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STABLE GROWTH IN THE NONLINEAR COMPONENTS-OF-CHANGE MODEL OF INTERREGIONAL POPULATION GROWTH AND DISTRIBUTION

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Preface

Interest in human settlement systems and policies has been a critical part of urban-related work at IIASA since its inception. Recently this interest has given rise to a concentrated research effort focusing on migration dynamics and settlement patterns. Four sub-tasks form the core of this research effort:

I. the study of spatial population dynamics;

II. the definition and elaboration of a new research area called demometrics and its application to migration analysis and spatial population forecasting;

III. the analysis and design of migration and settlement policy;

IV. a comparative study of national migration and settlement patterns and policies.

This paper, the sixteenth in the dynamics series, studies the long-term properties of the nonlinear model of interregional population growth and distribution proposed by McGinnis and Henry. Intended as an alternative to the linear model which underlies a large number of earlier IIASA publications, this model displays peculiar properties which hinder its usefulness in the study of the dynamics of multiregional population systems.

Related papers in the dynamics series, and other publications of the migration and settlement study, are listed on the back page of this report.

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Abstract

In this paper, a general components-of-change model for a multiregional demographic system is proposed. Characterized by independently derived retention probabilities, it subsumes two of the previously proposed models of population growth and distribution: the linear model studied by Rogers and the nonlinear model put forward by McGinnis and Henry. These two special cases are shown to be symmetrical variants of the proposed general model for a similar consideration of the independently derived retention probabilities.

The long-term behavior of the nonlinear model, partially looked at by McGinnis and Henry, is further examined here and then contrasted with the long-term behavior of the linear model. Unfortunately, the existence of a long-term equilibrium could not be formally proved. However, the derivation of various properties concerning the stable state of the system made possible the development of a methodology permitting the a priori determination of all acceptable equilibrium distributions. The ZPG (zero population growth) and non-ZPG specifications are separately examined, because the non-ZPG case is not as straight forward an extension of the ZPG case as in the linear model.

The long-term properties of the linear and nonlinear models are contrasted by applying these properties to the analysis of migration between the four U.S. Census regions over the period 1965-1970.

Because of its peculiar properties, we conclude that the nonlinear model cannot be a useful substitute for the linear model in the study of the dynamics of multiregional population systems.

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INTRODUCTION

The demographic components-of-change model has been applied to the problem of interregional population growth and distribution by Rogers (1968) and Liaw (1975). Both of these scholars have used a linear formulation characterized by an allocation of outmigrants from any region in constant proportions among possible destination regions. Such a feature has been criticized on the grounds that outmigrants distribute themselves among regions in proportion to economic opportunities offered by these regions (Lowry, 1966). This has led to the development of a nonlinear formulation of the model, which resembles the classical gravity model (McGinnis/Henry, 1973).

Our purpose is to analyze further the long-term features of the nonlinear formulation partially looked at by McGinnis/Henry and to contrast its features with those of the well established linear formulation. This will be carried out in four sections.

Section I, briefly describes the general formulation of the components-of-change model and posits the requirement of independently determined retention probabilities to generate adequate stable growth patterns. It then derives both the linear and nonlinear formulations of the model as "dual" variants of this general model, and goes on with a summary of the long-term properties of the linear formulation.

Section II, is a thorough empirical analysis of the nonlinear model; its results support the existence of a long-term convergence toward stability, similar to the linear case.

Section III, concentrates on the search for acceptable equilibrium solutions in the ZPG (zero population growth) case*,

*In this paper the ZPG system is, by our definition, characterized by zero regional rates of natural increase.

extending the analysis initiated by McGinnis/Henry (1973).

Section IV also deals with the same problem, but for the non-ZPG case, whose complexity makes it difficult to present a level of analysis as complete as in the ZPG case.

In the course of our explorations, we have also examined alternative specifications of the components-of-change model, in which retention probabilities are not independently determined, thereby generating undesirable problems. The analysis of the growth pattern of these specifications is included in Appendices 1 through 3.

I. BACKGROUND SECTION

In order to clarify the contrasts between the linear and non-linear formulations of the components-of-change model of inter-regional population growth and distribution, we begin with several important generalities.

The Components-of-Change Model: Generalities

Suppose there are n regions in a closed multiregional population system. Let $w_i(t)$ and $w_i(t+1)$ be the population sizes of the i^{th} region at times t and t + 1; $M_{ij}(t) \geq 0$ be the number of people present in region j at time t + 1 and in region i at time t; and $N_i(t)$ be the population change due to natural growth in region i during the unit time interval (t,t + 1). The flow equations of the multiregional population system can then be written as

$$w_i(t+1) = w_i(t) + N_i(t) + \sum_{j \neq i} M_{ji}(t) - \sum_{j \neq i} M_{ij}(t) \quad , \quad \forall i = 1,\ldots,n \quad . \tag{1}$$

This equation states that the population size in region i at time (t + 1) is obtained from the population present in region i at time t by adding net population change due to natural increase growth over the period (t,t + 1) to the flows of inmigration from all other regions, and by subtracting the flows of outmigration to all other regions.

In what follows, $N_i(t)$ is assumed to vary with the size of the at-risk population $w_i(t)$, i.e.,

$$N_i(t) = n_i(t)\ w_i(t) \quad , \quad \forall i = 1,\dots,n \quad . \tag{2}$$

The migration flows are assumed to depend on population sizes at the origin and destination as well as a relational term standing for the intervening obstacles between origin and destination regions:

$$M_{ij}(t) = a_{ij}(t)\ w_i(t)\ w_j(t) \quad , \quad \forall i,j = 1,\dots,n \quad , \quad j \neq i \quad , \tag{3}$$

in which $a_{ij}(t)$ is the relational term linking regions i and j.

Substitution of (2) and (3) into the flow equation (1) then yields

$$w_i(t+1) = [1 + n_i(t)]\ w_i(t) + w_i(t)\ [\sum_{j \neq i} a_{ji}(t)\ w_j(t)] - [\sum_{j \neq i} a_{ij}(t)\ w_j(t)]\ w_i(t) \quad , \quad \forall i = 1,\dots,n \quad . \tag{4}$$

This may be rewritten in a more compact format as:

$$\{w(t+1)\} = [\underset{\sim}{I} + \underset{\sim}{N}(t)]\ \{\underset{\sim}{w}(t)\} + \underset{\sim}{w}(t)\ [\underset{\sim}{A}(t) - \underset{\sim}{A}'(t)]\{w(t)\} \quad , \tag{5}$$

in which

- $\{w(t)\}$ is a vector whose typical elements is $w_i(t)$;
- $\underset{\sim}{w}(t)$ is a diagonal matrix whose typical element is $w_i(t)$;
- $\underset{\sim}{I}$ is the identity matrix;
- $\underset{\sim}{N}(t)$ is a diagonal matrix of natural increase rates;
- $\underset{\sim}{A}(t)$ is a matrix of relational terms between each pair of regions; and
- $\underset{\sim}{A}'(t)$ is the transpose of $\underset{\sim}{A}(t)$.

Note that in $\underset{\sim}{A}(t)$ all diagonal elements are equal to zero.

$$\underset{\sim}{A}(t) = \begin{bmatrix} 0 & a_{21}(t) & \cdots & a_{n1}(t) \\ a_{12}(t) & \ddots & & \\ \vdots & & 0 & \\ & & & \ddots \\ a_{1n}(t) & & & 0 \end{bmatrix}$$

Clearly, equation (5) makes it possible to iteratively calculate the population distribution of the system at any future point in time from prior knowledge of $\underset{\sim}{N}(t)$ and $\underset{\sim}{A}(t)$. However, after a sufficiently long period of time, the pattern of population growth and distribution implied by the implementation of the projection process embodied in (5) may create unfortunate problems. For example, if we suppose that the matrix $\underset{\sim}{A}(t)$ is stationary, under certain circumstances, we can obtain negative populations! Moreover, Appendix 2, which deals with this special case shows that regional populations need not be negative to obtain problems: it may happen that the number of migrants out of a region is higher than the number of people living in the region at the beginning of the time period considered and that the population of this region remains positive because the number of inmigrants is greater than the number of outmigrants. The occurrence of such problems stems from the assumptions concerning migration flows included in (5) according to which stayers are obtained as residuals (by subtracting total outmigration flows from the beginning of period populations), which does not guarantee their positivity. The conclusion is that a meaningful formulation of the components-of-change model must ensure that the total migration out of a region is less than the population of this region. Therefore, we suppose that the retention probabilities are given independently, as a property of the regions themselves, i.e.,

$$\frac{M_{ii}(t)}{w_i(t)} = p_{ii}(t) \quad , \quad \forall i = 1,\ldots,n \quad , \tag{6}$$

in which

$M_{ii}(t)$ is the flow of stayers in region i; and

$p_{ii}(t)$ is the probability of being in region i at time t + 1 for an individual present in region i at time t.

Since we have the following relationship between the number of stayers and migrants:

$$M_{ii}(t) = w_i(t) - \sum_{j \neq i} M_{ij}(t) \quad , \quad \forall i = 1,\ldots,n \quad , \tag{7}$$

the result is that we can rewrite (1) as

$$w_i(t+1) = M_{ii}(t) + N_i(t) + \sum_{j \neq i} M_{ji}(t) \quad , \quad \forall i = 1,\ldots,n \quad ,$$

or, after substituting (2), (3) and (6),

$$w_i(t+1) = [p_{ii}(t) + n_i(t)] \; w_i(t) + [\sum_{j \neq i} a_{ji}(t) \; w_j(t)] \; w_i(t) \quad ,$$

$$\forall i = 1,\ldots,n \quad . \tag{8}$$

We can rewrite (8) more compactly as

$$\{w(t+1)\} = [\underset{\sim}{P}_d(t) + \underset{\sim}{N}(t)] \; \{w(t)\} + \underset{\sim}{w}(t) \; \underset{\sim}{A}(t) \; \{w(t)\} \quad , \tag{9}$$

in which $\underset{\sim}{P}_d(t)$ is a diagonal matrix of retention probabilities. Indeed, a price has to be paid for the choice of an independent derivation of $\underset{\sim}{P}_d(t)$: $\underset{\sim}{A}(t)$ now depends on $\underset{\sim}{P}_d(t)$ as shown by the following equality linking two alternative expressions of the outmigration flows

$$w_i(t) \; [\sum_{j \neq i} a_{ij}(t) \; w_j(t)] = [1 - p_{ii}(t)] \; w_i(t) \quad ,$$

$$\forall i = 1,\ldots,n \quad ,$$

which can be expressed more concisely as

$$\underset{\sim}{w}(t)\ \underset{\sim}{A}'(t)\ \{w(t)\} = [\underset{\sim}{I} - \underset{\sim}{P}_d(t)]\ \{w(t)\} \quad . \tag{10}$$

The "Duality" of the Linear and Nonlinear Models

To allow for the variations of the relational term $a_{ij}(t)$, we posit with Alonso (1973, 1977)

$$a_{ij}(t) = \gamma_i(t) d_{ij}\ \beta_j(t) \quad ,$$

in which

- d_{ij} is a conductance term linking regions i and j (e.g., the distance between i and j);
- $\gamma_i(t)$ a term characteristic of region i related to its "pushing" power (population); and
- $\beta_j(t)$ a term characteristic of region j related to the ex-ante number of migrants to region j per unit of "pull" (population).

In matrix format, we thus have

$$\underset{\sim}{A}(t) = \underset{\sim}{\beta}(t)\ \underset{\sim}{D}\ \underset{\sim}{\gamma}(t) \quad ,$$

in which

- $\underset{\sim}{\beta}(t)$ and $\underset{\sim}{\gamma}(t)$ are diagonal matrices, and
- $\underset{\sim}{D}$ is a matrix whose $(i,j)^{th}$ element is the conductance factor d_{ji}.

Clearly, for any prior choice of $\underset{\sim}{P}_d(t)$, $\underset{\sim}{\beta}(t)$ and $\underset{\sim}{\gamma}(t)$ are to be obtained from (10). However, the vector equation (10) contains only n scalar equations which make it impossible to determine the 2n non-zero scalars contained in $\underset{\sim}{\beta}(t)$ and $\underset{\sim}{\gamma}(t)$. The result is that the linkage of $\underset{\sim}{A}(t)$ and $\underset{\sim}{D}$ must take the form of either

$$\tilde{A}(t) = \tilde{\beta}(t)\ \tilde{D} \quad , \tag{11}$$

or,

$$\tilde{A}(t) = \tilde{D}\ \tilde{\gamma}(t) \quad . \tag{12}$$

In the former case, substituting (11) into (10) yields

$$\tilde{w}(t)\ \tilde{D}\ \tilde{\beta}(t)\ \{w(t)\} = (\tilde{I} - \tilde{P}_d)\ \{w(t)\} \quad .$$

Supposing the $w_i(t) \neq 0 (\forall i)$, we then have

$$\tilde{D}\ \tilde{\beta}(t)\ \{w(t)\} = [\tilde{I} - \tilde{P}_d(t)]\{i\} \ ,$$

in which $\{i\}$ is a column vector of ones.

Let us now suppose that $\tilde{P}_d(t)$ is independent of time, i.e., $\tilde{P}_d(t) = \tilde{P}_d$. Then $\tilde{\beta}(t)\ \{w(t)\}$ is a constant vector:

$$\tilde{\beta}(t)\ \{w(t)\} = \tilde{\beta}(0)\ \{w(0)\} = \tilde{D}^{-1}\ (\tilde{I} - \tilde{P}_d)\ \{i\} \quad ,$$

so that the place-to-place migration flows

$$M_{ij}(t) = a_{ij}(t)\ w_i(t)\ w_j(t)$$

can be expressed as

$$M_{ij}(t) = d_{ij}\ [\beta_j(t)\ w_j(t)]\ w_i(t) = d_{ij}\ [\beta_j(0)\ w_j(0)]\ w_i(t) \quad ,$$

$$= p_{ij}\ w_i(t) \quad , \quad \forall i,j = 1,\ldots,n \quad , \quad j \neq i \quad ,$$

in which p_{ij} is a constant. Then the projection process reduces to

$$\{w(t + 1)\} = \tilde{G}\ \{w(t)\} \quad , \tag{13}$$

in which $\underset{\sim}{G}$ is a constant growth operator matrix, that is the sum of $\underset{\sim}{N}$ and a constant matrix of transition probabilities $\underset{\sim}{P} = \underset{\sim}{P}_d + \underset{\sim}{w}(0)\underset{\sim}{A}$.

The adjustment of $a_{ij}(t)$ by a multiplicative factor $\beta_j(t)$ relating to the destination region thus leads to the usual linear formulation of the components-of-change model (Rogers, 1968 and Liaw, 1975).

Alternatively, if we choose to take the multiplicative adjustment in relation to the origin region $\gamma_i(t)$, we have

$$\underset{\sim}{w}(t) \;\underset{\sim}{\gamma}(t)\; \underset{\sim}{D}\; \{w(t)\} = (\underset{\sim}{I} - \underset{\sim}{P}_d)\; \{w(t)\} \quad ,$$

or, in scalar terms,

$$w_i(t)\; \gamma_i(t) \left[\sum_{j \neq i} d_{ij}\; w_j(t) \right] = (1 - p_{ii})\; w_i(t) \quad ,$$

so that the place-to-place migration flows can be expressed as,

$$M_{ij}(t) = (1 - p_{ii})\; \frac{d_{ij}\; w_i(t)\; w_j(t)}{\sum_{k \neq i} d_{ik}\; w_k(t)} \quad , \quad \forall i,j = 1,\ldots,n \quad , \quad j \neq i \quad . \qquad (14)$$

This is precisely the specification of the nonlinear model proposed by McGinnis and Henry (1973).

In conclusion, the classic linear and nonlinear specifications of the components-of-change model are special cases of the version in which retention probabilities are independently determined. Moreover they appear as "dual" variants in that they correspond to similar types of adjustment for the relational elements: the difference between both variants lies in the choice of this adjustment that relates to destination (linear specification) or origin (nonlinear specification).

The Linear Model: Summary of Properties and Results

The specification of this model (13) makes clear that we can iteratively calculate the population distribution at any future point in time given structural matrices of $\underset{\sim}{N}$ and $\underset{\sim}{P}$, and an initial distribution $\{w(0)\}$.

With the data provided by the 1970 U.S. Census of Population it is possible to compute the probability transition matrix $\underset{\sim}{P}$ relating to the system of the four U.S. Census regions (North East, North Central, South and West) observed during the period 1965 - 1970 (see Table 1). Natural increase data for the same system (Table 2) allows the estimation of $\underset{\sim}{G}$ relating to the same period.

Table 1. U.S. regions 1965 - 1970: the $\underset{\sim}{P}$ matrix

1965 / 1970	NORTH EAST	NORTH CENTRAL	SOUTH	WEST
NORTH EAST	0.95294	0.00809	0.01164	0.00848
NORTH CENTRAL	0.01065	0.94573	0.01872	0.01921
SOUTH	0.02518	0.02615	0.95380	0.02699
WEST	0.01122	0.02003	0.01585	0.94532

Table 2. U.S. regions 1965 - 1970: the matrix of natural increase rates

1965 / 1970	NORTH EAST	NORTH CENTRAL	SOUTH	WEST
NORTH EAST	0.00599	0	0	0
NORTH CENTRAL	0	0.00780	0	0
SOUTH	0	0	0.00910	0
WEST	0	0	0	0.01003

Applying the matrix $\underset{\sim}{G}$ to the initial population distribution of the system (given by the first row figures of Table 3) permits one to calculate the regional population distribution for 1970 (given by the second row of the same table). From there, using the aforementioned iterative process, one can calculate the regional shares at any future point. Table 3 indicates that these regional shares tend to stabilize after a sufficiently long period of time:

- the North East region constitutes 16.43 percent of the total population in the stable state versus 24.20 percent initially.
- A similar decrease in importance is experienced by the North Central region--23.76 percent in the long-term versus 28.08 percent initially.
- In contrast, the South and West regions increase their shares from 30.82 to 36.50 percent and 16.90 to 23.31 percent, respectively.

However, the simplicity of (13) makes the iterative generation of the system's stable state unnecessary. It is possible to derive an analytical solution to the model by applying Laplace transformations to (13). Supposing that the eigenvalues of $\underset{\sim}{G}$ are distinct (which is generally the case) we have (Liaw, 1975):

$$\{w(t)\} = [\sum_{i=1} \lambda_i^t \underset{\sim}{B}_i] \{w(0)\} \quad , \qquad (15)$$

where λ_i is one of the n-distinct roots of the characteristic equation $|\underset{\sim}{G} - \lambda \underset{\sim}{I}| = 0$ and $\underset{\sim}{B}_i = \lim_{\lambda \to \lambda_i} (\lambda - \lambda_i)(\underset{\sim}{G} - \lambda \underset{\sim}{I})^{-1}$ whose nonzero columns are all characteristic vectors of the structural matrix $\underset{\sim}{G}$ associated with the characteristic value λ_i: Suppose λ_1 is the largest characteristic root of the system's structural matrix, then (15) may be rewritten as (Liaw, 1975):

$$\{w(t)\} = \lambda_1^t \underset{\sim}{B}_1 \{w(0)\} + \sum_{i=2}^{n} \left(\frac{\lambda_i}{\lambda_1}\right)^t \underset{\sim}{B}_i \{w(0)\} \quad , \qquad (16)$$

Table 3. Linear model-non-ZPG formulation - U.S. regions - ex ante simulation

Period	λ	Regional Shares of Total Population (Percentage)			
		North East	North Central	South	West
1	0.00000	24.197	28.082	30.820	16.901
2	1.00814	23.739	27.710	31.223	17.328
3	1.00816	23.308	27.369	31.597	17.726
4	1.00818	22.903	27.056	31.944	18.097
5	1.00820	22.521	26.769	32.267	18.442
6	1.00822	22.162	26.505	32.568	18.765
7	1.00824	21.825	26.263	32.847	19.066
8	1.00826	21.507	26.041	33.106	19.346
9	1.00827	21.208	25.838	33.347	19.607
10	1.00829	20.927	25.651	33.570	19.851
15	1.00834	19.751	24.930	34.473	20.846
20	1.00839	18.884	24.468	35.097	21.550
25	1.00842	18.245	24.176	35.530	22.050
30	1.00844	17.773	23.993	35.829	22.405
35	1.00845	17.424	23.882	36.036	22.658
40	1.00847	17.166	23.815	36.179	22.839
45	1.00847	16.976	23.777	36.278	22.969
50	1.00848	16.835	23.756	36.347	23.062
55	1.00848	16.731	23.746	36.394	23.129
60	1.00849	16.653	23.742	36.427	23.177
65	1.00849	16.596	23.741	36.450	23.212
70	1.00849	16.554	23.743	36.466	23.238
75	1.00849	16.522	23.745	36.477	23.256
80	1.00849	16.499	23.747	36.484	23.270
85	1.00849	16.482	23.749	36.490	23.279
90	1.00849	16.469	23.751	36.493	23.287
95	1.00849	16.459	23.753	36.496	23.292
100	1.00849	16.452	23.754	36.498	23.296
150	1.00849	16.433	23.760	36.502	23.306
200	1.00849	16.432	23.760	36.502	23.306
250	1.00850	16.432	23.760	36.502	23.306
300	1.00850	16.432	23.760	36.502	23.306
350	1.00850	16.432	23.760	36.502	23.306
400	1.00850	16.432	23.760	36.502	23.306
450	1.00850	16.432	23.760	36.502	23.306
500	1.00850	16.432	23.760	36.502	23.306

Since $|\frac{\lambda_i}{\lambda}| < 1$, we have $\{w(t)\} \sim \lambda_1^t \underset{\sim}{B}_1 \{w(0)\}$ as $t \to \infty$ where $\underset{\sim}{B}_1 = [c_1\{x\}_1, c_2\{x\}_1, \ldots, c_n\{x\}_1]$ is a positive characteristic matrix in which $\{x\}_1$ is the right characteristic vector associated with λ_1.

If $\lambda_1 < 1$, then $\{w(t)\} \to \{0\}$ (case of a vanishing system); whereas, if $\lambda_1 \geq 1$, then $\{w(t)\} \sim \lambda_1^t [\sum c_i w_i(0)]\{x\}_1$ (case of an exploding system tending towards a positive long-run proportional distribution that is independent of the initial population size and distribution). Specifically, the long-run proportional distribution is given by the characteristic vector $\{x\}_1$ of the structural matrix $\underset{\sim}{G}$ associated with the largest characteristic root λ_1

$$[\underset{\sim}{G} - \lambda_1 \underset{\sim}{I}] \; \{x\}_1 = \{0\} \quad .$$

Note that if $\underset{\sim}{N} = \underset{\sim}{0}$, then the interregional population system represented by (13) is a ZPG-system such as

$$\{w(t + 1)\} = \underset{\sim}{P} \; \{w(t)\} \quad . \tag{17}$$

The structural matrix $\underset{\sim}{P}$ is the stochastic matrix of a regular Markov chain which has a unit characteristic root that exceeds all other characteristic roots in magnitude and furthermore, has a stochastic characteristic matrix associated with this characteristic root that has identical columns. The ZPG-system thus approaches the positive equilibrium distribution

$$\{w(t)\} \to \underset{\sim}{B}_1 \; \{w(0)\} = [\sum_i w_i(0)] \; \{x\}_1 = K(0) \; \{x\}_1 \quad ,$$

where $K(0)$ is the initial total population of the system and $\{x\}_1$ the normalized right characteristic of $\underset{\sim}{P}$ (associated with λ_1).

Clearly, in such circumstances, the total population of the system remains equal to the initial population and the process studied is one of population distribution between regions.

II. THE NONLINEAR MODEL: EMPIRICAL ANALYSIS

Extensive projection exercises, carried out with several sets of data, allow us to conclude that the projection of a spatially disaggregated population using the nonlinear model also leads to a stable situation. However, the more complex formulation of the nonlinear model makes it difficult to establish a formal proof of this convergence. This analysis, therefore, is limited to the presentation of nonlinear projections and contrasts to their linear counterparts, and is continued in the next sections, with a search for acceptable equilibrium solutions.

Long-term Behavior of the Nonlinear Model: Empirical Evidence

The nonlinear specification of the components-of-change model consists of the flow equation (5) [or alternatively (9)] and the constraint equation (10) in which $\underset{\sim}{N}(t)$ and $\underset{\sim}{P}_d(t)$ are constant matrices and $\underset{\sim}{A}(t)$ is given by (12).

Since there is the following relationship between $\underset{\sim}{A}(t)$ and $\underset{\sim}{A}(0)$ (later denoted as $\underset{\sim}{A}$),

$$\begin{aligned} \underset{\sim}{A}(t) &= \underset{\sim}{A}(0)\ \underset{\sim}{\gamma}(0)^{-1}\ \underset{\sim}{\gamma}(t) \\ &= \underset{\sim}{A}\ \underset{\sim}{\alpha}(t) \quad , \end{aligned}$$

it follows that (5) and (10) can be rewritten, respectively, as

$$\begin{aligned} \{w(t+1)\} = (\underset{\sim}{I} + \underset{\sim}{N})\ \{w(t)\} + w(t)\ [\underset{\sim}{A}\ \underset{\sim}{\alpha}(t) \\ - \underset{\sim}{\alpha}(t)\ \underset{\sim}{A}]\ \{w(t)\} \quad , \end{aligned} \tag{18}$$

and

$$\underset{\sim}{w}(t)\ \underset{\sim}{\alpha}(t)\ \underset{\sim}{A}\ \{w(t)\} = [\underset{\sim}{I} - \underset{\sim}{P}_d]\ \{w(t)\} \quad . \tag{19}$$

Also note that (9) becomes:

$$\{w(t+1)\} = (\underset{\sim}{P}_d + \underset{\sim}{N})\ \{w(t)\} + \underset{\sim}{w}(t)\ \underset{\sim}{A}\ \underset{\sim}{\alpha}(t)\ \{w(t)\} \quad . \tag{20}$$

Clearly, given structural matrices $\underset{\sim}{N}$, $\underset{\sim}{P}_d$ and $\underset{\sim}{A}$, and an initial distribution $\{w(0)\}$, we can iteratively calculate the population distribution at any future point in time by obtaining $\underset{\sim}{\alpha}(t)$ from (19); and then inserting the estimate thus calculated into (18) or (20).

As an illustration, this iterative calculation has been performed for the system of the four U.S. Census regions already considered in Section I. Apart from $\underset{\sim}{N}$ and $\underset{\sim}{P}_d$ (a diagonal matrix whose diagonal is taken as the diagonal of $\underset{\sim}{P}$) whose actual values were given earlier in Section I, we have observed the matrix of relational elements which appears in Table 4 below. (All elements have been multiplied by 10^5).

Table 4. U.S. regions 1965 - 1970: the matrix of relational elements

1970 \ 1965	NORTH EAST	NORTH CENTRAL	SOUTH	NORTH
NORTH EAST	0	0.01915	0.02755	0.02007
NORTH CENTRAL	0.02172	0	0.03818	0.03919
SOUTH	0.04681	0.04861	0	0.05017
NORTH	0.03804	0.06788	0.05371	0

The successive regional shares obtained by the application of the method mentioned earlier are in Table 5*, which indicates the tendency of these shares to stabilize after a sufficiently long time period. Note the tendency of the North East region to empty and of the West region to augment its share to a proportion slightly less than the share of the South region.

*During the first ten or fifteen forecasting periods, the regional shares obtained from both specifications remain quite close (compare Tables 3 and 5).

Table 5. Nonlinear model-non-ZPG formulation - U.S. regions - ex ante simulation

Period	λ	Regional Shares of Total Population (Percentage)			
		North East	North Central	South	West
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2	1.00814	23.739	27.710	31.223	17.328
3	1.00816	23.292	27.353	31.605	17.750
4	1.00818	22.856	27.008	31.969	18.166
5	1.00821	22.431	26.677	32.315	18.578
6	1.00823	22.016	26.357	32.643	18.984
7	1.00825	21.612	26.048	32.956	19.384
8	1.00827	21.217	25.751	33.253	19.780
9	1.00829	20.831	25.464	33.536	20.169
10	1.00831	20.455	25.187	33.805	20.553
15	1.00840	18.702	23.940	34.973	22.385
20	1.00848	17.141	22.891	35.904	24.064
25	1.00855	15.746	22.004	36.664	25.586
30	1.00861	14.495	21.252	37.302	26.952
35	1.00867	13.369	20.612	37.852	28.168
40	1.00872	12.353	20.067	38.338	29.243
45	1.00876	11.433	19.603	38.776	30.188
50	1.00880	10.598	19.208	39.175	31.018
55	1.00884	9.839	18.872	39.544	31.745
60	1.00887	9.146	18.587	39.884	32.383
65	1.00890	8.513	18.345	40.200	32.942
70	1.00892	7.932	18.141	40.493	33.434
75	1.00895	7.399	17.970	40.763	33.868
80	1.00897	6.908	17.828	41.013	34.251
85	1.00899	6.456	17.710	41.243	34.591
90	1.00900	6.038	17.614	41.454	34.895
95	1.00902	5.652	17.536	41.647	35.165
100	1.00903	5.293	17.475	41.823	35.408
150	1.00913	2.824	17.417	42.899	36.861
200	1.00917	1.546	17.745	43.267	37.442
250	1.00919	0.855	18.087	43.370	37.688
300	1.00920	0.474	18.346	43.384	37.796
350	1.00920	0.264	18.519	43.374	37.844
400	1.00920	0.147	18.627	43.361	37.866
450	1.00921	0.081	18.692	43.350	37.876
450	1.00921	0.081	18.692	43.350	37.876
500	1.00921	0.045	18.730	43.343	37.882
550	1.00921	0.025	18.752	43.339	37.884
600	1.00921	0.014	18.764	43.337	37.885
650	1.00921	0.008	18.771	43.335	37.886
700	1.00921	0.004	18.775	43.334	37.886
750	1.00921	0.002	18.777	43.334	37.887
800	1.00921	0.001	18.779	43.333	37.887
850	1.00921	0.001	18.779	43.333	37.887
900	1.00921	0.000	18.780	43.333	37.887

The nonlinear projection process thus tends toward an equilibrium characterized by a constant regional allocation, say $\{y\}$. Thus, near stability, two consecutive population vectors satisfy the following relationship:

$$\{w(t+1)\} = \lambda \, \{w(t)\}$$

Substituting this equality into (20) yields:

$$\lambda\{w(t)\} = (\underset{\sim}{P}_d + \underset{\sim}{N}) \, \{w(t)\} + \underset{\sim}{w}(t) \, \underset{\sim}{A} \, \underset{\sim}{\alpha}(t) \, \{w(t)\} \quad .$$

From (19) it is clear that $\underset{\sim}{\alpha}(t) \, \underset{\sim}{w}(t)$ is a homogenous function in $\underset{\sim}{w}(t)$ of degree zero. Therefore, the constant regional allocation $\{y\}$ is given by

$$\lambda\{y\} = (\underset{\sim}{P}_d + \underset{\sim}{N}) \, \{y\} + \underset{\sim}{y} \, \underset{\sim}{A} \, \underset{\sim}{\alpha}(\infty) \, \{y\} \quad , \tag{21}$$

so that

$$\underset{\sim}{y} \, \underset{\sim}{\alpha}(\infty) \, \underset{\sim}{A}'\{y\} = (\underset{\sim}{I} - \underset{\sim}{P}_d) \, \{y\} \quad . \tag{22}$$

Linear and Nonlinear Projections: An Empirical Comparison

We begin our comparison of the linear and nonlinear projections by contrasting their equilibrium distributions.

Equilibrium Distributions Contrasted

Apparently, the nonlinear formulation of the components-of-change model always leads to a long-term convergence. None of the various experiments made with this model has proved this wrong. Although we could not establish any formal proof of this property [in spite of the recent developments in the balanced growth of nonlinear systems, Nikaido (1968)], we can safely claim that the nonlinear model always admits a limiting distribution as the linear model.

Striking differences between the limiting distributions of the alternative models are suggested by the figures of Table 6; these point to:

- the possible occurrence of empty regions at stability in the nonlinear model; and
- the tendency of the nonlinear model to exaggerate the long-term tendencies displayed by the linear model. On one hand, the population share of the North East region, which is initially 24.20 percent, increases to 16.43 percent in the long-term equilibrium of the linear model and vanishes in the long-term equilibrium of the nonlinear model; on the other hand, the share of the West region (16.90 percent initially) increases to 23.31 percent in the long-term equilibrium of the linear model and 37.89 percent in the limiting distribution of the nonlinear model.

Table 6. U.S. regions - initial and equilibrium distributions contrasted,* (all regional shares in percentages)

	Initial Net Inmigration Rate	Initial Distribution	Regional Shares		Percentage Change in Regional Shares	
			Linear	Nonlinear	Linear	Nonlinear
NORTH EAST	- 0.01693	24.20	16.43(17.16)	0(0)	- 32.8	- 100.0
NORTH CENTRAL	- 0.01299	28.08	23.76(23.90)	18.78(20.28)	- 15.3	- 33.1
SOUTH	+ 0.01221	30.82	36.50(36.18)	43.33(42.73)	+ 18.4	+ 40.2
WEST	+ 0.02356	16.90	23.31(22.76)	37.89(36.99)	+ 38.4	+ 124.1

*Figures in parentheses correspond to the equilibrium distribution of the non-ZPG formulation of the linear and nonlinear models.

Overall, the less conservative character of the limiting distribution of the nonlinear model is clear: the changes in the region's population shares are more radical in the nonlinear case than in the linear case. For example, the increase in the share of the West region is 124.1 percent in the nonlinear case and only 38.4 percent in the linear case.

Table 6 also indicates that, whatever the model (linear or nonlinear), the relative changes in the regional allocation of the U.S. population (between the initial period and the long run) are related to the initial net inmigration rates of each region: regions having initially positive net inmigration rates see their relative shares increase, while those regions with initially negative inmigration rates see their importance decrease.

Relation Between Initial and Equilibrium Distributions

Another important difference between the models which does not appear in the figures of Table 6 relates to the independence of the limiting regional distribution of population vis-a-vis the initial distribution. While the limiting behavior of the linear model is not affected by $\{w(0)\}$--the equilibrium state of the nonlinear model may, in some ways, be affected by $\{w(0)\}$.

If the projection process, characterized by the $\tilde{A}$, $\tilde{P}_d$ and $\tilde{N}$ matrices of our four region system of the U.S., always leads to some equilibrium solution (whatever the initial regional distribution), it may happen that different equilibrium solutions will be obtained (an illustration of such a situation will be given in Section III). Apparently, for each choice of the $\tilde{A}$, $\tilde{P}_d$ and $\tilde{N}$ matrices, there exists one or several equilibrium solutions, completely characterized by the elements of $\tilde{A}$, $\tilde{P}_d$ and $\tilde{N}$ and independent of $\{w(0)\}$; the initial distribution $\{w(0)\}$ affects the long-term behavior of the system only in that, when there exists more than one equilibrium solution it determines which one of the possible alternative stable equilibriums will be attained.

It is possible to gain further insights into the alternative models by comparing the evolution of out- and inmigration rates over the projection process.

Evolution of Out- and Inmigration Rates

Indeed, since both models assume constant retention probabilities, the path to equilibrium is characterized by total outmigration rates that remain constant. However, whereas in the linear case place-to-place outmigration rates also remain constant (by assumption), they tend*, in the nonlinear case, to vary in direct proportion to the population size of the destination region. This is confirmed by the figures of Table 7, which show that the place-to-place outmigration rates decrease if the destination is the North East or North Central, stabilize if the destination is the South (except in the West region outmigration), and increase if the destination is the West.

Inmigration rates, however, do not follow such a clear path toward equilibrium. If no natural increase occurs (ZPG case), place-to-place inmigration rates vary in such a way as to ensure the long-term equality of total migration flows into and out of each region. Note that this implies the equality of total outmigration and inmigration rates only in regions that do not vanish in the long-run.** Thus, in the linear ZPG case (in which no region can vanish), total inmigration rates of each region tend to increase (in regions in which there is initially a negative net inmigration) or to decrease (in regions initially displaying a positive net inmigration), in order to equal outmigration rates. Since the place-to-place inmigration rates are proportional to the ratio of the population sizes in the destination and origin regions, they generally tend to decrease if the origin is the North East or North Central, and to increase if the origin is the South or West.

In the nonlinear case, the occurrence of vanishing regions in the ZPG system is made possible by the impossibility of the total inmigration rate of these regions to be equal to the total

*The constant of adjustment entering $a_{ij}(t)$ relates to the origin and does not affect the relative importance of the place-to-place migration rates out of a region.

**Regions will be said to vanish when their populations decline to zero.

outmigration rate. For instance, Tables 7 and 8 show that the total migration rate into the North East region (0.03810) is less than the total migration rate out of that region (0.04706). As expected, a "dying out" region is characterized by a negative net inmigration rate. This feature of the nonlinear model is very useful to determine a priori the long-term equilibria and permits, as we will see later on, the narrowing down of the number of acceptable equilibrium solutions. Place-to-place migration rates, which vary as a direct proportion to the population size of the origin region and to its associated constant of adjustment, tend to decrease over the projection process if the origin is the North East or North Central region, and to increase if it is the West region.

There is a clear tendency for the place-to-place out- and inmigration rates of the same origin-destination regions to vary in the same direction. The relative pace of their variations depends only on the initial relative position of the net inmigration rates of these regions.

A similar analysis can also be performed in the non-ZPG case. The difference is that, the long term equilibrium is no longer characterized by the equality of out- and inmigration flows. Instead, we have the following:

Inmigration + Natural Increase-Outmigration = $(\lambda - 1) \times$ population in which λ is the ratio, common to each region, of the population sizes in two consecutive periods at equilibrium. Then at equilibrium, a nonvanishing region will be characterized by a net inmigration rate equal to $\lambda-1-n_i$ (where n_i is the natural increase rate of this region), while the vanishing region will have a net inmigration rate of less than $\lambda-1-n_i$.

The Aggregation Problem

The aggregation capabilities of the linear and nonlinear formulations provide another point of departure in the study of the components-of-change model. Suppose that we transform our four region system of the U.S. into various three region systems

Table 7. U.S. regions - initial and stable outmigration rates (ZPG case)*

Region / To	North East	North Central	South	West
North East	0	0.00809 0.00809 0	0.01164 0.01164 0	0.00848 0.00848 0
North Central	0.01065 0.01065 0.00539	0	0.01872 0.01872 0.01296	0.01921 0.01921 0.01479
South	0.02518 0.02518 0.02446	0.02615 0.02615 0.02457	0	0.02699 0.02699 0.03989
West	0.01122 0.01122 0.01721	0.02003 0.02003 0.02970	0.01585 0.01585 0.03324	0
Total	0.04706 0.04706 0.04706	0.05427 0.05427 0.05427	0.04620 0.04620 0.04620	0.05468 0.05468 0.05468

Table 8. U.S. regions - initial and stable inmigration rates (ZPG case)*

Region / To	North East	North Central	South	West
North East	0	0.00918 0.00764 0	0.01977 0.01194 0	0.01607 0.00846 0
North Central	0.00939 0.01127 0.00460	0	0.02383 0.01728 0.01166	0.03328 0.02103 0.01629
South	0.01482 0.02454 0.01969	0.02054 0.02833 0.02730	0	0.02890 0.02519 0.03840
West	0.00592 0.01125 0.01381	0.01156 0.01830 0.02698	0.01480 0.01698 0.03454	0
Total	0.03013 0.04706 0.03810	0.04128 0.05427 0.05427	0.05841 0.04620 0.04620	0.07824 0.05468 0.05468

*In both tables, the three figures in each box represent the outmigration or inmigration rates in the initial population and the stable population (linear and nonlinear cases) respectively.

obtained by the aggregation of two contiguous regions. We then perform the projection process on these alternative systems, using both the linear and nonlinear models. The comparison of the resulting limiting regional shares (Table 9), shows that in the linear case the timing of aggregation has little influence on the stable state. It does not make much difference whether aggregation takes place before or after the projection process. However, in the nonlinear model, the timing of aggregation has a large impact. For instance, the region obtained by aggregating the South and West regions accounts for 81.22 percent of the equilibrium population in the three region system thus obtained, versus 65.30 percent if calculated by aggregating the South and West shares of the four region system.

A Special Case of the Nonlinear Model: Specification and Limiting Behavior

An interesting special case of this model (denoted as nonlinear model II) is obtained by supposing no impact from the relational elements, i.e.

$$d_{ij} = 1 \quad , \quad \text{for all } i,j = 1,\dots,n \quad , \; j \neq i \quad .$$

In such circumstances, (14) reduces to*

$$M_{ij}(t) = (1 - p_{ii}) \frac{w_i(t) w_j(t)}{\sum_{k \neq i} w_k(t)} \quad , \quad \forall i,j = 1,\dots,n \quad , \quad j \neq i \quad ,$$

so that outmigrants distribute themselves among regions in proportion to the population size of destination regions.

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*This model is similar to the aggregate version of the model considered by Feeney (1973). The difference comes from the constant term which only relates to the origin region in the present model, but relates to both origin and destination regions in Feeney's case. However, the above specification is preferable since Feeney's formulation does not ensure that the total number of outmigrants out of region i is less than the population in region i.

Table 9. Linear and nonlinear models (ZPG formulation): comparison of aggregative capabilities using the four region system of the U.S., (all figures in percentages)

	LINEAR MODEL		NONLINEAR MODEL	
	Three Region System	Four Region System Aggregated	Three Region System	Four Region System Aggregated
North East	16.55	16.43	0	0
North Central	23.74	23.76	34.70	18.78
South/West	59.71	59.81	65.30	81.22
North East	16.14	16.43	0	0
North Central/South	60.22	60.26	65.89	62.11
West	23.64	23.31	34.11	37.89
N.East/N.Central	37.71	40.19	10.21	18.78
South	36.57	36.50	45.53	43.33
West	25.72	23.31	44.26	37.89
North East/South	54.14	52.93	57.79	43.33
North Central	23.02	23.76	10.92	18.78
West	22.84	23.31	31.29	37.89
North East	16.43	16.43	0	0
North Central/West	47.07	47.07	56.96	56.67
South	36.50	36.50	43.04	43.33

Carrying out the projection process on such assumptions also leads to a stable equilibrium; it is quite different, however, from the one obtained in the previous case. The successive regional shares obtained by means of this special model appear in Table 10. Briefly, we find that,

1. in opposition to the full model, accounting for differential elements, the present model leads to an equilibrium characterized by no empty regions; and

2. regional shares at equilibrium do not differ that much from initial ones (the greatest discrepancy is observed in the North Central region, 18.95 percent at equilibrium, compared to 28.08 percent initially).

Note that the position of the North East region in the equilibrium distribution of this model is stronger than in the stable state of the full model. This region actually increases its relative share from 24.20 percent to 27.77 percent, whereas it decreases in the case of the full model.

Table 10. Nonlinear model II - non-ZPG formulation - U.S. regions - ex ante simulation

Period	λ	Regional Shares of Total Population (Percentage)			
		North East	North Central	South	West
1	0.00000	24.197	28.082	30.820	16.901
2	1.00814	24.285	27.862	30.884	16.968
3	1.00814	24.371	27.649	30.945	17.034
4	1.00814	24.455	27.443	31.003	17.099
5	1.00814	24.536	27.243	31.059	17.162
6	1.00814	24.615	27.050	31.111	17.224
7	1.00814	24.692	26.862	31.161	17.285
8	1.00814	24.767	26.679	31.209	17.344
9	1.00814	24.840	26.502	31.255	17.403
10	1.00814	24.911	26.331	31.298	17.460
15	1.00815	25.239	25.542	31.486	17.732
20	1.00815	25.526	24.857	31.635	17.982
25	1.00815	25.779	24.257	31.752	18.211
30	1.00815	26.001	23.730	31.846	18.423
35	1.00815	26.198	23.265	31.920	18.617
40	1.00816	26.371	22.852	31.980	18.797
45	1.00816	26.524	22.484	32.028	18.964
50	1.00816	26.659	22.156	32.067	19.118
55	1.00816	26.778	21.863	32.099	19.260
60	1.00816	26.884	21.599	32.125	19.391
65	1.00816	26.978	21.362	32.147	19.513
70	1.00816	27.062	21.148	32.165	19.626
75	1.00817	27.136	20.955	32.180	19.730
80	1.00817	27.201	20.780	32.193	19.826
85	1.00817	27.260	20.622	32.204	19.915
90	1.00817	27.312	20.478	32.213	19.997
95	1.00817	27.358	20.347	32.222	20.073
100	1.00817	27.400	20.229	32.229	20.143
150	1.00818	27.634	19.487	32.274	20.605
200	1.00818	27.716	19.180	32.296	20.809
250	1.00818	27.746	19.050	32.307	20.898
281	1.00818	27.754	19.010	32.310	20.925
300	1.00818	27.758	18.994	32.312	20.937
350	1.00818	27.763	18.969	32.315	20.953
400	1.00818	27.765	18.959	32.316	20.961
450	1.00818	27.765	18.954	32.316	20.964
500	1.00818	27.766	18.952	32.316	20.965
550	1.00818	27.766	18.952	32.317	20.966
600	1.00818	27.766	18.951	32.317	20.966
650	1.00818	27.766	18.951	32.317	20.966
700	1.00818	27.766	18.951	32.317	20.966
750	1.00818	27.766	18.951	32.317	20.966

III. THE NONLINEAR MODEL (ZPG FORMULATION): SEARCH FOR EQUILIBRIUM SOLUTIONS

Because it was not possible to complete a formal proof of the convergence of the nonlinear model, the theoretical analysis of it becomes an a priori search for acceptable equilibrium solutions. This is first carried out in the ZPG case which allows for an easier and more complete study.

In the ZPG case natural increase rates are zero:

$$\underset{\sim}{N} = \underset{\sim}{0} \quad ,$$

so that the resulting model is described by:

$$\{w(t+1)\} = \{w(t)\} + \underset{\sim}{w}(t) \ [\underset{\sim}{A} \ \underset{\sim}{\alpha}(t) - \underset{\sim}{\alpha}(t) \ \underset{\sim}{A}'] \ \{w(t)\} \quad , \qquad (23)$$

or alternatively by:

$$\{w(t+1)\} = \underset{\sim}{P}_d \ \{w(t)\} + \underset{\sim}{w}(t) \ \underset{\sim}{A} \ \underset{\sim}{\alpha}(t) \ \{w(t)\} \quad , \qquad (24)$$

in which $\underset{\sim}{\alpha}(t)$ is still defined by (19).

Preliminary Results

We begin the analysis by establishing a preliminary property regarding the occurrence of zero levels of population before equilibrium is reached:

Property 1

If no region is initially empty and $p_{ii} > 0$, $\forall i$, *there exists no absorbing state, i.e., no region can become empty except in the long run. In other words,* $\{x(t)\} > 0$, *for all finite values of* t.

To prove this, we rewrite each scalar equation of (24) as:

$$w_i(t+1) = [p_{ii} + \sum_{j \neq i} a_{ji} \ \alpha_j(t) \ w_j(t)] \ w_i(t) \quad .$$

Since $\alpha_j(t)$ is nonnegative as suggested by (19), the terms between brackets are at least equal to p_{ii}. We then have

$$w_i(t+1) \geq p_{ii}\, w_i(t) \quad , \quad \forall t \quad ,$$

In other words,

$$w_i(1) \geq p_{ii}\, w_i(0) \quad ,$$

and, more generally,

$$w_i(t) \geq p_{ii}^t\, w_i(0) \quad .$$

Since we suppose that no region is initially empty, and that the retention probabilities are strictly positive, we have $w_i(t)\,(\forall i = 1,\ldots,n,)$ which is strictly positive for all finite values of t. No region can become empty except in the long run.

The Equations of the Stationary State

We now turn to properties relating to the equilibrium solutions of (23) [or (24)] accompanied by the constraint (19). If it exists, a long-term equilibrium is obtained as a solution of (21) (in which $\underset{\sim}{N} = \underset{\sim}{0}$) and (22).

In the present case $\lambda = 1$. Then (21) can be rewritten as:

$$(I - \underset{\sim}{P}_d)\ \{y\} = \underset{\sim}{y}\ \underset{\sim}{A}\ \underset{\sim}{\alpha}(\infty)\ \{y\} \quad , \tag{25}$$

a relationship which expresses the equality of out- and inmigration flows at stability.

Note that (25) may be alternatively presented as

$$[\underset{\sim}{y}\ \underset{\sim}{A}\ \underset{\sim}{\alpha}(\infty)\ \underset{\sim}{y} - (\underset{\sim}{I} - \underset{\sim}{P}_d)\ \underset{\sim}{y}]\ \{i\} = \{0\} \quad ,$$

or, after transposition,

$$\{i\}'[\underset{\sim}{y}\ \underset{\sim}{A}'\underset{\sim}{\alpha}(\infty)\ \underset{\sim}{y} - (\underset{\sim}{I} - \underset{\sim}{P}_d)\underset{\sim}{y}] = \{0\}'$$

The matrix between brackets in the above equation is such that its premultiplication by $\{i\}'$ yields the constraint equation and its postmultiplication by $\{i\}$ yields the equilibrium equation.

Returning to the equilibrium equation, it appears that the comparison of (22) with (25) yields an alternative formulation:

$$\underset{\sim}{y}[\underset{\sim}{\alpha}(\infty)\ \underset{\sim}{A}' - \underset{\sim}{A}\ \underset{\sim}{\alpha}(\infty)]\{y\} = \{0\} \quad , \tag{26}$$

that will be useful to establish a particular property of the model.

Finally, as suggested by juxtaposing (25) and (26), an acceptable equilibrium solution $\{y\}$ must verify:

$$\underset{\sim}{y}\ \underset{\sim}{\alpha}(\infty)\ \underset{\sim}{A}'\{y\} = \underset{\sim}{y}\ \underset{\sim}{A}\ \underset{\sim}{\alpha}(\infty)\{y\} = (\underset{\sim}{I} - \underset{\sim}{P}_d)\{y\} \quad .$$

Equilibrium Solutions with Nonvanishing Regional Populations

We initiate our search for equilibrium solutions by looking for those characterized by nonzero regional shares.

Characterization of Equilibrium Solutions with Nonvanishing Regional Populations

The following property was derived by McGinnis and Henry (1973):

If there exists an equilibrium solution with nonzero regional shares, it is unique and is obtained as the characteristic vector of the matrix $\underset{\sim}{C} = [\underset{\sim}{A}^{-1}(\underset{\sim}{I} - \underset{\sim}{P}_d)\{i\}]_{dg}$ $(\underset{\sim}{I} - \underset{\sim}{P}_d)^{-1}\underset{\sim}{A}'$, corresponding to the unit characteristic root.

The demonstration can be summarized as follows:

Supposing that all elements of $\underset{\sim}{y}$ are strictly positive, allows one to premultiply each side of (25) by $\underset{\sim}{y}^{-1}$. Since $\underset{\sim}{I} - \underset{\sim}{P}_d$ is a diagonal matrix, it follows that

$$(\underset{\sim}{I} - \underset{\sim}{P}_d)\{i\} = \underset{\sim}{A}\ \underset{\sim}{\alpha}(\infty)\{y\} \quad ,$$

or, after premultiplication by $\underset{\sim}{A}^{-1}$,

$$\underset{\sim}{\alpha}(\infty)\{y\} = \underset{\sim}{A}^{-1}(\underset{\sim}{I} - \underset{\sim}{P}_d)\{i\} \quad ,$$

in which $\{i\}$ is a column vector of ones. Then, the matrix product $\underset{\sim}{\alpha}(\infty)\underset{\sim}{y}$ is a diagonal matrix whose i-th diagonal element is the i-th element of the vector $\underset{\sim}{A}^{-1}[\underset{\sim}{I} - \underset{\sim}{P}_d]\{i\}$:

$$\underset{\sim}{\alpha}(\infty)\underset{\sim}{y} = [\underset{\sim}{A}^{-1}[\underset{\sim}{I} - \underset{\sim}{P}_d]\{i\}]_{dg} \quad . \tag{27}$$

Substituting this into the constraint equation (22) yields

$$[\underset{\sim}{A}^{-1}(\underset{\sim}{I} - \underset{\sim}{P}_d)\{i\}]_{dg}\ \underset{\sim}{A}'\{y\} - (\underset{\sim}{I} - \underset{\sim}{P}_d)\{y\} = 0 \quad ,$$

a relationship that can be rewritten as

$$[\underset{\sim}{C} - \underset{\sim}{I}]\{y\} = \{0\} \quad ,$$

in which

$$\underset{\sim}{C} = [\underset{\sim}{A}^{-1}(\underset{\sim}{I} - \underset{\sim}{P}_d)\{i\}]_{dg}(\underset{\sim}{I} - \underset{\sim}{P}_d)^{-1}\underset{\sim}{A}' \quad .$$

Observing that

$$\{i\}'\underset{\sim}{C}^{1} = \{i\}'\underset{\sim}{A}^{-1}[\underset{\sim}{I} - \underset{\sim}{P}_d][\underset{\sim}{A}^{-1}(\underset{\sim}{I} - \underset{\sim}{P}_d)\{i\}]_{dg} = \{i\}' \quad ,$$

it follows that $\underset{\sim}{C}^{-1}$ and, consequently, $\underset{\sim}{C}$ are matrices admitting a unit characteristic root. However, since $\underset{\sim}{C}$ need not be stochastic (nonnegative), it might well be that the vector {y} does not have all its components strictly positive.

For example, in the case of our four region system, the $\underset{\sim}{C}$ matrix includes three negative entries as shown by Table 11 below.

Table 11. Nonlinear model - ZPG formulation - U.S. regions - the $\underset{\sim}{C}$ matrix

	North East	North Central	South	West
$\underset{\sim}{C}$ =	0	-0.19523	-0.42067	-0.34187
	0.15717	0	0.39891	0.55708
	0.45001	0.62380	0	0.87746
	0.21435	0.63346	0.81094	0

The normalized vector {y} presents a negative entry corresponding to the North East region, the region which appeared to be empty in the projection process described previously (see Table 5).

Table 12. Nonlinear model - ZPG formulation - U.S. regions - the {y} vector

$$\{y\} = \begin{Bmatrix} -0.48594 \\ 0.42976 \\ 0.51993 \\ 0.53625 \end{Bmatrix}$$

A Necessary and Sufficient Condition

It is actually not necessary to explicitly calculate the characteristic vector of $\underset{\sim}{C}$ corresponding to the unit characteristic root in order to determine whether all components of this vector are strictly positive. The occurrence of empty regions in the equilibrium situation can be found a priori by the application of the following theorem:

> A necessary and sufficient condition for the characteristic vector corresponding to the unit characteristic root to have strictly positive entries is that $\underset{\sim}{A}^{-1}[\underset{\sim}{I} - \underset{\sim}{P}_d]\{i\} > 0$.

Suppose $\underset{\sim}{A}^{-1}[\underset{\sim}{I} - \underset{\sim}{P}_d]\{i\} > 0$, then the matrix $[\underset{\sim}{A}^{-1}(\underset{\sim}{I} - \underset{\sim}{P}_d)\{i\}]_{dg}$ is strictly positive as well as $\underset{\sim}{C}$. We find, therefore, that the characteristic vector of $\underset{\sim}{C}$ is non-negative in such circumstances. Note that the interpretation of the above condition simply lies in the possibility of finding a positive value of $\alpha_i(\infty)$ for all i satisfying equation (27).

In the case of our four region system for the U.S., the calculation of the vector $\{z\} = \underset{\sim}{A}^{-1}[\underset{\sim}{I} - \underset{\sim}{P}_d]\{i\}$ leads to a vector whose first component (corresponding to the North East region) is negative (see Table 13).

Table 13. Nonlinear model - ZPG formulation - U.S. regions - the {z} vector

$$\{z\} = \begin{Bmatrix} -0.42290 \\ 0.44537 \\ 0.75477 \\ 0.88384 \end{Bmatrix} \times 10^8$$

We can conclude, without calculating the characteristic vector {y}, from the simple calculation of {z}, that this system does not admit any acceptable equilibrium solutions with nonzero entries.

Conversely, if {y} is a vector with strictly positive entries, we see from (22) that $\underset{\sim}{\alpha}(\infty)$ is strictly positive. It then follows from (26) that $\underset{\sim}{A}^{-1}[\underset{\sim}{I} - \underset{\sim}{P}_d]\{i\}$ is a strictly positive vector, which completes the proof of the necessary and sufficient condition.

Contrasting Systems with Odd and Even Numbers of Regions

The existence of a characteristic vector of $\underset{\sim}{C}$ corresponding to the unit characteristic root, and having strictly positive entries does not ensure that it is an acceptable equilibrium solution of the ZPG nonlinear model, because it does not necessarily lead to a positive value of $\underset{\sim}{\alpha}(\infty)$.

It is simple to construct an example in which the values of {z} and {y} are strictly positive but do not lead to a value of $\underset{\sim}{\alpha}(\infty)$ verifying (27). Table 14 presents a four-region system in which the values of {z} and {y}, strictly positive, fail to yield a value of $\underset{\sim}{\alpha}(\infty)$, owing to the nonzero value of the determinant of $[\underset{\sim}{\alpha}(\infty)\underset{\sim}{A} - \underset{\sim}{A}\underset{\sim}{\alpha}(\infty)]$. This property can be immediately generalized as follows:

> If the system is not initially stationary, there generally exists no equilibrium solution characterized by an even number of regions (higher than two) of nonempty regions. Major exceptions occur when $\underset{\sim}{A}$ is a symmetric matrix.

Table 14. Nonlinear model - ZPG formulation - constructed example 1

$$\underset{\sim}{P}_d = \begin{bmatrix} 0.6 & 0 & 0 & 0 \\ 0 & 0.48 & 0 & 0 \\ 0 & 0 & 0.21 & 0 \\ 0 & 0 & 0 & 0.05 \end{bmatrix} \qquad \underset{\sim}{A} = \begin{bmatrix} 0 & 1.2 & 0.2 & 0.2 \\ 1.2 & 0 & 0.2 & 0.2 \\ 1 & 1.3 & 0 & 2 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$$\underset{\sim}{C} = \begin{bmatrix} 0 & 1.14737 & 0.95614 & 0.95614 \\ 0.65182 & 0 & 0.70614 & 0.54318 \\ 0.07217 & 0.07217 & 0 & 0.36087 \\ 0.00425 & 0.00425 & 0.04247 & 0 \end{bmatrix}$$

$$\{z\} = \begin{Bmatrix} 0.38246 \\ 0.28246 \\ 0.28509 \\ 0.02018 \end{Bmatrix} \qquad \{y\} = \begin{Bmatrix} 0.52754 \\ 0.39643 \\ 0.06916 \\ 0.00686 \end{Bmatrix}$$

We can rewrite (26) as:

$$[\underset{\sim}{E} - \underset{\sim}{E}']\{i\} = \{0\} \quad ,$$

in which

$$\underset{\sim}{E} = \underset{\sim}{y}\,\underset{\sim}{A}\,\underset{\sim}{\alpha}\,(\infty)\,\underset{\sim}{y} \quad .$$

Clearly, for given values of $\underset{\sim}{A}$ and $\underset{\sim}{y}$, this last equation yields a value of $\underset{\sim}{\alpha}$ (∞), only if the determinant of $\underset{\sim}{E} - \underset{\sim}{E}'$ is equal to zero.

In a restrictive manner, this condition requires that $\underset{\sim}{E} = \underset{\sim}{E}'$, i.e.

$$\underset{\sim}{\alpha}(\infty)\underset{\sim}{A}' = \underset{\sim}{A}\,\underset{\sim}{\alpha}\,(\infty) \quad , \tag{28}$$

an equality that can be satisfied if:

1. the system is initially stationary [because $\underset{\sim}{\alpha}(\infty) = \underset{\sim}{\alpha}(0)$ such that $\underset{\sim}{\alpha}(0)\underset{\sim}{A}' = \underset{\sim}{A}\,\underset{\sim}{\alpha}(0)$]; or

2. if the number of regions in the system is equal to two: (28) reduces to twice the scalar equation $\alpha_1(\infty)\ a_{12} = \alpha_2(\infty)\ a_{21}$ for which there exists a solution if none of the off-diagonal elements are zero; or

3. if the $\underset{\sim}{A}$ matrix is symmetric.

Aside from these particular cases, the condition $|\underset{\sim}{E} - \underset{\sim}{E}'| = 0$ is satisfied if n is an odd number but does not generally hold if n is an even number (unless the more restrictive condition $\underset{\sim}{E} - \underset{\sim}{E}' = 0$ holds). This immediately follows from the fact that a skew symmetric matrix $\underset{\sim}{E} - \underset{\sim}{E}'$ [because $\underset{\sim}{E} - \underset{\sim}{E}' = -(\underset{\sim}{E}' - \underset{\sim}{E})'$] has a zero determinant if the number of its columns (or rows) is an odd number, and is generally different from zero if the number of columns is even.

Property 2

Now, summarizing the above results, we have the following property:

A demographic system, initially nonstationary, character *ized by a matrix* $\underset{\sim}{P}_d$ *of retention probabilities and a matrix* $\underset{\sim}{A}$ *of relational terms, admits a strictly positive equilibrium vector* $\{y\}$ *if and only if:*

$\underset{\sim}{A}^{-1}\ (\underset{\sim}{I} - \underset{\sim}{P}_d)\{i\} > 0$ *, and*

the number of regions is not an even number higher than two. (This second condition is, however, not required if $\underset{\sim}{A}$ *is a symmetric matrix.) Moreover,* $\{y\}$ *is obtained as the characteristic vector of* $\underset{\sim}{C} = [\underset{\sim}{A}^{-1}(\underset{\sim}{I} - \underset{\sim}{P}_d)\{i\}]_{dg}$ $(\underset{\sim}{I} - \underset{\sim}{P}_d)^{-1}\underset{\sim}{A}'$ *corresponding to the characteristic root* 1.

Having determined the conditions of existence (or non-existence) of a strictly positive equilibrium solution, we now turn to the search for equilibrium solutions including one or several empty regions, say k regions.

Equilibrium Solutions with Vanishing Regional Populations

Searching for Equilibrium Solutions with Vanishing Regional Populations

To determine whether the system leads to an equilibrium solution with a predetermined set of k vanishing regions, we set $y_i = 0$ for these regions in (25) and look for solutions of the resulting equation:

$$\{\underset{\sim}{I} - \bar{\underset{\sim}{P}}_d\}\{\bar{y}\} = \bar{\underset{\sim}{y}}\ \bar{\underset{\sim}{\alpha}}(\infty)\bar{\underset{\sim}{A}}^{-1}\{\bar{y}\} \quad , \tag{29}$$

in which $\bar{\underset{\sim}{P}}_d$, $\bar{\underset{\sim}{A}}$, $\bar{\underset{\sim}{\alpha}}(\infty)$ and $\bar{\underset{\sim}{y}}$ are respectively obtained from $\underset{\sim}{P}_d$, $\underset{\sim}{A}$, $\underset{\sim}{\alpha}(\infty)$ and $\underset{\sim}{y}$ by removing the k columns and k rows corresponding to the k vanishing regions. Since (29) is similar to (22)(it only differs from the latter by the number of regions n - k instead n), we can apply Property 2 to the system characterized by $\bar{\underset{\sim}{A}}$ and $\bar{\underset{\sim}{P}}_d$. We may conclude that if (n - k) is an even number higher than two, no equilibrium vector exists such that it contains k zero elements corresponding to as many empty regions (unless $\bar{A}$ is a symmetric matrix). We may also conclude that if (n - k) is equal to two or is an odd number, there exists at most one equilibrium solution, whose set of nonzero elements is described by the characteristic vector corresponding to the unit characteristic root of

$$\bar{C} = [\bar{A}^{-1}(I - \bar{P}_d)\{i\}]_{dg}(I - \bar{P}_d)^{-1}\bar{A}' \quad ,$$

in which $\bar{A}$ and $\bar{P}_d$ are (n - k) submatrices of A and P_d such that

$$[A^{-1}(\bar{I} - \bar{P}_d)]\{i\} \geq 0 \quad .$$

<u>The Maximum Number of Acceptable Solutions</u>

<u>Property 3</u>:

The ZPG formulation of the nonlinear model admits a maximum of $2^n - (n + 1)$ *equilibrium solutions characterized by at least two nonempty regions; this number reduces to* $2^{n-1} + \frac{n(n-3)}{2}$ *if the initial matrix* A *contains no off-diagonal element such as* $a_{ij} = a_{ji}$.

Since there exists a unique equilibrium solution for each predetermined choice of the vanishing regions, it suffices to calculate the number of alternative sets of vanishing regions that the system can admit in order to obtain the maximum number of equilibrium solutions.

There exist $\binom{n}{k}$ different ways of constructing an $\bar{A}$ matrix by dropping k columns and k rows of A, so that the McGinnis/ Henry model admits, at the most,

$\binom{n}{0} = 1$ solution with no zero entry

$\binom{n}{1} = n$ solutions with one zero entry

$\binom{n}{2} = \frac{n(n-1)}{2}$ solutions with two zero entries

⋮

$\binom{n}{k} = \frac{n!}{(n-k)!k!}$ solutions with k zero entries

⋮

$\binom{n}{n-1} = n$ solutions with (n - 1) zero entries.

Then, there exists a maximum of $\binom{n}{0} + \binom{n}{1} \ldots + \binom{n}{n-1} = 2^n - 1$ equilibrium solutions. However, equilibrium solutions with (n - 1) zero entries (the whole population concentrated in one region) cannot occur since this leads to an undefined value of $\bar{\underset{\sim}{\alpha}}(\infty)$. Thus the maximum number of equilibrium solutions, all characterized by at least two nonempty regions, reduces $2^n - 1 - \binom{n}{n-1} = 2^n - (n + 1)$.

Consequently, a system of two regions yields at the most $2^2 - 3$ equilibrium solutions, i.e., a unique equilibrium solution, while a system of three and four regions admits no more than four and eleven equilibriums, respectively.

Also, if the matrix $\underset{\sim}{A}$ has no off diagonal element such that $a_{ij} = a_{ji}$, the ZPG system does not lead to any equilibrium solution characterized by an even number of regions greater than two. In such a case, the maximum number of equilibrium solutions is equal to

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \binom{n}{6} + \ldots + \binom{n}{n-2}, \text{ if n is an odd number}$$

$$\binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \binom{n}{4} + \ldots + \binom{n}{n-3} + \binom{n}{n-2}, \text{ if n is an even number.}$$

Because of the properties of the number of combinations $\binom{n}{k}$ it can be established that this maximum number of equilibrium solutions is equal to $2^{n-1} + \frac{n(n - 3)}{2}$ in all cases (n odd or even).

It thus follows that the number of maximum solutions is respectively, 1, 4, 10, 21, 41, for n = 2, 3, 4, 5 and 6. The restriction imposed on $\underset{\sim}{A}$ causes the number of solutions to drop from 11 to 10 (if n = 4) and from 57 to 41 (if n = 6).

<u>Determining all Solutions of (27) in the Four Region System System of the U.S.</u>

In the case of our four region system we have a maximum of 10 equilibrium solutions characterized by one or two empty regions. We have thus calculated all 10 solutions of (27) admitting empty entries (see Table 15). It appears:

- that only two of the four characteristic vectors containing a unique zero are nonnegative, and

- that all of the six characteristic vectors containing two zeros are nonnegative.

Table 15. Nonlinear model - ZPG formulation - U.S. regions - the {z} and {y} vectors

Solution No.	{z}	{y}	Solution No.	{z}	{y}
(1)	$\begin{bmatrix}0.30187\\0.65972\\1.24960\\0\end{bmatrix} \times 10^8$	$\begin{bmatrix}0.19026\\0.32860\\0.48114\\0\end{bmatrix}$	(6)	$\begin{bmatrix}0.98697\\0\\1.70830\\0\end{bmatrix} \times 10^8$	$\begin{bmatrix}0.49540\\0\\0.50460\\0\end{bmatrix}$
(2)	$\begin{bmatrix}-0.17570\\0.90402\\0\\1.48210\end{bmatrix} \times 10^8$	$\begin{bmatrix}-0.12450\\0.57811\\0\\0.54639\end{bmatrix}$	(7)	$\begin{bmatrix}1.43750\\0\\0\\2.34490\end{bmatrix} \times 10^8$	$\begin{bmatrix}0.53748\\0\\0\\0.46252\end{bmatrix}$
(3)	$\begin{bmatrix}-0.01395\\0\\1.02800\\0.93381\end{bmatrix} \times 10^8$	$\begin{bmatrix}-0.01285\\0\\0.54786\\0.46499\end{bmatrix}$	(8)	$\begin{bmatrix}0\\0.40071\\0.59929\\0\end{bmatrix} \times 10^8$	$\begin{bmatrix}0\\0.45983\\0.54017\\0\end{bmatrix}$
(4)	$\begin{bmatrix}0\\0.23996\\0.71486\\0.68831\end{bmatrix} \times 10^8$	$\begin{bmatrix}0\\0.20285\\0.42725\\0.36990\end{bmatrix}$	(9)	$\begin{bmatrix}0\\0.80555\\0\\1.38470\end{bmatrix} \times 10^8$	$\begin{bmatrix}0\\0.50189\\0\\0.49811\end{bmatrix}$
(5)	$\begin{bmatrix}2.49830\\2.45700\\0\\0\end{bmatrix} \times 10^8$	$\begin{bmatrix}0.53560\\0.46440\\0\\0\end{bmatrix}$	(10)	$\begin{bmatrix}0\\0\\1.01820\\0.92079\end{bmatrix} \times 10^8$	$\begin{bmatrix}0\\0\\0.54205\\0.45795\end{bmatrix}$

Narrowing Down the Number of Acceptable Solutions

A nonnegative vector $\{y\}$ with k zero entries is an acceptable equilibrium solution only if the vector $\{\bar{z}\} = [\bar{\underset{\sim}{A}}^{-1}(\underset{\sim}{I} - \bar{\underset{\sim}{P}}_d)]^{-1}\{i\}$, and the matrices $\bar{\bar{\underset{\sim}{A}}}$ and $\bar{\bar{\underset{\sim}{P}}}$, respectively obtained by removing the (n - k) rows corresponding to the nonzero entries are such that

$$\bar{\bar{A}}\{\bar{z}\} \leq [\underset{\sim}{I} - \bar{\bar{\underset{\sim}{P}}}_d]\{i\} \quad .$$

So far, we have searched for equilibrium solutions admitting zero entries, but we have not examined the likelihood of occurrence of such solutions.

Clearly, a necessary condition for any region l to become empty in the long run is that its inmigration be equal to or less than its outmigration as t becomes large, i.e.,

$$w_l(t)\,[\sum_{j\neq l} a_{jl}\alpha_j(t)] \leq (1 - p_{ll})w_l(t) \quad ,$$

for any finite value of $t > T$. Because $w_l(t)$ is strictly positive for any finite value of t, the above condition becomes:

$$\sum_{j\neq l} a_{jl}\alpha_j(t)w_j(t) \leq (1 - p_{ll}) \quad ,$$

or in compact form,

$$\bar{\bar{\underset{\sim}{A}}}\ \bar{\underset{\sim}{\alpha}}(t)\{\bar{w}(t)\} < [\underset{\sim}{I} - \bar{\bar{\underset{\sim}{P}}}_d]\{i\} \quad , \tag{30}$$

for any finite value of $t > T$, in which the double bar relates to sections of the $\underset{\sim}{A}$ and $\underset{\sim}{P}_d$ matrices obtained by removing the n - k rows corresponding to the non zero entries and the k columns corresponding to the zero entries of $\{y\}$.

As $t \to \infty$, $\bar{\alpha}(t)\{\bar{w}(t)\} \to \{\bar{z}\} = \bar{A}^{-1}[I - \bar{P}_d]\{i\}$. Then a necessary condition for any characteristic vector $\{y\}$ (including zero entries) to be an adequate equilibrium solution is that

$$\bar{\bar{A}}\{\bar{z}\} \leq [I - \bar{\bar{P}}_d]\{i\} \quad . \tag{31}$$

Returning to our four region example, it appears that among the eight nonnegative characteristic vectors $\{y\}$ derived above, only one (solution number 4 in Table 16) verifies condition (31). The multiregional system of the United States consisting of the four Census Regions thus yields a unique acceptable equilibrium solution in which the North East region is empty and the other regions contain 14.32 percent of the total population (north Central), 43.26 percent (South) and 42.43 percent (West).*

Table 16. Nonlinear model - U.S. regions - comparison of stationary inmigration and outmigration rates relating to the vanishing regions

Solution Number	$\bar{\bar{A}}\{\bar{z}\}$	$[I - \bar{\bar{P}}_d]\{i\}$	Solution Number	$A\{\bar{z}\}$	$[I - P_d]\{i\}$
(1)	$\{0.12339\}$	$\{0.05468\}$	(7)	$\begin{Bmatrix}0.12313\\0.18494\end{Bmatrix}$	$\begin{Bmatrix}0.05427\\0.04620\end{Bmatrix}$
(4)	$\{0.03810\}$	$\{0.04706\}$	(8)	$\begin{Bmatrix}0.05736\\0.14086\end{Bmatrix}$	$\begin{Bmatrix}0.04706\\0.05468\end{Bmatrix}$
(5)	$\begin{Bmatrix}0.23638\\0.26183\end{Bmatrix}$	$\begin{Bmatrix}0.04620\\0.05468\end{Bmatrix}$	(9)	$\begin{Bmatrix}0.04322\\0.10864\end{Bmatrix}$	$\begin{Bmatrix}0.04706\\0.04620\end{Bmatrix}$
(6)	$\begin{Bmatrix}0.08667\\0.12930\end{Bmatrix}$	$\begin{Bmatrix}0.05427\\0.05468\end{Bmatrix}$	(10)	$\begin{Bmatrix}0.04652\\0.07496\end{Bmatrix}$	$\begin{Bmatrix}0.04706\\0.05427\end{Bmatrix}$

*Note that this limiting distribution was the one which we obtained in Section I by iteratively projecting the initial population.

The Uniqueness Versus the Non Uniqueness of the Stationary State: An Illustration

Note that uniqueness of the stationary state is not a general property of the ZPG formulation of the nonlinear model. For instance, we have constructed an example (for which the $\underset{\sim}{A}$ and $\underset{\sim}{P}_d$ matrices are shown in Table 17) that offers two acceptable equilibrium solutions.

Table 17. Nonlinear model - constructed example 2 - the $\underset{\sim}{A}$ and $\underset{\sim}{P}_d$ matrices

$$\underset{\sim}{A} = \begin{bmatrix} 0.00000 & 1.20000 & 0.20000 & 0.20000 \\ 1.20000 & 0.00000 & 0.20000 & 0.20000 \\ 1.00000 & 1.30000 & 0.00000 & 2.00000 \\ 1.00000 & 1.00000 & 1.00000 & 0.00000 \end{bmatrix}$$

$$\underset{\sim}{P}_d = \begin{bmatrix} 0.60000 & 0 & 0 & 0 \\ 0 & 0.48000 & 0 & 0 \\ 0 & 0 & 0.21000 & 0 \\ 0 & 0 & 0 & 0.05000 \end{bmatrix}$$

From Property 3 we know that since $\underset{\sim}{A}$ has no symmetrical off-diagonal elements, there exist at the most ten equilibrium solutions for this system. The calculation of the ten equilibrium vectors corresponding to the unit characteristic root, having no more than (n - 2) zero entries* reveals that only eight of them are nonnegative vectors. They include:

- all six characteristic vectors with two non-zero components, and
- two out of the four characteristic vectors with one zero component.

However, only two of these vectors are acceptable in that they verify (31). One is a vector with one zero entry and the other with two zero entries; both of them are shown in Table 18.

*Since the number of regions in the system is even, at least one region has to be empty at equilibrium.

Table 18. Nonlinear model - constructed example 2 - the two acceptable equilibrium solutions

$$\{y\}_1 = \begin{bmatrix} 0 \\ 0 \\ 0.54598 \\ 0.45402 \end{bmatrix} \qquad \{y\}_2 = \begin{bmatrix} 0.54107 \\ 0.41073 \\ 0.04819 \\ 0 \end{bmatrix}$$

Note that, when carrying out the ZPG projection process characterized by the $\underset{\sim}{A}$ and $\underset{\sim}{P}_d$ matrices defined in Table 17, the two alternative equilibrium states shown in Table 18 are actually obtained. In fact, the first equilibrium characterized by two empty regions is obtained much more often than the alternative one. Only when the relative share of the fourth region is initially small is the alternative equilibrium obtained. For example, when setting the initial population of regions 1, 2 and 3 to 100,000, the first stable equilibrium is reached every time the initial population of region 4 is higher than 107. On the other hand, the alternative equilibrium is obtained when the initial population of that chosen region is less than 106. Unfortunately, it was not possible to carry out this result further in order to determine a priori which stable equilibrium would be obtained for any predetermined choice of initial population.

Returning to our general analysis of equilibrium solutions with zero entries, we could also show without difficulty that, if for a given choice of k regions $\{\bar{z}\}$ is nonnegative, then (31) is also a sufficient condition for the corresponding characteristic vector {y} to be an acceptable equilibrium solution of the ZPG nonlinear model.

<u>Property 4</u>:

Summarizing the above results, we can now state the following:

A necessary and sufficient condition for an n region system, initially nonstationary, to admit one or several equilibrium distributions containing no more than (n - 2) empty regions is that -

(1) *there exist one or several partial matrices* $\bar{A}_d$ *and* $\bar{P}_d$, *obtained by removing* k $(0 \leq k \leq n-2)$ *columns and* k *rows of* $\underset{\sim}{A}$ *and* $\underset{\sim}{P}_d$, *respectively, such that* $\{\bar{z}\} = \bar{A}^{-1}[\underset{\sim}{I} - \bar{P}_d]\{i\} > 0$,

(2) *the vectors* $\{\bar{z}\}$ *satisfy the condition* $\bar{\bar{A}}\{\bar{z}\} \leq (\underset{\sim}{I} - \bar{\bar{P}}_d)\{i\}$, *expressing that, at equilibrium, regional inmigration rates must be less than their outmigration counterparts, and*

(3) $n - k$, *the number of nonvanishing regions, is not an even number higher than two. (This third condition is, however, not required if* $\bar{A}$ *is a symmetric matrix.)*

Thus, in our search for acceptable equilibrium solutions in the ZPG case, we have set forth a methodology permitting the a priori calculation of all acceptable equilibrium solutions (see Appendix 4 for a formal exposition of this methodology).

We now continue the study of the ZPG formulation by examining particular cases.

Particular Cases

We will now examine in detail the case of systems consisting of three regions* and then analyze the long-term behavior of the nonlinear model II in which the influence of the relational factors between pairs of regions is ruled out.

Case of a Three Region System

Firstly, we demonstrate that the vector $\{z\} = \underset{\sim}{A}^{-1}[\underset{\sim}{I} - \underset{\sim}{P}_d]\{i\}$ has at least two nonnegative components.

Suppose that $\{z\}$ contains two negative components, say, the second and third components. Then, the first element of the

*In the case of two regions, the nonlinear model reduces to the linear one.

vector

$$\tilde{A}\{z\} = \begin{bmatrix} 0 & a_{21} & a_{31} \\ a_{12} & 0 & a_{32} \\ a_{13} & a_{23} & 0 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \\ z_3 \end{Bmatrix}$$

is negative, which is impossible since $\tilde{A}\{z\}$ is equal to $[\tilde{I} - \tilde{P}_d]\{i\}$, that is, a nonnegative vector.

<u>Secondly</u>, we show that if {z} has no negative component, there exists a unique acceptable equilibrium presenting no zero entry.

Suppose that {z} has no negative component, then from Property 2 there exists an acceptable equilibrium with no zero entries. Are there, however, equilibriums with zero entries? If we assume that there exists an equilibrium solution in which the third component is zero, the new value of {z}, say {z´}, is given by

$$\bar{\tilde{A}}\{\bar{z}'\} = (\tilde{I} - \bar{\tilde{P}}_d)\{i\} \quad , \quad \text{i.e.,}$$

$$\begin{bmatrix} 0 & a_{21} \\ a_{12} & 0 \end{bmatrix} \begin{Bmatrix} z_1' \\ z_2' \end{Bmatrix} = \begin{Bmatrix} 1 - p_{11} \\ 1 - p_{22} \end{Bmatrix}$$

Since $a_{21}\ z_2 + a_{31}\ z_3 = 1 - p_{11}$ and $a_{12}\ z_1 + a_{32}\ z_3 = 1 - p_{22}$ in which z_1, z_2 and z_3 are positive by assumption, it can be clearly seen that $z_1' > z_1$ and $z_2' > z_2$. Furthermore, we have

$$a_{13}'\ z_1' + a_{23}\ z_2' > a_{13}\ z_1 + a_{23}\ z_2 = (1 - p_{33}) \quad ,$$

an inequality which states that, as the system tends toward equilibrium, the inmigration rate is higher than the outmigration rate in region 3. However, this is impossible since the third region has been hypothesized to become empty so that the opposite inequality between inmigration and outmigration rates should hold. Thus a three region system characterized by the existence of an equilibrium with no zero entries cannot have any other equilibrium.

Thirdly, we demonstrate that if $\{z\}$ has one negative component, there exists a unique acceptable equilibrium with a zero entry for the region having the negative entry in $\{z\}$.

Suppose now that $\{z\}$ has a negative component, say the third one. Then, we know that the three region system admits no equilibrium solution with strictly positive entries but has at least an acceptable solution in which one region is empty.

To obtain this result we show that the characteristic vector of $[\bar{\underset{\sim}{A}}^{-1}[\underset{\sim}{I} - \bar{\underset{\sim}{P}}_d]\{i\}]^{-1}_{dg}\ \bar{\underset{\sim}{A}}'^{-1}(\underset{\sim}{I} - \bar{\underset{\sim}{P}}_d)$, in which $\bar{\underset{\sim}{A}}$ and $\bar{P}_d$ are submatrices of $\underset{\sim}{A}$ and $\underset{\sim}{P}_d$ obtained by removing the third row and column, is an acceptable solution and that there exists no other equilibrium.

The new values z_1'' and z_2'' of the non-zero elements of $\{\bar{z}\}$ are less than z_1 and z_2 so that $a_{13}z_1'' + a_{23}z_2''$ is less than $(1 - p_{33})$.

Then the two region subsystem consisting of the regions for which the components of $\{z\}$ are nonnegative is an equilibrium solution since the vector $\begin{Bmatrix} z_1'' \\ z_2'' \\ 0 \end{Bmatrix}$ satisfies the necessary and sufficient condition of Property 4.

For example, the three region system of the U.S. obtained by aggregating the South and West regions of the four region system previously used has a $\{z\}$ vector whose first component (North East) is zero, thus concentrating the equilibrium population in the other two regions (North Central and South/West). Note that, in such instances, the allocation of the equilibrium population among the two regions denoted by i and j can be simply obtained by observing that $M_{ij}(\infty) = M_{ji}(\infty)$, i.e., $(1 - p_{ii})y_i = (1 - p_{jj})y_j$: the share of each region is inversely proportional to the total outmigration rate $(1 - p_{ii})$. Numerically, it appears that the equilibrium solution implies that a constant 35.14 percent of the whole U.S. population will ultimately live in the North Central region versus 64.86 percent in the region constituted by the U.S. Census South and West Regions.*

Moreover, because of the occurrence of a zero entry at equilibrium (the third entry), the following inequality holds:

$$\frac{a_{13}}{a_{12}} (1 - p_{22}) + \frac{a_{23}}{a_{21}} (1 - p_{11}) < 1 - p_{33} \quad . \qquad (32)$$

Suppose now that there exists a second acceptable equilibrium solution characterized by a zero component, say, in region 1. Then the following inequality, similar to (32), must hold

$$\frac{a_{21}}{a_{23}} (1 - p_{33}) + \frac{a_{31}}{a_{32}} (1 - p_{22}) < 1 - p_{11} \quad . \qquad (33)$$

Multiplying (32) by $\frac{a_{21}}{a_{23}}$ and adding the resulting inequality to (33) we get

$$\frac{a_{13}}{a_{12}} \frac{a_{21}}{a_{23}} + \frac{a_{31}}{a_{32}} (1 - p_{22}) < 0 \quad ,$$

which is clearly contrary to the hypothesis that $p_{22} < 1$.

Finally, a three region system admits a unique equilibrium distribution characterized either by strictly positive entries (if $\underset{\sim}{A}^{-1}[\underset{\sim}{I} - \underset{\sim}{P}_d]\{i\}$ is nonnegative) or by two strictly positive entries accompanied by a zero entry (corresponding to the entry of $\underset{\sim}{A}^{-1}[\underset{\sim}{I} - \underset{\sim}{P}_d]\{i\}$ which is negative). In the latter case, moreover, the population shares of the two nonvanishing regions are

*For the sake of comparison, we remind the reader that the four region system had an equilibrium solution in which the sum of the shares of the South and West regions was 85.68 percent.

inversely proportional to their total outmigration rates.

Case of a System in which Distance has no Influence (Nonlinear Model II)

In such an instance, the matrix $\underset{\sim}{A}$ is symmetric so that in contrast to the general case, (25) does not raise any problem if n is an even number. The general Properties 2 and 4 are thus valid here without the restriction attached to the number of regions in the system. Therefore, the maximum number of equilibrium solutions is $2^n - (n + 1)$, i.e., 11 in the case of our four region system of the U.S.

Table 19 displays the values of the 11 characteristic vectors of the new $\underset{\sim}{C}$ matrix obtained for this system. Note that we have not reported here the values of {z} since the normalized vector {z} is identical to {y}. (This stems from the fact that if A is symmetric, the value of α_i is $(1 - p_{ii})$ for all nonempty regions.) All the 11 characteristic vectors are nonnegative; however, none of the solutions with zero entries are such

Table 19. Nonlinear model II - ZPG case - U.S. regions - the {y} vectors

Solution No.	{y}	Solution No.	{y}	Solution No.	{y}	Solution No.	{y}
(1)	0.30186 0.19483 0.31459 0.18872	(4)	0.36383 0 0.37543 0.26073	(7)	0.49540 0 0.50460 0	(10)	0 0.50189 0 0.49811
(2)	0.36206 0.26425 0.37369 0	(5)	0 0.30042 0.40448 0.29511	(8)	0.53748 0 0 0.46252	(11)	0 0 0.54205 0.45795
(3)	0.39675 0.30427 0 0.29898	(6)	0.53560 0.46440 0 0	(9)	0 0.45983 0.54017 0		

that the difference between inmigration and outmigration rates is negative when $t \to \infty$. Therefore, the system offers a unique equilibrium characterized by non-empty regions.* We note that the North East region is comparatively much larger at equilibrium (30.19 percent) than initially (27.77 percent), while the share of the North Central region is much smaller (19.48 percent versus 28.08 percent). By contrast, the population shares of the South and West regions are similar in both initial and stationary populations.

IV. THE NONLINEAR MODEL (NON-ZPG FORMULATION): SEARCH FOR EQUILIBRIUM SOLUTIONS

In this case, regions are exposed to natural increase $\underset{\sim}{N} \neq \underset{\sim}{0}$, and the projection process is entirely defined by (18) [or, alternatively, by (20)] and accompanied by the constraint equation (19).

Preliminary Property

In order to avoid any potential problem concerning the sign $w_i(t)$, we put down the following restriction about the $\underset{\sim}{P}_d$ and $\underset{\sim}{N}$ matrices:**

$$\underset{\sim}{P}_d + \underset{\sim}{N} > 0 \quad .$$

Then we can establish the following property.

Property 5

If no region is initially empty, then no region can become empty except in the long run. In other words, $\{w(t)\} \geq 0$ *for all finite values of* t.

*This equilibrium solution was the one obtained as the limiting allocation of the population of this system when projecting iteratively the 1970 population.

**Note that imposing such a constraint is not very restrictive for usual applications of multiregional systems: $p_{ii} + n_i$ is highly positive $\forall i$.

To prove this, we write equation (20) in scalar terms as:

$$w_i(t + 1) = [p_{ii} + n_i + \sum_{j \neq i} a_{ji}\alpha_j(t)w_j(t)]w_i(t) \quad .$$

Then, we have

$$w_i(t + 1) \geq (p_{ii} + n_i)w_i(t) \quad ,$$

so that $w_i(t) \geq (p_{ii} + n_i)^t w_i(0)$ (if $p_{ii} + n_i \geq 0$). Therefore, if $w_i(0) \neq 0 \; \forall i$, $w_i(t)$ is strictly positive for all i whenever t remains finite.

Equilibrium Solutions With Nonvanishing Regional Populations

Turning to the search for equilibrium solutions, we first establish a property extending the one derived by McGinnis and Henry (1973) in the ZPG case.

Toward the Derivation of an Acceptable Equilibrium Solution

If it exists, an equilibrium solution to the non-ZPG formulation of the nonlinear model, characterized by all strictly positive entries, is unique and is obtained as the characteristic vector of the matrix

$$\underset{\sim}{D} = \underset{\sim}{A}'^{-1}[\underset{\sim}{A}^{-1}\{i\}]_{dg}^{-1}\left[[\underset{\sim}{A}^{-1}(\underset{\sim}{P}_d + \underset{\sim}{N})\{i\}]_{dg}\underset{\sim}{A}' + \underset{\sim}{I} - \underset{\sim}{P}_d\right]$$

corresponding to the largest characteristic root λ (provided that λ is equal to or larger than 1).*

---

*Alternatively {y} can be derived as $\underset{\sim}{A}'^{-1}[\underset{\sim}{A}^{-1}\{i\}]_{dg}^{-1}\{u\}$ in which {u} is the characteristic vector of the matrix $\underset{\sim}{F} = [\underset{\sim}{A}^{-1}(\underset{\sim}{P}_d + \underset{\sim}{N})\{i\}]_{dg} + (\underset{\sim}{I} - \underset{\sim}{P}_d)\underset{\sim}{A}'^{-1}$ corresponding to the largest characteristic root of λ($\underset{\sim}{D}$ and $\underset{\sim}{F}$ have the same characteristic roots). This alternative was used in setting up a computer program to calculate the estimates of {y} (see Appendix 4).

To prove this we rearrange the steady state equation (21) and obtain the following generalization of (25)

$$[\lambda \underset{\sim}{I} - (\underset{\sim}{P}_d + \underset{\sim}{N})]\{y\} = \underset{\sim}{y}\, \underset{\sim}{A}\, \underset{\sim}{\alpha}(\infty)\{y\} \quad . \tag{34}$$

This relationship may also be rewritten as:

$$[\underset{\sim}{y}\, \underset{\sim}{A}\, \underset{\sim}{\alpha}(\infty) + \underset{\sim}{N} - (\underset{\sim}{I} - \underset{\sim}{P}_d)]\{y\} = (\lambda - 1)\{y\} \quad ,$$

which expresses that each regional increase of population (at stability) is obtained by subtracting the outmigration flow from the sum of the inmigration and natural increase flows.*

Premultiplying both sides of (34) by $\underset{\sim}{A}^{-1}\underset{\sim}{y}^{-1}$ (which is possible here as a consequence of the assumption that {y} admits no zero entries) yields:

$$\underset{\sim}{\alpha}(\infty)\{y\} = \underset{\sim}{A}^{-1}[\lambda \underset{\sim}{I} - (\underset{\sim}{P}_d + \underset{\sim}{N})]\{i\} \quad .$$

Then we can rewrite the matrix product $\underset{\sim}{\alpha}(\infty)\underset{\sim}{y}$ as

$$\underset{\sim}{\alpha}(\infty)\underset{\sim}{y} = \lambda[\underset{\sim}{A}^{-1}\{i\}]_{dg} - [\underset{\sim}{A}^{-1}(\underset{\sim}{P}_d + \underset{\sim}{N})\{i\}]_{dg} \quad , \tag{35}$$

in which the two terms between brackets represent diagonal matrices whose general diagonal elements are equal to the general terms of the vectors appearing inside the brackets. Substituting (35) into the constraint equation (22) then yields:

$$\left[\lambda[\underset{\sim}{A}^{-1}\{i\}]_{dg} - [\underset{\sim}{A}^{-1}(\underset{\sim}{P}_d + \underset{\sim}{N})\{i\}]_{dg}\right]\underset{\sim}{A}'\{y\} - (\underset{\sim}{I} - \underset{\sim}{P}_d)\{y\} = 0 \quad , \tag{36}$$

---

*Note that (34) may be alternatively presented as:

$$[\underset{\sim}{y}\, \underset{\sim}{A}\, \underset{\sim}{\alpha}(\infty)\underset{\sim}{y} - (\underset{\sim}{I} - \underset{\sim}{P}_d)\underset{\sim}{y}]\{i\} = [(\lambda - 1)\underset{\sim}{I} - \underset{\sim}{N}]\{y\} \quad ,$$

which contrasts with the constraint equation:

$$\{i\}'[\underset{\sim}{y}\, \underset{\sim}{A}\, \underset{\sim}{\alpha}(\infty)\underset{\sim}{y} - (\underset{\sim}{I} - \underset{\sim}{P}_d)\underset{\sim}{y}] = \{0\}' \quad .$$

a relationship that can be rewritten, after premultiplication by $\underset{\sim}{A}'^{-1}[\underset{\sim}{A}^{-1}\{i\}]_{dg}$ as

$$(\underset{\sim}{D} - \lambda\underset{\sim}{I})\{y\} = \{0\} \quad , \tag{37}$$

in which

$$\underset{\sim}{D} = \underset{\sim}{A}'^{-1}[\underset{\sim}{A}^{-1}\{i\}]_{dg}^{-1}[(\underset{\sim}{A}^{-1}(\underset{\sim}{P}_d + \underset{\sim}{N})\{i\}]_{dg}\underset{\sim}{A}' + (\underset{\sim}{I} - \underset{\sim}{P}_d)] \quad .$$

Then, an equilibrium solution {y}, if it exists, is a characteristic vector of the matrix $\underset{\sim}{D}$ corresponding to one of its real characteristic roots. Thus, if it exists, the stable state of the nonlinear model is identical to the stable state of the linear model in which

$$\{w(t + 1)\} = \underset{\sim}{D}\,\{w(t)\} \quad . \tag{38}$$

Since the stable state of this system is unique and corresponds to the largest real characteristic root of $\underset{\sim}{D}$, the result is that, if it exists, {y} is unique and is obtained as the characteristic vector of $\underset{\sim}{D}$ corresponding to its largest real characteristic root λ, provided that λ is greater than one (if $\lambda < 1$, $w_i(t + s) = \lambda^s\, w_i(t) \to 0$ as $s \to \infty$, i.e., the system vanishes).

Note that $\underset{\sim}{D}$ is not necessarily nonnegative and that the characteristic vector of $\underset{\sim}{D}$ corresponding to its largest real characteristic root may admit negative entries. Unfortunately, unlike in the ZPG case, it is impossible here to derive a necessary and sufficient condition permitting one to determine a priori whether there exists an acceptable equilibrium solution with nonnegative entries.

To determine the existence (or nonexistence) of an equilibrium solution with strictly positive entries, we must carry out the projection process embodied in the linear system (38) and thus find out whether it leads to an acceptable estimate of {y}.*

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*The algorithm used to calculate applied estimates of {y} is presented in Appendix 4.

The application of this method to our four region system leads to a vector {y} in which the first component (North East) is negative (see Table 20). Then, the ZPG formulation of that system admits at least a vanishing region.

Table 20. Nonlinear model - non ZPG formulation - U.S. regions - equilibrium solution with no vanishing regions

$$F^* = \begin{bmatrix} 1.13192 & 0.08774 & 0.01652 & 0.04968 \\ -0.07686 & 0.87715 & 0.05494 & 0.00616 \\ -0.01317 & 0.02697 & 0.90471 & 0.02092 \\ -0.04250 & 0.01387 & 0.03105 & 0.92832 \end{bmatrix} \quad \{y\} = \begin{Bmatrix} -1.03592 \\ 0.80150 \\ 0.51516 \\ 0.71926 \end{Bmatrix} \quad \lambda = 1.06566$$

As a digression, observe that the matrix D of the non-ZPG case is related to the matrix C of the ZPG case by the following relationship:

$$D = A'^{-1}[A^{-1}\{i\}]_{dg}^{-1}[(I - P_d)(I - C) + [A^{-1}(I + N)\{i\}_{dg}A'] \quad .$$

It follows that

$$D - \lambda I = A'^{-1}[A^{-1}\{i\}]_{dg}^{-1}[(I - P_d(I - C)$$

$$+ [A^{-1}[(1-\lambda)I + N]\{i\}]_{dg}A'] \quad ,$$

so that $[D - \lambda I]\{y\} = \{0\}$ can be rewritten as

$$\{A'^{-1}[A^{-1}\{i\}]_{dg}^{-1}[(I - P_d)(I - C) + [A^{-1}N\{i\}]_{dg}] + (1-\lambda)I\}\{y\} = \{0\} \quad . \tag{39}$$

In the ZPG case, $N = 0$, and $\lambda = 1$ and (39) reduces to

$$(I - C)\{y\} = \{0\} \quad , \tag{40}$$

which is precisely the steady state equation of the ZPG case.

*As indicated in a previous footnote, the stable state of the non-ZPG case is obtained through the calculation of a matrix F which has the same characteristic roots as D.

As the comparison of (39) and (40) suggests, the non ZPG case is not a simple extension of the ZPG case. In the non ZPG case there appears to be no simple theorem determining a priori the existence, or non-existence, of equilibrium distributions characterized by nonvanishing regions. Actually, this statement generally holds only if the system initially contains two or an odd number of regions. As in the ZPG case, a system having an even number of regions greater than two, has at least one vanishing region in the long run.

Contrasting Systems with Odd and Even Numbers of Regions

If a non-ZPG system is initially not stable, there generally exists no equilibrium solution characterized by an even number of regions (higher than two) of non-empty regions.* Major exceptions occur when A is symmetric. The equilibrium equation (34) can be written here as

$$[\underset{\sim}{y}\ \underset{\sim}{A}\ \underset{\sim}{\alpha}(\infty)\underset{\sim}{y} - \underset{\sim}{y}\ \underset{\sim}{\alpha}(\infty)\underset{\sim}{A}'\underset{\sim}{y}]\{i\} = [(\lambda - 1)\underset{\sim}{I} - \underset{\sim}{N}]\{y\} \quad ,$$

or

$$[\underset{\sim}{E} - \underset{\sim}{E}']\{i\} = [(\lambda - 1)\underset{\sim}{I} - \underset{\sim}{N}]\{y\} \quad , \tag{41}$$

in which $\underset{\sim}{E}$ is equal to $\underset{\sim}{y}\ \underset{\sim}{A}\ \underset{\sim}{\alpha}(\infty)\underset{\sim}{y}$.

Suppose that we premultiply (41) by a row vector of ones $\{i\}'$, then we have

$$\{i\}'[\underset{\sim}{E} - \underset{\sim}{E}']\{i\} = \{i\}'[(\lambda - 1)\underset{\sim}{I} - \underset{\sim}{N}]\{y\} \quad .$$

Since the population system considered is closed, the sum of the regional outmigration flows and the sum of the regional inmigration flows are equal. Therefore,

*The demonstration of this property, slightly more difficult to establish than in the ZPG case, is in fact very general and includes the demonstration proposed in the ZPG case as a special case.

$$\{i\}'[\underset{\sim}{E} - \underset{\sim}{E}'\{i\}] = 0 \quad .^{*} \tag{42}$$

If we assume that $\underset{\sim}{\alpha}(\infty)$ and $\underset{\sim}{y}$ are known, (42) must be a system of n linearly dependent equations with the determinant of $\underset{\sim}{E} - \underset{\sim}{E}'$ required to be zero. However, as shown earlier, $\underset{\sim}{E} - \underset{\sim}{E}'$ has a zero determinant if $\underset{\sim}{A}$ is symmetric. If $\underset{\sim}{A}$ is not symmetric, $\underset{\sim}{E} - \underset{\sim}{E}'$ has a zero determinant when the number of regions in the system is equal to two or is an odd number.

Property 6

Summarizing the above properties, it appears that a demographic system, initially nonstable, characterized by a matrix $\underset{\sim}{P}_d$ *of retention probabilities and a matrix* $\underset{\sim}{A}$ *of relational elements does not admit a strictly positive equilibrium vector* $\{y\}$ *if the number of regions in the system is an even number greater than two. If the number of regions is two or an odd number, it may have equilibrium vector that is unique and is obtained as the characteristic vector of* $\underset{\sim}{D} = \underset{\sim}{A}'^{-1}[\underset{\sim}{A}^{-1}\{i\}]_{dg}^{-1}[\underset{\sim}{A}^{-1}(\underset{\sim}{P}_d + \underset{\sim}{N})\{i\}]_{dg}\underset{\sim}{A}' + (\underset{\sim}{I} - \underset{\sim}{P}_d)]$ *corresponding to its largest real characteristic root (provided that this characteristic root is larger than one).*

Equilibrium Solutions with Vanishing Regional Populations

Searching for Equilibrium Solutions with Vanishing Regional Populations

In order to find whether there exist equilibrium solutions characterized by a given set of k vanishing regions, it suffices to set $y_i = 0$ for these k regions in (34), which yields a new vector equation:

$$\bar{y}\ \bar{\alpha}(\infty)\ \underset{\sim}{\bar{A}}\{\bar{y}\} = [\lambda I - (\underset{\sim}{\bar{P}}_d + \underset{\sim}{\bar{N}})]\{\bar{y}\} \quad , \tag{43}$$

*There exists the following relationship between λ and $\{y\}$

$$\lambda = 1 + \{i\}'\underset{\sim}{N}\{y\} \quad .$$

in which $\bar{P}_d$, $\bar{A}$, $\bar{\alpha}(\infty)$ and $\bar{y}$ are respectively obtained from P_d, A, $\alpha(\infty)$ and $\bar{y}$ by removing the k columns and k rows corresponding to the vanishing regions. Since (43) is similar to (34), we can apply Property 6 to the system characterized by $\bar{A}$, $\bar{P}_d$ and $\bar{N}$. We conclude that:

(1) if (n - k) is an even number higher than two, there does not exist any equilibrium vector containing k zero elements corresponding to as many empty regions, except if $\bar{A}$ is a symmetric matrix, and

(2) if (n - k) is equal to two or odd numbers, there exists at most one equilibrium solution corresponding to the predetermined choice of vanishing regions.

Moreover, the application of Property 2 to (39) suggests that the possible equilibrium solutions of the ZPG system are characteristic vectors of

$$\bar{D} = \bar{A}^{-1}[\bar{A}^{-1}\{i\}]_{dg}^{-1}[\bar{A}^{-1}(\bar{P}_d + \bar{N})\{i\}]_{dg}\bar{A} + (I - \bar{P}_d)] \quad ,$$

corresponding to their largest characteristic root (if higher than one).

The Maximum Number of Acceptable Solutions

Because there is not more than one equilibrium solution for each choice of vanishing region sets, we find that Property 3, concerning the maximum number of acceptable solutions, holds in the non-ZPG case.

Determining all Solutions of (39) in the Four Region System of the U.S.

The non-ZPG formulation of the nonlinear model applied to the four region system of the United States, thus offers ten possible equilibrium solutions characterized by one or two empty regions. The calculation of the characteristic roots and characteristic vectors corresponding to the ten $\bar{D}$ matrices which are possible to construct for this system reveals that:

- two of the four characteristic vectors containing a unique zero component are nonnegative, and
- all of the six characteristic vectors containing two zeros are nonnegative (see Table 21).

Table 21. Nonlinear model - non-ZPG case - U.S. regions - alternative stable equilibriums

Solution Number	λ	{y}	Solution Number	λ	{y}
(1)	1.00816	{0.05375, 0.35054, 0.49570, 0}	(6)	1.00758	{0.51279, 0, 0.48721, 0}
(2)	1.15054	{-52.27200, 31.81302, 0, 21.45898}	(7)	1.00790	{0.47233, 0, 0, 0.52767}
(3)	1.44917	{-121.33634, 0, 55.33747, 66.99887}	(8)	1.00850	{0, 0.54299, 0.45701, 0}
(4)	1.00921	{0, 0.18773, 0.43323, 0.37903}	(9)	1.00892	{0, 0.50305, 0, 0.49695}
(5)	1.00684	{0.46863, 0.53137, 0, 0}	(10)	1.00953	{0, 0, 0.45975, 0.54025}

Narrowing Down the Number of Acceptable Solutions

The equilibrium solutions determined above are again limiting distributions of the non-ZPG formulation of the nonlinear model if conditions concerning the regions assumed to vanish hold. We shall demonstrate that a nonnegative vector $\{y\}$ with k zero entries is an acceptable equilibrium solution only if the vector $\{\bar{z}\} = [\lambda \underset{\sim}{I} - (\underset{\sim}{\bar{P}}_d + \underset{\sim}{\bar{N}})]^{-1}\{i\}$ and the matrices $\bar{\bar{A}}$, $\bar{\bar{P}}_d$ and $\bar{\bar{N}}$, respectively, obtained from $\underset{\sim}{A}$, $\underset{\sim}{P}_d$ and $\underset{\sim}{N}$ by removing the (n - k) rows corresponding to the non-zero entries are such that

$$\underset{\sim}{\bar{\bar{A}}}\{\bar{z}\} - [\underset{\sim}{I} - \underset{\sim}{\bar{\bar{P}}}_d]\{i\} \leq [(\lambda - 1)\underset{\sim}{I} - \underset{\sim}{\bar{\bar{N}}}]\{i\} \quad .$$

Clearly, a necessary condition for any one region to become empty in the long run is that the sum of its net migration and natural increase be equal to or less than $(\lambda - 1)w_1(t)$ as t becomes large, i.e.,

$$w_1(t)[\sum_{j\neq 1} a_{j1}\alpha_j(t)] - (1 - p_{11})w_1(t) + n_1 w_1(t) \leq (\lambda - 1)w_1(t) \quad ,$$

for any finite value of $t > T$. Because $w_1(t)$ is strictly positive for any finite value of t, the above condition becomes

$$\sum_{j\neq 1} a_{j1}\alpha_j(t)w_j(t) - (1 - p_{11}) \leq (\lambda - 1 - n_1)$$

or in compact form,

$$\underset{\sim}{\bar{\bar{A}}\bar{\alpha}}(t)\{w(t)\} - (\underset{\sim}{I} - \underset{\sim}{\bar{\bar{P}}}_d)\{i\} \leq [(\lambda - 1)\underset{\sim}{I} - \underset{\sim}{\bar{\bar{N}}}]\{i\} \quad ,$$

in which the double bar indicates that the matrices $\underset{\sim}{\bar{\bar{A}}}$, $\underset{\sim}{\bar{\bar{P}}}_d$ and $\underset{\sim}{\bar{\bar{N}}}$ are sections of $\underset{\sim}{A}$, $\underset{\sim}{P}_d$ and $\underset{\sim}{N}$ obtained by removing the n - k rows corresponding to the zero entries of $\{y\}$.

As $t \to \infty, \underset{\sim}{\bar{\alpha}}(t)\{w(t)\} \to \{\bar{z}\} = \lambda[\underset{\sim}{\bar{A}}^{-1}\{i\}]_{dg} - [\underset{\sim}{\bar{A}}^{-1}(\underset{\sim}{\bar{P}}_d + \underset{\sim}{\bar{N}})\{i\}]_{dg}$, so that a necessary condition for any characteristic vector $\{y\}$ with zero entries to be an acceptable equilibrium solution is that:

$$\bar{\bar{A}}\{\bar{z}\} - (\underset{\sim}{I} - \underset{\sim}{\bar{\bar{P}}}_d)\{i\} \leq [(\lambda - 1)\underset{\sim}{I} - \underset{\sim}{\bar{\bar{N}}}]\{i\} \quad . \qquad (44)$$

In the case of our four region example, among the eight nonnegative characteristic vectors {y} derived above, only one (solution number (4) in Table 21) verifies condition (44). The multiregional system of the United States consisting of the four U.S. Census Regions gives a unique acceptable solution in which the North East region is empty and the other regions contain respectively 18.77 percent (North Central), 43.32 percent (South) and 37.90 percent (West) of the U.S. population.*

As in the ZPG case, it can be shown that the condition (44) is also sufficient and then the following property can be stated,

Table 22. Nonlinear model - non-ZPG case - U.S. regions - comparison of stable inmigration and outmigration rates relating to the vanishing regions

Solution number	$\bar{\bar{A}}\{\bar{z}\} - (I - \bar{\bar{P}}_d)\{i\}$	$(\lambda - 1)\{i\} - \{n\}$	Solution number	$\bar{\bar{A}}\{\bar{z}\} - (I - \bar{\bar{P}}_d)\{i\}$	$(\lambda - 1)\{i\} - \{n\}$
(1)	0.06870	-0.00187	(7)	0.06886 0.13874	0.00009 -0.00120
(4)	-0.00896	0.00322	(8)	0.01030 0.08618	0.00252 -0.00153
(5)	0.19018 0.20715	-0.00226 -0.00319	(9)	-0.00384 0.06244	0.00294 -0.00018
(6)	0.03240 0.07462	-0.00022 -0.00245	(10)	-0.00053 0.02069	0.00354 0.00173

*Again, note that if we project the future multiregional population of the U.S. using the non ZPG formulation of the nonlinear model based on 1965-70 data, we observe such a limiting distribution.

summarizing the results of this section:

Property 7

Necessary and sufficient conditions for an initially nonstationary n *region system, to offer one or several equilibrium distributions containing no more than* (n - 2) *empty regions are:*

(1) *that there exist one or several partial matrices* $\bar{A}_d$, $\bar{P}_d$ *and* $\bar{N}$ *obtained by removing* k ($0 \leq k \leq n - 2$) *columns and* k *rows of* A, P_d *and* N, *such that* $\{\bar{z}\} = \bar{A}^{-1}[\lambda I - (\bar{P}_d + \bar{N})]\{i\} > 0$ *(in which* λ, *the largest real characteristic root of the matrix*

$$\bar{D} = \bar{A}'^{-1}[\bar{A}^{-1}\{i\}]_{dg}^{-1}[[\bar{A}^{-1}(\bar{P}_d + \bar{N})\{i\}]_{dg}\bar{A}' + (I - \bar{P}_d)] \quad ,$$

is necessarily higher than one),

(2) *that the vectors* $\{\bar{z}\}$ *satisfy the condition*

$$\bar{\bar{A}}\{\bar{z}\} - [I - \bar{\bar{P}}_d]\{i\} \leq [(\lambda - 1)I - \bar{\bar{N}}]\{i\} \quad ,$$

expressing that, at equilibrium, regional net migration rates must be less than the difference between the common stable growth rate (λ - 1) *and the regional natural increase rate, and*

(3) *that* (n - k), *the number of nonvanishing regions, is not an even number higher than two. (This third condition is, however, not required if* $\bar{A}$ *is a symmetric matrix.)*

Like the ZPG formulation, the non-ZPG formulation does not necessarily yield a unique equilibrium solution as appears to be the case in the above example. Again, there might be several equilibrium distributions whose regional shares depend on the matrices A, P_d and N but not on the initial population $\{w(0)\}$; as suggested by numerical experiments, the initial population influences the stable equilibrium in that it determines which one of the acceptable equilibrium distributions will be reached. Also, note the possibility, as in the linear system, of a vanishing system if there exists no acceptable solution with a value of λ larger than one.

Thus, in our search for acceptable equilibrium solutions, we have set forth a methodology permitting us to determine a priori all acceptable equilibrium solutions of the non ZPG-case (see Appendix 4 for a formal and concise exposition of this methodology).

Particular Cases

Case of a Three Region System

In contrast to the ZPG case, the non-ZPG case does not lend itself to establishing the existence of at most one equilibrium solution. However, as in the ZPG case, it is possible to determine the equilibrium shares in the case of one region vanishing.

Let us suppose that the third region is empty at equilibrium. Then, we can simply express $\bar{\tilde{D}}$ in terms of $\bar{\tilde{A}}$, $\bar{\tilde{P}}$ and $\bar{\tilde{N}}$. Since

$$\bar{\tilde{A}} = \begin{bmatrix} 0 & a_{21} \\ a_{12} & 0 \end{bmatrix} \quad \text{and} \quad \bar{\tilde{A}}^{-1} = \begin{bmatrix} 0 & \frac{1}{a_{12}} \\ \frac{1}{a_{21}} & 0 \end{bmatrix}$$

we thus have:

$$\bar{\tilde{D}} = \begin{bmatrix} 0 & \frac{1}{a_{21}} \\ \frac{1}{a_{12}} & 0 \end{bmatrix} \begin{bmatrix} a_{12} & 0 \\ 0 & a_{21} \end{bmatrix} \left\{ \begin{bmatrix} \frac{p_{22}+n_2}{a_{12}} & 0 \\ 0 & \frac{p_{11}+n_1}{a_{21}} \end{bmatrix} \begin{bmatrix} 0 & a_{12} \\ a_{21} & 0 \end{bmatrix} + \begin{bmatrix} 1 - p_{11} & 0 \\ 0 & 1 - p_{22} \end{bmatrix} \right\}$$

which after simplification reduces to:

$$\underset{\sim}{\bar{D}} = \begin{bmatrix} p_{11} + n_1 & 1 - p_{22} \\ \\ 1 - p_{11} & p_{22} + n_2 \end{bmatrix} ;$$

λ is then the larger root of the following second-degree polynomial

$$\lambda^2 - \lambda(p_{11} + n_1 + p_{22} + n_2) + (p_{11} + n_1)(p_{22} + n_2) - (1 - p_{11})(1 - p_{22}) = 0 \quad ,^*$$

i.e.,

$$\lambda = \frac{1}{2}[(p_{11} + n_1 + p_{22} + n_2) + \sqrt{[(p_{11} + n_1) - (p_{22} + n_2)]^2 + 4(1 - p_{11})(1 - p_{22})}] \tag{45}$$

Since an equilibrium solution is given by

$$[\lambda - (p_{11} + n_1)]y_1 = [1 - p_{22}]y_2 \quad ,$$

and

$$[1 - p_{11}]y_1 = [\lambda - (p_{22} + n_2)]y_2 \quad ,$$

it follows by subtracting the second equation from the first that:

$$[\lambda - (1 + n_1)]y_1 = [1 + n_2 - \lambda]y_2 \quad , \tag{46}$$

---

*Since the discriminant $\Delta = (p_{11} + n_1 + p_{22} + n_2)^2 - 4[(p_{11} + n_1)(p_{22} + n_2) - (1 - p_{11})(1 - p_{22})] = [p_{11} + n_1 - (p_{22} + n_2)]^2 + 4(1 - p_{11})(1 - p_{22})$ and the sum of the roots $p_{11} + n_1 + p_{22} + n_2$ are positive, there exist two solutions, the higher of which is positive.

so that the normalized shares of regions 1 and 2 are, respectively,

$$y_1 = \frac{1 + n_2 - \lambda}{\lambda - (1 + \frac{n_1+n_2}{2})} ,$$

and

$$y_2 = \frac{\lambda - (1 + n_1)}{\lambda - (1 + \frac{n_1+n_2}{2})} ,$$

in which λ is given by (45).*

To summarize, if one region of a three region system (non-ZPG case) vanishes, the two other regions generally take on limiting shares directly proportional to the values of their net migration rates.

Case of a System in which Distance Has No Influence (Nonlinear Model II)

Again, $\underset{\sim}{A}$ is symmetric so that the general Properties 3, 6 and 7 are valid without the restriction attached to the number of regions in the system. Then, the four region system of the U.S. admits a maximum of 11 equilibrium distributions. Using the methodology set above, we have thus derived the eleven possible matrices $\underset{\sim}{\bar{F}}$, calculated their largest characteristic root using the aforementioned methodology, and determined the corresponding $\{y\}$ vectors.

---

*Note that if $\lambda = 1 + \frac{n_1+n_2}{2}$, (46) may be rewritten as $(n_2 - n_1)y_1 = (n_2 - n_1)y_2$ so that the equilibrium population is equally distributed between regions 1 and 2 [except if $n_2 = n_1 = n$ in which case the stable state is given as in the ZPG formulation by $(1 - p_{11})y_1 = (1 - p_{22})y_2$].

Table 23. Nonlinear model II - non ZPG case - U.S. regions - the stable equilibrium

$$\underset{\sim}{F} = \begin{bmatrix} 0.85980 & 0.04706 & 0.04706 & 0.04706 \\ 0.05427 & 0.86157 & 0.05427 & 0.05427 \\ 0.04620 & 0.04620 & 0.84961 & 0.84961 \\ 0.05468 & 0.05468 & 0.05468 & 0.85529 \end{bmatrix} \quad y = \begin{Bmatrix} 0.27766 \\ 0.18951 \\ 0.32137 \\ 0.20966 \end{Bmatrix} \quad \lambda = 1.00818$$

After verifying whether these equilibrium solutions meet the existence condition concerning vanishing regions, we found that only one equilibrium solution was an acceptable limiting distribution. Table 23 indicates that this limiting distribution is characterized by non-empty regions and that the regional population shares remain closer to the initial shares, as in the ZPG case, than in the full model.

CONCLUSION

This paper, devoted to the examination of the limiting distributions of alternative specifications of the interregional components-of-change model, made clear that retention probabilities must be independently determined to avoid the type of problems mentioned in Section I and illustrated by Appendix 1. It also demonstrated that the classic linear formulation (Rogers, 1968, Liaw, 1975) and the nonlinear formulation of McGinnis/Henry (1973), are close variants of the components-of-change model characterized by independently determined retention probabilities: they were labelled as "dual", because their point of departure stems from symmetric implementations of the constraint imposed by the independent choice of the retention probabilities.

The main contribution of this paper was to examine some of the long-term mathematical properties of the McGinnis/Henry model and to develop a methodology for determining a priori all the acceptable equilibrium solutions of the model. Unfortunately, we were not able to complete a proof of either the existence of an acceptable solution of the state equation or of the long-term convergence of the model. However, in consideration of the results of our numerous experiments with the model, it appears legitimate to accept the long-term convergence property of the model as granted and to leave its formal proof to mathematicians.

Contrasting the long-term properties of the linear and nonlinear models has revealed the less conservative character of the nonlinear model and the less favorable characteristics of its stable state, such as the occurrence of empty regions and the possible existence of more than one equilibrium distribution. Numerical experiments have also shown that, in the case of systems consisting of a large number of regions, the nonlinear model may display a few alternative equilibrium distributions, always characterized by a small number of nonvanishing regions (provided that the stable growth rate is positive).

These unfortunate long-term properties of the nonlinear model thus prevent its use as a substitute for the linear model, in order to gain insights into the dynamics of multiregional population systems.

Note that the nonlinear model examined in this paper as well as the linear model make use of data relating to a unique time period. Also following a suggestion of Vining Jr. (1975), we have examined generalizations of these models (especially the linear model) based on the observation of data during two consecutive time periods. Unfortunately, these generalized models can eventually lead to problems when projected indefinitely (see Appendix 3). Thus, at the present time, no model other than the classic linear model of population growth and distribution based on data for a single time period seems better suited for examining the dynamics of multiregional population systems.

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Appendix 1

Long-term Behavior of the Unconstrained Model

Starting from the general formulation of the components-of-change model (5), the most natural way to study its long-term behavior is to suppose that

$$\underset{\sim}{N}(t) = \underset{\sim}{N} \text{ and } \underset{\sim}{A}(t) = \underset{\sim}{A} \quad . \tag{A.1}$$

Equation (5) in which these assumptions are introduced makes it possible to calculate the regional population distribution at any future point in time.

An application of the resulting model to the four-region system of the U.S. based on 1965-70 data shows the existence of an equilibrium (in both the ZPG and non-ZPG cases) in which all regions except the West region are empty.

However, the application of the same model to other examples does not always lead to such an acceptable long-term behavior. It might well happen that the population of a region becomes negative or that the total migration out of a region is greater than the population of this region. This undesirable feature is indeed the consequence of the fact that retention probabilities are treated here as residuals and may thus take on inadequate values.

Problems associated with this feature are, in fact, well known and have been described in the biology literature in which appears a population model of interacting biological species, called the Volterra model, identical to the components-of-change model, defined by (5) and (A.1). (For an extensive review of the Volterra model of interacting populations, see Goel et al., 1971.)*

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*Goel, N., et al. (1971), On the Volterra and other Non-linear Models of Interacting Populations, *Reviews of Modern Physics*, 43, No. 2, Part I, 231-276.

We have, nevertheless, attempted to study the long-term behavior of the model embodied in (5) such that the assumption (A.1) holds. A demonstration similar to the one appearing in Section IV of this paper, leads to the conclusion that equilibrium distributions cannot contain an even number of non-empty regions (in the present case even if $n = 2$) and necessarily has an odd number of non-empty regions that may well be one.

In the case of a three-region system, it is clear that there are four alternative equilibrium solutions: three of them correspond to a concentration of the population, while the fourth one $\{y\}$ presents all non-zero entries. This population equilibrium $\{y\}$ is obtained as the solution of (4) in which $\{w(t-1)\} = \lambda\{w(t)\}$, i.e.,

$$\underset{\sim}{y}[\underset{\sim}{A} - \underset{\sim}{A}']\{y\} = [(\lambda - 1)\underset{\sim}{I} - \underset{\sim}{N}]\{y\} \quad ,$$

or by premultiplying by $\underset{\sim}{y}^{-1}$, which is possible since $\underset{\sim}{y}$ has no zero entry by assumption

$$[\underset{\sim}{A} - \underset{\sim}{A}']\{y\} = [(\lambda - 1)\underset{\sim}{I} - \underset{\sim}{N}]\{i\} \quad .$$

It can easily be established that in the non-ZPG case

$$\lambda = 1 + \frac{(a_{21} - a_{12})n_3 - (a_{31} - a_{13})n_2 + (a_{32} - a_{23})n_1}{(a_{21} - a_{12}) - (a_{31} - a_{13}) + (a_{32} - a_{23})}$$

in which n_1, n_2 and n_3 are the natural increase rates of each one of the regions of the system.

In the ZPG case ($\underset{\sim}{N} = 0$ and $\lambda = 1$), it is moreover possible to obtain the normalized equilibrium vector $\{y\}$ as:

$$\{y\} = \frac{1}{(a_{21} - a_{12}) - (a_{31} - a_{13}) + (a_{32} - a_{23})} \begin{Bmatrix} a_{32} - a_{23} \\ -(a_{31} - a_{13}) \\ a_{21} - a_{12} \end{Bmatrix}$$

Clearly, a necessary and sufficient condition for y to be stochastic (i.e. $0 < y_i < 1 \; \forall i$) is that $(a_{21} - a_{12})$, $-(a_{31} - a_{13})$ and $(a_{32} - a_{23})$ have the same sign. Noticing that these quantities are simply related to the initial net migration flows observed between each pair of regions, we can make the following conclusion. If one (and therefore each) region originally experiences a positive net inmigration with another region and a negative one with the third region, there exists an equilibrium solution characterized by $0 \leq (y_i)_n \leq 1$. (However, this equilibrium solution is not stable since $\{\frac{\partial^2 y}{\partial t}\} = [\underset{\sim}{A} - \underset{\sim}{A}'] \{i\}$ admits at least a strictly positive component.) If at least one region has net migration balances with the other two regions of the same sign, the stationary equilibrium solutions contain at least one entry $(y_i)_n$ such that $(y_i)_n \leq 0$.

In fact, numerical applications indicate that in both cases, if the iterative process is continued sufficiently long enough, diverging tendencies and negative populations are likely to appear.

Appendix 2

Outmigration and Inmigration Models

In the main body of this paper, we supposed that place-to-place migration flows were proportional to the product $a_{ij}w_i(t)w_j(t)$ and that the adjustment constant was either dependent on the origin or on the destination, which permitted us to derive the usual linear and nonlinear formulations as "dual" variants of the components-of-change model with independent retention probabilities.

Note that alternative models of population distribution can be formulated by considering other special cases of our components-of-change model: they are simply obtained by replacing either one of the two population variables of $a_{ij}w_i(t)w_j(t)$ by one. We thus derive two models that we label outmigration and inmigration models.

Outmigration Model

The substitution of one for $w_j(t)$ corresponds to the case in which $M_{ij}(t)$ is proportional to $a_{ij}w_i(t)$. Clearly, whenever the adjustment constant is related to the region of origin or destination, the place-to-place migration flows are to be expressed as

$$M_{ij}(t) = p_{ij}w_i(t) \qquad \forall i,j = 1,\dots,n \quad , \quad j \neq i \quad ,$$

in which p_{ij}, independent of time, is such that $\sum_{j\neq i} p_{ij}$ is equal to a predetermined value $1 - p_{ii}$.

Indeed, the outmigration model thus obtained is the classic linear model examined in Section I.

Inmigration Model

The substitution of one for $w_i(t)$ results in $M_{ij}(t)$ varying with the size of the destination population $w_j(t)$

$$M_{ij}(t) = [a_{ij}w_i(0)]w_j(t) = b_{ij}w_j(t) \quad . \tag{A.2}$$

Substituting (A.2) into the basic flow equation of the ZPG model, yields

$$w_i(t+1) = [1 + \sum_{j\neq i} b_{ji}]w_i(t) - \sum_{j\neq i} b_{ij}w_j(t)$$

$$\forall i = 1,2,\ldots,n \quad . \tag{A.3}$$

We can rewrite (A.3) in a more compact form as

$$\{w(t+1)\} = \underset{\sim}{Q}\{w(t)\} \quad ,^{*}$$

in which $\underset{\sim}{Q}$ is given by

$$\underset{\sim}{Q} = \begin{bmatrix} 1 + \sum_{j\neq 1} b_{ji} & -b_{12} & \cdots\cdots & -b_{1n} \\ -b_{21} & 1 + \sum_{j\neq 2} b_{j2} & & \\ \vdots & & & \\ -b_{1n} & & & 1 + \sum_{j\neq n} b_{jn} \end{bmatrix}$$

*$\underset{\sim}{Q}$, unlike $\underset{\sim}{P} = (p_{ij})$ in the outmigration case, is not a matrix of transition probabilities, although the column elements add up to one. The diagonal elements are higher than one and the off-diagonal elements are negative.

Clearly, a limiting distribution {y} of this model verifies

$$[\underset{\sim}{Q} - \underset{\sim}{I}]\{y\} = 0 \quad .$$

This equation yields a non-trivial solution, the entries of which are all strictly positive. However, the corresponding equilibrium is not stable. This can be seen from the fact that at least one component of $\frac{d^2}{dt}$ {w(t)} is positive.

$$\frac{d\{w(t)\}}{dt} = (\underset{\sim}{Q} - \underset{\sim}{I})\{w(t)\} \text{ so that } \frac{d^2\{w(t)\}}{dt} = \{\underset{\sim}{Q} - \underset{\sim}{I}\}\{i\} \quad .$$

Since $\{i\}(\underset{\sim}{Q} - \underset{\sim}{I}) = \{0\}$, the sum of all the components of $[\underset{\sim}{Q} - \underset{\sim}{I}]\{i\}$ is equal zero. Thus, $[\underset{\sim}{Q} - \underset{\sim}{I}]\{i\} \leq 0$ only if $[\underset{\sim}{Q} - \underset{\sim}{I}]\{i\} = 0$, which is possible only if $\underset{\sim}{Q}$ is symmetric: the steady state equation of the above model offers a strictly positive solution which does not constitute a stationary equilibrium.

In fact, the above model does not impose any restriction on the retention probabilities and therefore its iterative projection generally runs into the types of problems already encountered in Appendix 1.

A feasible inmigration model must then ensure that the number of outmigrants out of a region is less than the population of that region. We must then introduce into (A.2) an adjustment term depending on either the region of origin or the region of destination.

Suppose that $M_{ij}(t) = \alpha_i(t)a_{ij}w_j(t)$, then the imposition of independent retention probabilities implies that

$$\alpha_i(t) = \frac{1 - p_{ii}}{\sum\limits_{j \neq i} a_{ij}w_j(t)} w_i(t) \quad ,$$

so that the resulting pattern of population distribution is that of the nonlinear model of McGinnis/Henry (1973).

Alternatively, we can suppose $M_{ij}(t) = a_{ij}\beta_j(t)w_j(t)$ and the imposition of independent retention probabilities then implies that

$$\underset{\sim}{A}'\underset{\sim}{\beta}(t)\{w\} = (\underset{\sim}{I} - \underset{\sim}{P}_d)\{w\} \quad . \tag{A.4}$$

The flow equation of the ZPG model in matrix form becomes

$$\{w_{t+1}\} = \{w_t\} + \underset{\sim}{w}\,\underset{\sim}{\beta}(t)\underset{\sim}{A}\{i\} - (\underset{\sim}{I} - \underset{\sim}{P}_d)\{w\} \quad ,$$

whose steady state solution {y} is such that

$$\underset{\sim}{y}\underset{\sim}{\beta}(\infty)\underset{\sim}{A}\{i\} = (\underset{\sim}{I} - \underset{\sim}{P}_d)\{y\} \quad .$$

On supposing $\underset{\sim}{y} \neq \underset{\sim}{0}$, the result is that $\underset{\sim}{\beta}(\infty)$ is strictly positive as seen from

$$\{{}^1/\underset{\sim}{\beta}(\infty)\} = (\underset{\sim}{I} - \underset{\sim}{P}_d)^{-1}\underset{\sim}{A}\{i\} \quad , \tag{A.5}$$

and that

$$\{w(t+1)\} = \underset{\sim}{R}\,\{w(t)\} \quad , \tag{A.6}$$

in which:

$$\underset{\sim}{R} = \underset{\sim}{P}_d + \underset{\sim}{A}'[(\underset{\sim}{I} - \underset{\sim}{P}_d)^{-1}\underset{\sim}{A}\{i\}]_{dg} \quad . \tag{A.7}$$

Since $\underset{\sim}{R}$ is a matrix of transition probabilities the elements of its columns sum to one. This inmigration model - in which the adjustment accounting for independent retention probabilities is made by reference to the destination region results in a classical linear model in which the transition probability matrix $\underset{\sim}{R}$ is slightly different from the original transition matrix $\underset{\sim}{P}$.

Appendix 3

Long-term Behavior of the Population Distribution Model Described by Nonstationary Transition Probabilities and a Constant Causative Matrix

As an alternative approach to the linear model of population distribution, Vining Jr. (1975), suggests the use of a nonstationary Markov process with a constant causative matrix, recently developed in the context of consumer behavior (Lipstein 1965).* This appendix attempts to explore the feasibility of such an approach to deal with population growth and distribution.

Formulation of the Model in the ZPG Case

In Section I, the linear model of interregional population distribution was specified as

$$\{w(t + 1)\} = \underset{\sim}{P}(t)\{w(t)\} \quad , \tag{A.8}$$

in which the transition probability matrix was stationary, i.e.,

$$\underset{\sim}{P}(t) = \underset{\sim}{P}(0) \text{ for all } t \geq 0 \quad . \tag{A.9}$$

We suppose now that the transition probability matrix $\underset{\sim}{P}(t)$ in (A.8) satisfies

$$\underset{\sim}{P}(t + 1) = \underset{\sim}{P}(t)\ \underset{\sim}{C} \text{ for all } t > 1 \quad , \tag{A.10}$$

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*Lipstein, B. (1965), A Mathematical Model of Consumer Behavior, *Journal of Marketing Research*, No. 2, 259-65.

Vining Jr., D.A. (1975), The Spatial Distribution of Human Populations and its Characteristic Evolution over Time; some recent evidence from Japan, *Papers of the Regional Science Association*, 35, 157-180.

in which $\underset{\sim}{C}$ is a constant matrix.*

The corresponding Markov chain is said to have a constant causative matrix (Harary et al., 1970)**

If $\underset{\sim}{C} = \underset{\sim}{I}$, $\underset{\sim}{P}(t) = \underset{\sim}{P}(0)$ for all t and the transition probabilities are stationary (the underlying distribution process is then the one of the linear case). However, if $\underset{\sim}{C} \neq \underset{\sim}{I}$, the transition probabilities are nonstationary.

Clearly, letting $\underset{\sim}{P}(0) = \underset{\sim}{Q}$, we have (Harary et al., 1970)

$$\underset{\sim}{P}(t) = \underset{\sim}{Q}\ \underset{\sim}{C} \text{ for all } t \geq 0 \quad . \qquad \text{(A.11)}$$

Such a formula allows for an easy calculation of the successive regional allocations of any multiregional population system, as illustrated by the following example.

Example

On December 31st, 1970, Poland had 32,659,000 inhabitants among whom 2,518,700 resided in Warsaw. During the year 1971, 30184 persons left the capital and migrated to the rest of the country while 19,756 moved from the rest of the country to Warsaw. The result is that on December 31st, 1970, a resident of Warsaw had a probability of living, exactly one year later, in the rest of the country, equal to 30,184/2,518,700 = .01198 and in Warsaw equal to 1. - 0.01198 = 0.98802. Similar calculations for a resident of the rest of the country led to:

*The consideration of population growth due to natural increase is not necessary for the development of the arguments to follow. Instead of a right constant causative matrix, it is possible to introduce a left constant causative matrix. In general, the left and right causative matrices are different, so that the two corresponding sequences of {w(t)} are generally different. However, it can be shown that if one sequence tends towards a limit, the other tends towards the same limit. Since this paper focuses on limiting behavior, it is thus sufficient to use right causative matrices for the remainder of this section.

**Harary, F., et al. (1970), A Matrix Approach to Non-Stationary Chains, *Operations Research*, 1168-1181.

$$\underset{\sim}{P}(0) = \underset{\sim}{Q} = \begin{bmatrix} 0.98802 & 0.00066 \\ 0.01198 & 0.99934 \end{bmatrix}$$

The 1972 data allows us to calculate the transition matrix of the next period

$$\underset{\sim}{P}(1) = \begin{bmatrix} 0.98660 & 0.00064 \\ 0.01340 & 0.99936 \end{bmatrix}$$

and to obtain the causative matrix $\underset{\sim}{C}$

$$\underset{\sim}{C} = \underset{\sim}{P}(0)^{-1}P(1) = \begin{bmatrix} 0.99856 & -0.00001 \\ 0.00144 & 1.00001 \end{bmatrix}$$

Note that $\underset{\sim}{C}$ presents some entries either negative or greater than 1, i.e., $\underset{\sim}{C}$ is not stochastic.

Then, application of the Formula (A.11) to the successive values of $\{w(t)\}$ makes it possible to derive the successive regional population allocations that the above nonstationary Markov chain implies.

Table A1 indicates that the part of the Polish population living in Warsaw tends to diminish and ultimately become equal to zero (such a result is obtained for year 2039), if the process described by (A.8) and (A.10) is maintained over time. In contrast, the stationary model, whose forecasts appear in the same table, displays a similar but more moderate decreasing tendency for the population of Warsaw which ultimately reaches a constant share of the Polish population (5.19 percent versus 7.71 initially).

Table A1: Stationary and nonstationary distribution models compared: percentage of total population residing in Warsaw in successive periods

Period	Stationary	Nonstationary	Period	Stationary	Nonstationary
1	7.71	7.71	25	7.05	4.42
2	7.68	7.68	30	6.93	3.55
3	7.65	7.63	35	6.83	2.74
4	7.62	7.58	40	6.72	2.03
5	7.59	7.43	45	6.63	1.44
6	7.56	7.34	50	6.54	0.96
7	7.53	7.24	55	6.46	0.59
8	7.50	7.14	60	6.38	0.31
9	7.47	7.02	65	6.31	0.10
10	7.44	6.90	68	6.27	<0
15	7.30	6.16	70	6.24	
20	7.17	5.31	∞	5.19	

Long-term Behavior in the ZPG Case

As illustrated in the above example, the nonstationary model does not converge towards a stochastic vector in all circumstances. Its convergence has been studied by Lipstein (1968),* Harary et al. (1970) and more recently by Pullman/Styan (1973).**

If one denotes by $\underset{\sim}{T}(t)$ the transition probability from the initial period t, the result is that:

$$\underset{\sim}{T}(t) = \prod_{x=0}^{t} \underset{\sim}{P}(x) = \prod_{x=0}^{t} (\underset{\sim}{Q}\ \underset{\sim}{C}^{t}) \quad . \qquad \text{(A.12)}$$

*Lipstein, B. (1968), Best Marketing a Perturbation in the Market Place, *Management Science*, Series B, 14, 437-48.

**Pullman, N., and P.H. Styan. (1973), The Convergence of Markov Chains with Non-stationary Transition Probabilities and Constant Causative Matrix, *Stochastic Processes and their Applications*, 279-85.

Then the limiting properties of the nonstationary process are linked to the convergence, as $t \to \infty$, of $\underset{\sim}{C}^t$ which itself depends on the characteristics of the matrix $\underset{\sim}{C}$. Harary et al. (1970) showed that $\underset{\sim}{C}$ has a characteristic root of unity and that, if all other roots were less than one in absolute value, $\underset{\sim}{C}^t$ converges to $\{1\}\{i\}'$ in which $\{1\}$ is the right hand characteristic vector of $\underset{\sim}{C}$ corresponding to the unit characteristic value and $\{i\}'$ a row vector of ones (the left-hand characteristic vector of $\underset{\sim}{C}$ for the same unit characteristic value. Lipstein (1968), suggested that, in such a case, $\underset{\sim}{T}(t)$ would also converge to $\{1\}\{i\}'$. This is true only for a stochastic $\underset{\sim}{C}$, however, if $\underset{\sim}{C}$ is not stochastic, it might happen that $\underset{\sim}{T}(t)$ and $\underset{\sim}{C}^t$ have the same limit. Lipstein (1968) proved this for two state chains and later Pullman and Styan (1973) proved it for chains with more states. Note that, in such circumstances $\underset{\sim}{T}(t)$ converges to $\{1\}\{i\}'$ so rapidly that $\sum_t \| \underset{\sim}{T}(t) - \{1\}\{i\}' \|$ converges.

The aforementioned authors seem to concentrate on the long-term behavior of $\underset{\sim}{C}^t$ and $\underset{\sim}{T}(t)$, and ignore the one of $\{w(t)\}$. We note that, if it exists, the limiting distribution $\{y\}$ of $\{w(t)\}$ is given by:

$$[\underset{\sim}{C} - \underset{\sim}{I}]\{y\} = \{0\} \quad , \qquad \text{(A.13)}$$

an equality which results from the comparison of

$$\{w(t+1)\} = \underset{\sim}{P}(t)\{w(t)\} = \underset{\sim}{P}(t-1)\ \underset{\sim}{C}\ \{w(t)\} \quad ,$$

and

$$\{w(t)\} = \underset{\sim}{P}(t-1)\{w(t-1)\} \quad .$$

When stability is reached

$$\{w(t+1)\} = \{w(t)\} = \{w(t-1)\} = \{y\} \quad .$$

Clearly, the limiting distribution {y} is the right characteristic vector {l} of $\underset{\sim}{C}$ corresponding to the unit characteristic root. As in the stationary case, it is independent of the initial conditions and only depends on the elements of $\underset{\sim}{C}$. However, in contrast to the stationary case in which the limiting distribution vector is stochastic, the vector {l} might be nonstochastic (if $\underset{\sim}{C}$ is not stochastic).

To summarize,

1. if $\underset{\sim}{C}$ is stochastic, $\underset{\sim}{C}^t$ is stochastic, $\underset{\sim}{T}(t)$ converges, and {w(t)} tends toward a stochastic limiting vector {l} defined as the right-hand characteristic vector of $\underset{\sim}{C}$ corresponding to its unit characteristic root. Moreover this vector is independent of the initial distribution of population; and

2. if $\underset{\sim}{C}$ is not stochastic, $\underset{\sim}{C}^t$ might be either

   i) stochastic in which case $\underset{\sim}{T}(t)$ converges and {w(t)} tends toward the right-hand characteristic vector of $\underset{\sim}{C}$ (which is not necessarily stochastic), or

   ii) nonstochastic in which case $\underset{\sim}{T}(t)$ might not converge.

Appendix 4

Search for Acceptable Equilibrium Solutions of the Nonlinear Model

1. The ZPG Case

As a first step, we determine all the solutions of the steady-state equation that appears on page 29.

For each possible set of vanishing regions (there are $2^n - (n + 1)$ sets if $\underset{\sim}{A}$ is not symmetric, $2^{n-1} + \frac{n(n - 3)}{2}$ if $\underset{\sim}{A}$ is symmetric as indicated by Property 3), we calculate the matrix:

$$\bar{\underset{\sim}{C}} = [\bar{\underset{\sim}{A}}^{-1}[\underset{\sim}{I} - \bar{\underset{\sim}{P}}_d]\{i\}]_{dg}(\underset{\sim}{I} - \bar{\underset{\sim}{P}}_d)^{-1}\bar{\underset{\sim}{A}}' \quad ,$$

in which $\bar{\underset{\sim}{A}}$ and $\bar{\underset{\sim}{P}}_d$ are submatrices of $\underset{\sim}{A}$ and $\underset{\sim}{P}_d$ obtained by removing the k rows and k columns corresponding to the vanishing regions. Since $\bar{\underset{\sim}{C}}$ admits one as a real characteristic root, the corresponding characteristic vector $\{\bar{y}\}$ can be simply obtained by solving for each set of vanishing regions, the vector equation:

$$[\bar{\underset{\sim}{C}} - \underset{\sim}{I}]\{\bar{y}\} = \{0\} \quad ,$$

whose solution is then normalized (scaled such that $\sum \bar{y}_i = 1$).

Once all solutions of the steady state equation have been determined we derive for each set of vanishing regions leading to a positive vector $\{\bar{y}\}$, the vectors of inmigration rates $\{\overline{\overline{in}}\}$:

$$\{\overline{\overline{in}}\} = \bar{\bar{\underset{\sim}{A}}}\ \bar{\underset{\sim}{A}}^{-1}[\underset{\sim}{I} - \bar{\underset{\sim}{P}}_d]\{i\} \quad ,$$

in which $\bar{\bar{\underset{\sim}{A}}}$ is a submatrix of $\underset{\sim}{A}$ obtained by removing the (n - k) rows corresponding to the nonvanishing regions. Those are then compared with the corresponding vectors of outmigration rates

$$\{\overline{\overline{out}}\} = [\underset{\sim}{I} - \underset{\sim d}{\overline{\overline{P}}}]\{i\} \quad .$$

Finally, the acceptable equilibrium solutions are those solutions of the steady-state equation such that $\{\overline{\overline{in}}\} < \{\overline{\overline{out}}\}$.

2. The Non-ZPG Case

Again, as a first step, we determine all the solutions of the state equation that appears on page 51.

For each possible set of vanishing regions (again there are $2^n - (n + 1)$ sets if $\underset{\sim}{A}$ is not symmetric, $2^{n-1} + \frac{n(n - 3)}{2}$ if $\underset{\sim}{A}$ is symmetric), we calculate the matrix:*

$$\underset{\sim}{\bar{F}} = [\underset{\sim}{\bar{A}}^{-1}(\underset{\sim d}{\bar{P}} + \underset{\sim}{\bar{N}})\{i\}]_{dg}[\underset{\sim}{\bar{A}}^{-1}\{i\}]_{dg}^{-1} + (\underset{\sim}{I} - \underset{\sim d}{\bar{P}})\underset{\sim}{\bar{A}}'^{-1}\{i\}]_{dg}^{-1} \quad .$$

Then, we compute the successive powers of $\bar{F}$, determining at each iteration the ratio $\lambda^{(n)}$ of the sum of the elements of the first column in the $(n + 1)^{th}$ and the n^{th} iterations:

$$\lambda^{(n)} = \frac{\sum_{j=1}^{n} \bar{f}_{ji}^{(n+1)}}{\sum_{j=1}^{n} \bar{f}_{ji}^{(n)}} \quad .$$

As n becomes large, $\lambda^{(n)}$ converges to the largest characteristic root of $\bar{F}$. We then obtain the value of λ when the iteration process leads to unchanged values of $\lambda^{(n)}$. In practice the iteration was stopped when

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*The algorithm used here relies on the calculation of matrices $\underset{\sim}{\bar{F}}$ different from the matrices $\underset{\sim}{\bar{D}}$ put forward in the main body of this paper. The rationale for this is in the fact that $\underset{\sim}{\bar{F}}$, which is simply related to $\underset{\sim}{\bar{D}}$, by the same characteristic roots, and characteristic vectors, permits an easier and more rapid calculation than if the algorithm is based on the use of the matrices $\underset{\sim}{\bar{D}}$.

$$- E < \frac{\sum_{j=1}^{n} f_{j2}^{(n+1)}}{\sum_{j=1}^{n} f_{ji}^{(n+1)}} - \frac{\sum_{j=1}^{n} f_{j2}^{(n)}}{\sum_{j=1}^{n} f_{ji}^{(n)}} < E \quad ,$$

with E = 0.000001.

The right characteristic vector of $\bar{\tilde{F}}$ associated with λ is proportional to any column of $\bar{\tilde{F}}^n$ for n large. At the end of the iteration process, we pick the first column of $\bar{\tilde{F}}^n$ as right characteristic vector, say $\{\bar{f}^{(n)}\}_1$, and obtain the vector $\{\bar{y}\}$ containing the non zero elements of the equilibrium distribution {y} from

$$\{\bar{y}\} = \bar{\tilde{A}}'^{-1}[\bar{\tilde{A}}^{-1}\{i\}]_{dg}\{\bar{f}^{(n)}\}_1 \quad .$$

For convenience, $\{\bar{y}\}$ is then scaled so that $\sum \bar{y}_i = 1$. Once the solutions of the state equation have been calculated, we derive, for each set of vanishing regions leading to a positive vector $\{\bar{y}\}$, the vectors of net migration rates from:

$$\{\overline{\overline{net}}\} = \bar{\bar{\tilde{A}}}[\lambda[\bar{\tilde{A}}^{-1}\{i\}]_{dg} - [\tilde{A}^{-1}[\bar{\bar{\tilde{P}}}_d + \bar{\bar{\tilde{N}}}]\{i\}]_{dg}\{i\} - (\tilde{I} - \bar{\bar{\tilde{P}}}_d)]\{i\} \quad ,$$

as well as the vectors $[(\lambda - 1)\tilde{I} - \bar{\bar{\tilde{N}}}]\{i\}$. Finally, the acceptable equilibrium solutions are those solutions of the state equation such that $\{\overline{\overline{net}}\} < [(\lambda - 1)\tilde{I} - \bar{\bar{\tilde{N}}}]\{i\}$.