)$ in K is equivalent to the following GLP problem [10]: find points $v^j \in V$, $k=\overline{1,l}$, $t \le l+1$ and real numbers p_j, $k=\overline{1,l}$, $j=\overline{1,t}$, minimizing

$$\sum_{j=1}^{t} q^0(v^j)p_j \tag{35}$$

subject to

$$\sum_{j=1}^{t} q^k(v^j)p_j \le 0,\ k=\overline{1,l}, \tag{36}$$

$$\sum_{j=1}^{t} p_j = 1,\ p_j \ge 0,\ j=\overline{1,t}. \tag{37}$$

Consider arbitrary points v^j, $j=\overline{1,l+1}$ (setting $t=l+1$), and for the fixed set $\{v^1,v^2,...,v^{l+1}\}$ find a solution $\bar{p}=(\bar{p}_1,\bar{p}_2,...,\bar{p}_{l+1})$ of problem (35) – (37) with respect to p. Assume that $\bar{p}$ exists and that $(\bar{u}_1,\bar{u}_2,...,\bar{u}_{l+1})$ are the corresponding dual variables. We know that if there exists a point v^* such that $q^0(v^*)-\sum_{k=1}^{l}\bar{u}_k q^k(v^*)-\bar{u}_{l+1}<0$, then the solution $\bar{p}$ could be improved by dropping one of the columns $(q^0(v^j),q^1(v^j),....,q^l(v^j),1)$, $j=\overline{1,l+1}$ from the basis and replacing it by the column $(q^0(v^*),q^1(v^*),....,q^l(v^*),1)$, following the revised simplex method. Point v^* could be defined by minimizing $q^0(v)-\sum_{k=1}^{l}\bar{u}_k q^k(v)$, $v\in V$.

This conceptual framework leads to various methods [18], [21] for solving not only (35)-(37) but also some more general classes of nonlinear (in probability) problems. The interesting important issue is the combination of the described procedure with simultaneous gradient type adjustments of x ensuring optimal solution of compound problem (31).

7.6. Duality relations. The duality relations for minimization of $Q^0(H)$ in K provide a more general approach. It can be shown that if V is compact, $q^k(v)$, $k=\overline{0,l}$, are continuous and $0\in \operatorname{int} co\,\{z: z=(q^0(v),q^1(v),...,q^l(v)\}$, $v\in V$, then

$$\min_{H\in K}\int f(x,v)dH(v)=\max_{u\in U^+}\min_{v\in V}\left[f(x,v)-\sum_{k=1}^{m}u_k q^k(v)\right] \tag{38}$$

for each $x\in X$, where $f(x,\cdot)$ is a continuous function. Hence, original infinite dimensional STO problem can then be reduced to finite-dimensional a maximin type problem as follows: maximize the function

$$\gamma(x,u)=\min_{v\in V}\left[f(x,v)-\sum_{k=1}^{m}u_k q^k(v)\right] \tag{39}$$

with respect to $x\in X$, $u\geq 0$. This allows developing a number of algorithms using GLP approach and algorithms based on solving directly maximin problem (39). A general scheme of such an algorithms is the following.

7.7. Stochastic procedure. According to (38), (39) the STO model with uncertain distribution is reduced to a finite-dimensional maximin problem with a possibly non-convex inner problem of minimization and a concave final problem of maximization. A vast amount of work has been done on maximin problems but virtually all of the existing methods fail if the inner problem is non-convex. The following approach allows to overcome this difficulty.

Consider a general maximin problem

$$\max_{x \in X} \min_{v \in V} g(x,v),$$

where $g(x,v)$ is a continuous function of (x,v) and a concave function of x for each $v \in V$, $X \subset R^n$, $V \subset R^m$. Although $G(x) = \min_{v \in V} g(x,v)$ is a concave function, to compute its value requires a solution $v(x)$ of non-convex problem. In order to avoid the difficulties involved in computing $v(x)$ one could try to approximate V by an ε - set.

But, in general, this would require a set containing a very large number of elements. An alternative is to use the following ideas [10]. Consider a sequence of sets V_s, $s = 0,1,...$ and the sequence of functions $G^s(x) = \min_{v \in V_s} g(x,v)$. It can be proven that, under natural assumptions concerning the behavior of sequence G^s, the sequence of points generated by the rule

$$x^{s+1} = x^s - \rho_s G_x^s(x^s),\ G_x^s(x^s) = g_x(x^s, v^s),\ s = 0,1,...,$$

where the step size ρ_s satisfies assumptions such as $\rho_s \geq 0$, $\rho_s \to 0$, $\sum_{s=0}^{\infty} \rho_s = \infty$, tends to follow the time-path of optimal solutions: for $s \to \infty$

$$\lim \left[G^s(x^s) - \max G^s(x) \right] = 0.$$

It was shown (see discussion in [10]) how V_s (which depends on x^s) can be chosen so that we obtain the convergence

$$\min G^s(x) \to \min G(x),$$

where V_s contains only a finite number $N_s \geq 2$ of random elements. The main idea is the following.

We start by choosing initial points x^0, v^0, a probability distribution μ on set V and an integer $N_0 \geq 1$. Suppose that after the s-th iteration we have arrived at points x^s, v^s. The next approximations x^{s+1}, v^{s+1} are then constructed in the following way. Choose $N_s \geq 1$ points $v^{s,1}$, $v^{s,2}$, …, v^{s,N_s}, which are sampled from the distribution μ, and determine the set

$$V_s = \left\{v^{s,1}, v^{s,2}, ..., v^{s,N_s}\right\} \bigcup v^{s,0},$$

where $v^{s,0} = v^s$. Take $v^{s+1} = Arg \max_{v \in V_s} g(x^s, v)$ and compute

$$x^{s+1} = \pi_X \left[x^s - \rho^s g_x(x^s, v^{s+1})\right], \ s = 0,1,...,$$

where ρ_s is the step size and π_X is the result of the projection operation on X. The convergence analysis of this method can be found in [10].

8. Concluding remarks

In the case of perfect information the best response of the follower to a decision x of the leader is the decision $y(x)$ maximizing his reward $A(x, y)$. Therefore, the best decision of the leader is $x = x^*$ maximizing his rewards $R(x, y(x))$. Since $y(x)$ is assumed to be a unique solution, then the follower has strong incentive to choose $y(x^*)$ afterwards, i.e., x^* is the Stackelberg equilibrium.

In the case of uncertainty the situation is different. The leader may again use the best response $y(x, \omega)$ of the follower according to his perception of uncertainty ω and rewards $A(x, y, \omega)$ of the follower. However, the decision $y(x, \omega)$ is no longer rational for the follower to choose afterwards. In addition, as Remark 7 indicates, it may produce degenerated solutions $y(x, \omega)$ resulting in discontinuities and instabilities. Therefore, in the case of uncertainty the proposed decision-theoretic approach relies on random extreme scenarios for the leader rather than random best case scenarios for the follower. This preserves convexities of models and it allows the introduction of concepts of robust solutions based on new type of non-Bayesian multicriteria STO models with uncertain probability distributions and multivariate extreme events. As section 7 shows, specific classes of such models can be solved by linear and nonlinear programming methods in the case of price-wise linear random functions. An important food security case study in [4] was analyzed by linear programming methods in an extended space of proposed two-stage multi-agent STO model. In general, adaptive fast Monte Carlo and SQG optimization methods can be used [20], [39] to solve arising STO models. Developments of tools for solving STO problems involving implicit dynamic dependencies of probabilities on decisions in section 5.3 demand special attention. Truly systemic security management is required for coping with systemic failures and extreme events generating disruptions of financial and economic systems, communication and information systems, food-water-energy supply networks.

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