MAXIMIZATION OF A CONVEX QUADRATIC FUNCTION UNDER LINEAR CONSTRAINTS

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1. Introduction

Since the appearance of a paper by H. Tui [14], maximization of convex function over a polytope has attracted much attention. In his paper, two algorithms were proposed: one cutting plane and the other enumerative. However, the numerical experiments reported in [16] on the naive cutting plane approach were discouraging enough to shift the researchers more to the direction of enumerative approaches ([7],[8],[17]).

In this paper, we will develop a cutting plane algorithm for maximizing a convex quadratic function subject to linear constraints. The basic idea is much the same as Tui's method. It also parallels some of the recent results by E. Balas and C-A. Burdet [2]. We will, however, use standard tools which are easier to understand and will fully exploit the special structure of the problem. The main purpose of the paper is to demonstrate that the full exploitation of special structure will enable us to generate a cut which is much deeper than Tui's cut and that the cutting plane algorithm can be used to solve a rather big problem efficiently.

We will first prove the equivalence of the original problem and an associated bilinear program (See [9] for details) and then exploit its special structure to obtain a 'deep' cut. The algorithm has been tested on CYBER 74 up to a problem size of 9 x 19 and the numerical results turned out to be quite encouraging. This work is closely related to [9] and its results will be frequently referred to without proof.

2. ε-Locally Maximum Basic Feasible Solution and Equivalent Bilinear Program

We will consider the following quadratic program:

$$\begin{aligned} \max\ & f(x) = c^t x + \tfrac{1}{2} x^t Q x \\ \text{s.t.}\ & Ax = b, \quad x \geq 0 \end{aligned} \tag{2.1}$$

where $c,\ x \in R^n$, $b \in R^m$, $A \in R^{m \times n}$ and $Q \in R^{n \times n}$ is a symmetric positive semi-definite matrix. We will assume that the feasible region

$$X = \{x \in R^n \mid Ax = b, \quad x \geq 0\} \tag{2.2}$$

is non-empty and bounded. It is well known that in this case (2.1) has an optimal solution among basic feasible solutions.

Given a feasible basis B of A, we will partition A as (B, N) assuming, without loss of generality, that the first m columns of A are basic. Partition x correspondingly, i.e. $x = (x_B, x_N)$. Premultiplying B^{-1} to the constraint equation $Bx_B + Nx_N = b$ and suppressing basic variables x_B, we get the following system which is totally equivalent to (2.1):

$$\begin{aligned} \max\ & \bar{f}(x_N) = \bar{c}_N x_N + \tfrac{1}{2} x_N \bar{Q} x_N + \phi_o \\ \text{s.t.}\ & B^{-1} N x_N \leq B^{-1} b, \quad x_N \geq 0 \quad , \end{aligned} \tag{2.3}$$

where $x^o \equiv (x_B^o, x_N^o) = (B^{-1}b, 0)$ and $\phi_o = f(x^o)$. Introducing the notations: $\ell = n - m$, $d = \bar{c}_N$, $y = x_N$, $F = B^{-1}N$, $f = B^{-1}b$, $D = \bar{Q}$, we will rewrite (2.3) as:

$$\begin{aligned} \max\ & g(y) = d^t y + \tfrac{1}{2} y^t D y + \phi_o \\ \text{s.t.}\ & Fy \leq f, \quad y \geq 0 \end{aligned} \tag{2.4}$$

and call this a 'canonical' representation of (2.1) relative to a feasible basis B. To express the dependence of vectors in (2.4) on B, we occasionally use the notation d(B) etc.

Definition 2.1. Given a basic feasible solution $x \in X$, let $N_x(x)$ be the set of adjacent basic feasible solutions which can be reached from x in one pivot step.

Definition 2.2. A basic feasible solution $x^* \in X$ is called an ε-locally maximum basic feasible solution of (2.1) if

(i) $d \leq 0$,

(ii) $f(x^*) > f(x) - \varepsilon, \ \forall x \in N_x(x^*)$

Let us intorduce here a bilinear program associated with (2.1), which is essential for the development of cutting planes:

$$\begin{aligned} \max \ \phi(x_1, x_2) &= c^t x_1 + c^t x_2 + x_1 Q x_2 \\ \text{s.t.} \quad Ax_1 &= b, \quad x_1 \geq 0 \\ Ax_2 &= b, \quad x_2 \geq 0 \end{aligned} \tag{2.5}$$

Theorem 2.1 [9]. If X is non-empty and bounded, then (2.5) has an optimal solution (x_1^*, x_2^*) where x_1^* and x_2^* are basic feasible solutions of X.

Moreover, two problems (2.1) and (2.5) are equivalent in the following sense:

Theorem 2.2. If x^* is an optiaml solution of (2.1), then $(x_1, x_2) = (x^*, x^*)$ is an optimal solution of (2.5). Conversely, if (x_1^*, x_2^*) is optimal for (2.5), then both x_1^*, x_2^* are optimal for (2.1).

Proof. Let x^* be optimal for (2.1) and (x_1^*, x_2^*) be optimal for (2.5). By definition $f(x^*) \geq f(x), \forall x \in X$. In particular,

$$f(x^*) \geq f(x_i^*) = \phi(x_i^*, x_i^*), \quad i = 1, 2$$

also

$$\phi(x_1^*, x_2^*) = \max\{\phi(x_1, x_2) \mid x_1 \in X, x_2 \in X\}$$

$$\geq \max\{\phi(x, x) \mid x \in X\} = f(x^*) \quad .$$

To establish the theorem, it suffices therefore to prove that

$$f(x_i^*) = \phi(x_1^*, x_2^*), \quad i = 1, 2 \tag{2.6}$$

because we then have $f(x_i^*) \geq f(x^*)$, $i = 1, 2$ and $\phi(x^*, x^*) = f(x^*)$ $= \phi(x_1^*, x_2^*)$. Let us now prove (2.6). Since (x_1^*, x_2^*) is optimal for (2.5), we have

$$0 \leq \phi(x_1^*, x_2^*) - \phi(x_1^*, x_1^*) = c^t(x_2^* - x_1^*) + (x_1^*)^t Q(x_2^* - x_1^*)$$

$$0 \leq \phi(x_1^*, x_2^*) - \phi(x_2^*, x_2^*) = c^t(x_1^* - x_2^*) + (x_2^*)^t Q(x_1^* - x_2^*)$$

Adding these two inequalities, we obtain

$$(x_1^* - x_2^*)^t Q\,(x_1^* - x_2^*) \leq 0$$

Since Q is positive semi-definite, this implies $Q(x_1^* - X_2^*) = 0$. Putting this into the inequality above, we get $c^t(x_1^* - x_2^*) = 0$. Hence $\phi(x_1^*, x_2^*) = \phi(x_1^*, x_2^*) = \phi(x_2^*, x_2^*)$ as was required. ||

As before, we will define a canonical representation of (2.5) relative to a feasible basis B:

$$\max \psi(y_1, y_2) = d^t z_1 + d^t z_2 + z_1^t D z_2 + \phi_o$$

$$\text{s.t.} \quad Fz_1 \le f, \quad z_1 \ge 0 \qquad (2.7)$$

$$Fz_2 \le f, \quad z_2 \ge 0$$

which is equivalent to (2.4). Also let

$$Y = \{y \in R^\ell \mid Fy \le f, \quad y \ge 0\} \quad . \qquad (2.8)$$

3. Cutting Plane at an ε-Locally Maximum Basic Feasible Solution

We will assume in this section that an ε-locally maximum basic feasible solution x^o and corresponding basis B_o have been obtained. Also, let ϕ_{max} be the best feasible solution obtained so far by one method or another.

Given a canonical representations (2.4) relative to B_o, we will proceed to introduce a 'valid' cutting plane in the sense that it

(i) does eliminate current ε-locally maximum basic feasible solution, i.e., the point $y = 0$,

(ii) does not eliminate any point y in Y for which $g(y) > \phi_{max} + \varepsilon$.

Theorem 3.1 [14]. Let θ_i be the larger root of the equation:

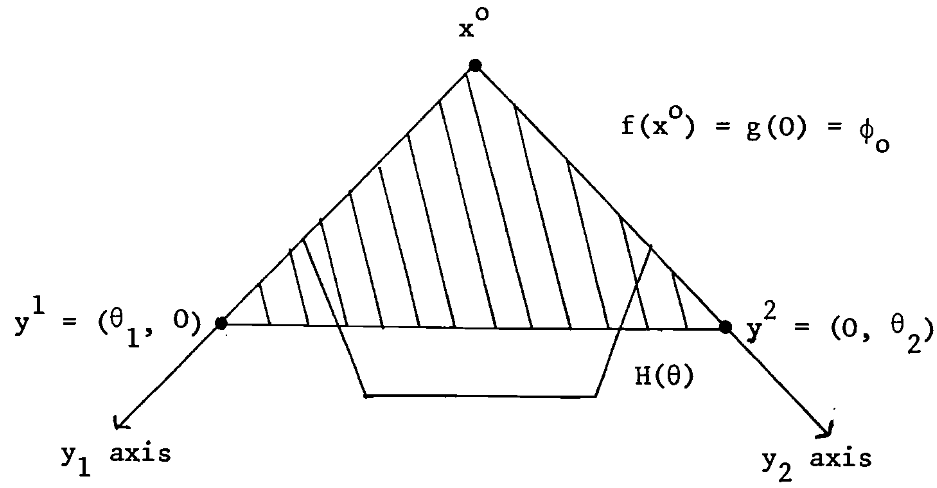
$$d_i\lambda + \tfrac{1}{2}d_{ii}\lambda^2 = \phi_{max} - \phi_o + \varepsilon \qquad (3.1)$$

Then the cut

$$H(\theta): \sum_{i=1}^{\ell} y_i/\theta_i \ge 1$$

is a valid cut.

This theorem is based upon the convexity of g(y) and the simple geometric observation illustrated below for two dimensional case.



$$g(y^1) = g(y^2) = \phi_{max} + \varepsilon$$

Figure 3.1

Though this cut is very easy to generate and attractive from geometric point of view, it tends to become shallower as the dimension increases and the results of numerical experiments reported in [16] were quite disappointing. In this section, we will demonstrate that if we fully exploit the structure, then we can generate a cut which is generally much deeper than Tui's cut.

Let us start by stating the results proved in [9], taking into account the symmetric property of the bilinear programming problem (2.7) associated with (2.4).

Theorem 3.2. Let θ_i be the supremum of λ for which

$$\max_{z_2} \max_{z_1}\{\psi(z_1, z_2) \mid 0 \leq z_{1i} \leq \lambda,\ z_{1j} = 0,\ j \neq \ ,\ z_2 \in Y\}$$

$$\leq \phi_{max} + \varepsilon \ .$$

Then the cut

$$H(\theta):\quad \sum_{j=1}^{\ell} y_j/\theta_j \geq 1$$

is a valid cut (relative to (2.4)).

Theorem 3.3. θ_i of Theorem 3.2 is given by solving a linear program:

$$\theta_i = \min\left[-d^t z + (\phi_{max} - \phi_o + \varepsilon) z_o\right]$$

$$\text{s.t.}\quad Fz - fz_o \leq 0$$

$$d_{i\cdot}^t z + d_i z_o = 1$$

$$z \geq 0,\ z_o \geq 0$$

where $d_{i\cdot}$ is the ith column vector of D.

For the proofs of these theorems, readers are referred to [9]. Also Theorem 3.3 is proved in [2] using the theory of outer polars. We will next proceed to the method to improve a given valid cut.

For a given positive vector $\theta = (\theta_1, \ldots, \theta_\ell) > 0$, let

$$\Delta(\theta) = \{y \in R^\ell \mid \sum_{j=1}^{\ell} y_j/\theta_j \leq 1,\ y_j \geq 0,\ j = 1, \ldots, \ell\} \ . \qquad (3.3)$$

Theorem 3.4. Let $\tau \geq \theta > 0$. If

$$\max\{\psi(z_1, z_2) \mid z_1 \in \Delta(\theta),\ z_2 \in Y\} \leq \phi_{max} + \varepsilon \tag{3.4}$$

and if

$$\max\{\psi(z_1, z_2) \mid z_1 \in \Delta(\tau),\ z_2 \in Y\setminus\Delta(\theta)\} \leq \phi_{max} + \varepsilon \tag{3.5}$$

then

$$H(\tau):\ \sum_{j=1}^{\ell} y_j/\tau_j \geq 1$$

is a valid cut (relative to (2.4)).

Proof. Let $Y_1 = \Delta(\theta) \cap Y$, $Y_2 = (\Delta(\tau)\setminus\Delta(\theta)) \cap Y$, $Y_3 = Y\setminus\Delta(\tau)$. Obviously $Y = Y_1 \cup Y_2 \cup Y_3$. By (3.3) and (3.4), we have that:

$$\max\{\psi(z_1, z_2) \mid z_1 \in Y_1,\ z_2 \in Y_1 \cup Y_2 \cup Y_3\} \leq \phi_{max} + \varepsilon$$

$$\max\{\psi(z_1, z_2) \mid z_1 \in Y_1 \cup Y_2,\ z_2 \in Y_2 \cup Y_3\} \leq \phi_{max} + \varepsilon$$

By symmetry of function ψ, we have that

$$\max\{\psi(z_1, z_2) \mid z_1 \in Y_2,\ z_2 \in Y_1\} = \max\{\psi(z_1, z_2) \mid z_1 \in Y_1,\ z_2 \in Y_2\}$$

and hence

$$\max\{\psi(z_1, z_2) \mid z_1 \in Y_1 \cup Y_2,\ z_2 \in Y_1 \cup Y_2\} \leq \phi_{max} + \varepsilon$$

Referring to Theorem 2.2, this implies that

$$\max\{g(y) \mid y \in Y_1 \cup Y_2\} \leq \phi_{max} + \varepsilon$$

This, in turn, implies that $H(\tau)$ is a valid cut. ||

This theorem gives us a technique to improve a given valid cut (e.g. Tui's cut or the cut defined in Theorem 3.2). Given a cut $H(\theta)$, let τ_i be

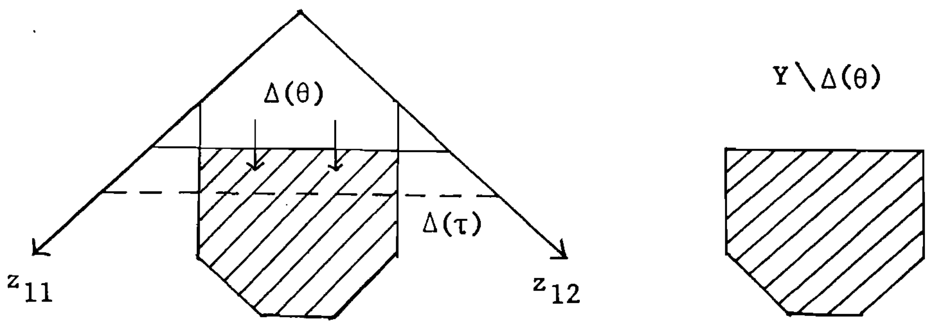


Figure 3.2

the maximum of λ for which

$$\max\{\psi(z_1, z_2) \mid 0 \leq z_{1i} \leq \lambda,\ z_{1j} = 0,\ j \neq i,\ z_2 \in Y \setminus \Delta(\theta)\} \leq \phi_{max} + \varepsilon$$

then $H(\tau)$ is also a valid cut as is illustrated in Figure 3.2.

It is easy to prove (See [9], Theorems 3.2 and 3.3) that τ_i defined above is equal to the optimal objective value of the following linear program:

$$\begin{aligned} \tau_i = \min & [-d^t z + (\phi_{max} - \phi_o + \varepsilon) z_o] \\ \text{s.t.} \quad & Fz - fz_o \leq 0 \\ & \sum_{j=1}^{\ell} d_{ij} z_j + d_i z_o = 1 \\ & \sum_{j=1}^{\ell} z_j/\theta_j - z_o \geq 0 \end{aligned} \tag{3.6}$$

Note that since $d \leq 0$ and $\phi_{max} - \phi_o + \varepsilon > 0$, $(z, z_o) = (0, 0)$ is a dual feasible solution with only one constraint violated and it usually takes only several pivots to solve this linear program starting from this dual feasible solution. Also it should be noted that the objective value is monotonically increasing during the dual simplex procedure and hence we can stop pivoting whenever the objective functional value exceeds some specified level.

Lemma 3.5.

(i) $\phi(x_1, x_2) \leq \max\{\phi(x_1, x_1), \phi(x_2, x_2)\}$, $\forall x_1 \in X, x_2 \in X$.

(ii) If Q is positive definite and $x_1 \neq x_2$, then

$$\phi(x_1, x_2) < \max\{\phi(x_1, x_1), \phi(x_2, x_2)\}$$

Proof.

(i) Assume not. Then

$$0 < \phi(x_1, x_2) - \phi(x_1, x_1) = c^t(x_2 - x_1) + x_1^t Q(x_2 - x_1)$$

$$0 < \phi(x_1, x_2) - \phi(x_2, x_2) = c^t(x_1 - x_2) + (x_1 - x_2)^t Q x_2$$

Adding these two inequalities, we obtain

$$-(x_1 - x_2)^t Q(x_1 - x_2) > 0$$

which is a contradiction since Q is positive semi-definite.

(ii) Assume not. As in (i) above, we get

$$-(x_1 - x_2)^t Q(x_1 - x_2) \geq 0$$

which is a contradiction to the assumption that $x_1 - x_2 \neq 0$ and that Q is positive definite.

Theorem 3.6. If Q is positive definite, then the iterative improvement procedure either generates a point $y \in Y$ for which $g(y) \geq \phi_{max} + \varepsilon$ or else generates a cut which is strictly deeper than corresponding Tui's cut.

Proof. Let $H(\theta)$ be Tui's cut and let $H(\tau)$ be the cut resulting from iterative improvement starting from a valid cut $H(\omega)$ where $\omega \geq 0$. Let

$$z_1^i = (0, \ldots, 0, \tau_i, 0, \ldots, 0), \quad i = 1, \ldots, \ell \quad .$$

By definition:

$$\psi(z_1^i, z_2^i) \equiv \max\{\psi(z_1^i, z_2^i) \mid z_2 \in Y \setminus \Delta(\theta)\} = \phi_{max} + \varepsilon \quad . \tag{3.7}$$

Case 1. $\psi(z_2^i, z_2^i) \geq \psi(z_1^i, z_1^i)$. It follows from Lemma 3.5 and (3.7) that

$$g(z_2^i) = \psi(z_2^i, z_2^i) \geq \psi(z_1^i, z_2^i) = \phi_{max} + \varepsilon \quad .$$

Note that $z_2^i \in Y$.

Case 2. $\psi(z_1^i, z_1^i) > \psi(z_2^i, z_2^i)$. Again by Lemma 3.5 and (3.7), we have

$$\psi(z_1^i, z_1^i) \geq \psi(z_1^i, z_2^i) = \phi_{max} + \varepsilon \quad .$$

We will prove that this inequality is indeed a strong one. Suppose that $\psi(z_1^i, z_1^i) = \psi(z_1^i, z_2^i)$, then

$$c^t(z_1^i - z_2^i) + z_1^{t}D(z_1^i - z_2^i) = 0 \quad .$$

From $\psi(z_1^i, z_2^i) > \psi(z_2^i, z_2^i)$ we obtain

$$c^t(z_2^i - z_1^i) + z_2^{t}D(z_2^i - z_1^i) > 0 \quad .$$

Adding these two, we have that $(z_1^i - z_2^i)^t D(z_1^i - z_2^i) < 0$, which is a contradiction. Thus we have established

$$g(z_1^i) > \phi_{max} + \varepsilon \quad ,$$

which, in turn, implies that $\tau_i > \theta_i$ since θ_i is defined (See (3.1)) as a point at which $g(\cdot)$ attains the value $\phi_{max} + \varepsilon$. ||

It turned out that this iterative improvement procedure quite often leads to a substantially deep cut. Figure 3.3 shows a typical example.

The deeper the cut $H(\theta)$ gets, the better is the chance that some of the non-negativity constraints $y_i \geq 0$, $i = 1, \ldots, \ell$ becomes redundant for specifying the reduced feasible region $Y \setminus \Delta(\tau)$. Such redundant constraints can be identified by solving the following linear program:

$$\min\{y_i \mid Fy \leq f,\ y \geq 0,\ \Sigma y_j/\tau_j \geq 1\}$$

If the minimal value of y_i is positive, then the constraint $y_i \geq 0$ is redundant and we can reduce the size of the problem. This procedure is certainly costly and its use is recommended only when there is a very good chance of success, i.e., when τ is sufficiently large.

4. Cutting Plane Algorithm and the Results of Experiments

We will describe below one version of cutting plane algorithm which has been coded in FORTRAN IV for CYBER 74.

ILLUSTRATIVE EXAMPLE OF ITERATIVE IMPROVEMENT

$$\begin{cases} \max \; -2z_1 - 3z_2 + 2z_1^2 - 2z_1z_2 + 2z_2^2 \\ \text{s.t.} \quad -z_1 + z_2 \le 1 \\ \quad z_1 - z_2 \le 1 \\ \quad -z_1 + 2z_2 \le 3 \\ \quad 2z_1 - z_2 \le 3 \\ \quad z_1 \ge 0, \; z_2 \ge 0 \end{cases}$$

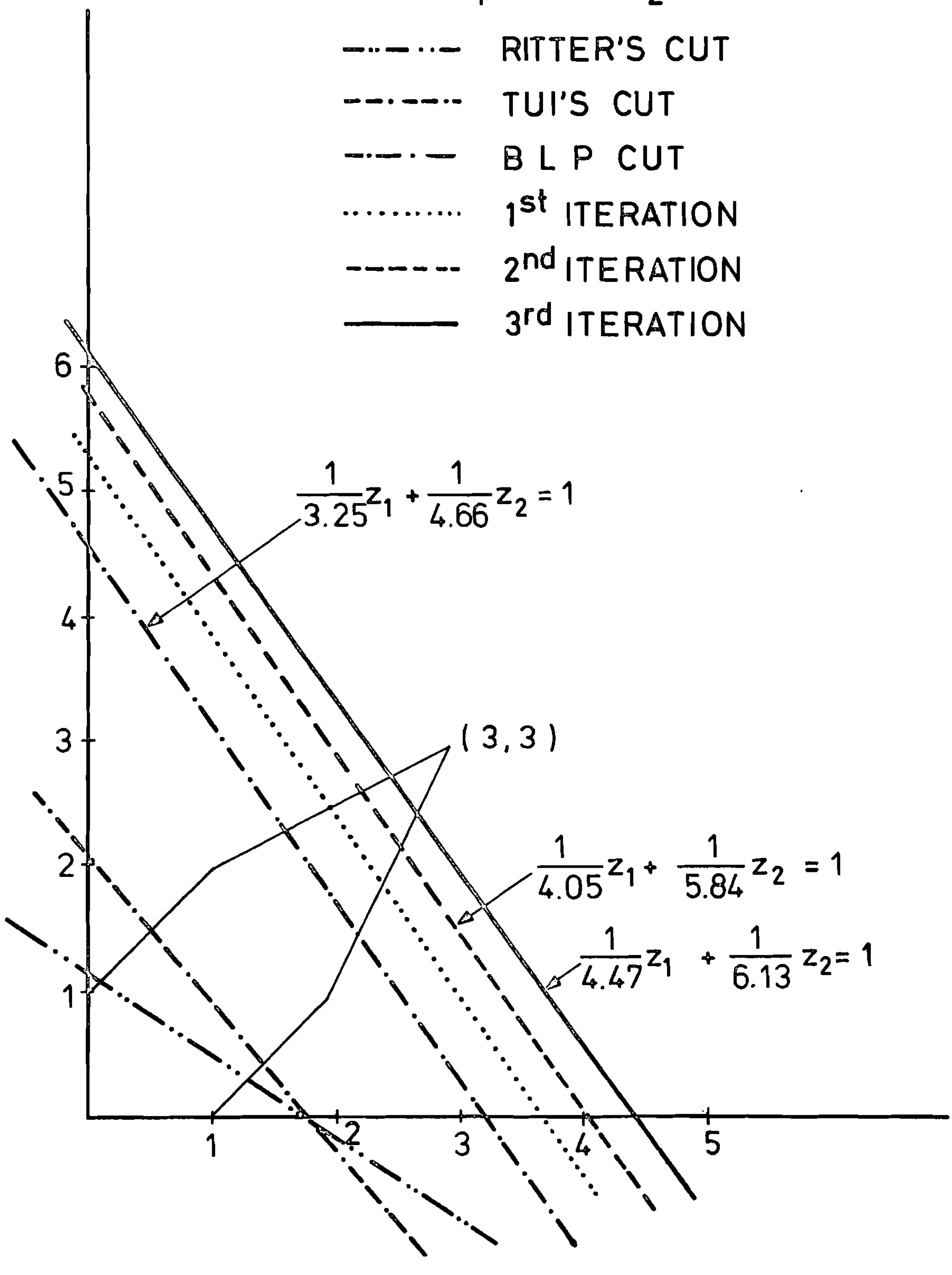


Figure 3.3

Cutting Plane Algorithm

Step 1. Let $\ell = 0$ and $X_o = X$.

Step 2. If $\ell > \ell_{max}$ then stop. Otherwise go to Step 3.

Step 3. Let $k = 0$ and let $x^o \in X_\ell$ be a basic feasible solution and let $\phi_{max} = f(x^o)$.

Step 4. Solve a subproblem: $\max\{\phi(z, x^k) \mid z \in X_\ell\}$ and let x^{k+1} and B^{k+1} be its optimal basic feasible solution and corresponding basis.

Step 5. Compute $d(B_{k+1})$, the coefficients of linear term of (2.7) relative to B_{k+1}. If $d(B_{k+1}) \not\leq 0$, then add 1 to k and go to Step 4. Otherwise let $B^* = B_{k+1}$, $x^* = x^{k+1}$ and go to Step 6.

Step 6. Compute matrix D in (2.7) relative to B^*. If x^* is an ε-locally maximum basic feasible solution (relative to X), then let $\phi_{max} := \max\{\phi_{max}, f(x^*)\}$, $\phi_o = f(x^*)$ and go to Step 7. Otherwise move to a new basic feasible solution $\hat{x}$ where $f(\hat{x}) = \max\{f(x) \mid x \in N_{X_\ell}(x^*)\}$. Let $k = 0$, $x^o = x$ and go to Step 4.

Step 7. Let $j = 0$ and let $Y^o_{\ell+1} = Y_\ell$.

Step 8. Compute $\theta(Y^j_{\ell+1})$ and let $Y^{j+1}_{\ell+1} = Y^j_{\ell+1} \setminus \Delta(\theta(Y^j_{\ell+1}))$. If $Y^{j+1}_{\ell+1} = \phi$ then stop. Otherwise go to Step 9.

Step 9. Let $\alpha = \|\theta(Y^{j+1}_{\ell+1}) - \theta(Y^j_{\ell+1})\|$. If $\alpha > \alpha_o$ then add 1 to j and go to Step 8. Otherwise let $X_{\ell+1}$ be the feasible region in X corresponding to $Y^{j+1}_{\ell+1}$. Add 1 to ℓ and go to Step 2.

When this algorithm stops in Step 8 with $Y^{j+1}_{\ell+1}$ becoming empty, then $x_{max} \in X$ corresponding to ϕ_{max} is actually an ε-optimal solution of (2.1).

Though this algorithm may stop in Step 2 rather than in Step 8 and thus may fail to identify an ε-optimal solution, the numerical experiments conducted on CYBER 74 are quite encouraging. Table 4.1 summarizes some of the results for smaller problems.

Table 4.1

Problem No.	Size of the Problem		ε/ϕ_{max}	No. of Local Maxima Identified	Approximate CPU time (sec)
	m	n			
1	3	6	0.0	1	0.2
2	5	8	0.0	2	0.6
3	6	11	0.0	1	0.3
4	7	11	0.0	1	0.5
5	9	19	0.0	2	3.0
6-1	6	12	0.05	5	2.5
6-2	6	12	0.01	6	3.0
6-3	6	12	0.0	6	3.0
7	11	22	0.1	8	28.0

Problems 1 ∿ 5 have no particular structure, while problems 6-1, 6-2, 6-3 and 7 have the following data structure:

$$\max\{c_m^t x + \tfrac{1}{2}x^t Q_m x \mid A_m x \leq b_m,\ x \geq 0\}$$

where

$$A_m = \begin{bmatrix} 1 & 2 & \cdots & m-1 & m \\ 2 & 3 & \cdots & m & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ m & 1 & \cdots & m-2 & m-1 \end{bmatrix}, \quad Q_m = \begin{bmatrix} 2 & -1 & & & \bigcirc \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & \bigcirc & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix}$$

$$c_m = (0, \ldots, 0)^t, \quad b_m = (m(m+1)/2, \ldots, m(m+1)/2)^t$$

They have m local maxima with same objective functional values. All of them are, in fact, global maxima.

The experiments for larger problems are now under way using a more sophisticated version of primal simplex (to be used in Step 4) and dual simplex algorithm (to be used in Step 8). These results will be reported subsequently.

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